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POSITIVE SOLUTIONS FOR PSEUDO-LAPLACIAN EQUATIONS WITH CRITICAL SOBOLEV EXPONENTS

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ABSTRACT. A sufficient condition for pseudo-Laplacian equations involving critical Sobolev exponents to have positive solutions is established.

1. Introduction

In this paper we deal with the existence of positive solutions of the quasilinear elliptic equation

$$\begin{aligned} &\text{-div}(|Du|^{p-2}Du) = Q(x)|u|^{p^*-2}u + f(x,u) \quad \text{in} \quad \Omega \\ (1) \qquad &u > 0 \quad \text{in } \Omega \\ &u = 0 \quad \text{on } \partial\Omega \end{aligned}$$

where Ω is a bounded open subset of R^N, $1 < p < N$, $p^* = Np/N-p$, the function $f(x,u) : \Omega \times [0, +\infty) \to R$, $Q(x) \geq 0$ is a bounded measurable function in Ω and $Q(x)$ satisfy the following property:

PROPERTY (P). *There is a maximum point x_0 of $Q(x)$ in Ω such that $|Q(x) - Q(x_0)| = o(|x-x_0|^{p-1})$ as $x \to x_0$.*

In [1], we studied for $Q(x) = 1$. Later ,we introduce the new normalized function different from the normalized function used in [1] to show the existence of solution. Also we make the following assumptions:

$$(2) \qquad f(x,u) : \Omega \times [0,\infty) \to R \quad \text{is measurable in } x,$$

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(3) continuous in u and $\sup_{\substack{x\in\Omega \\ 0\le u\le M}} |f(x,u)| < \infty$ for every $M > 0$;

(4) $$f(x,u) = a(x)|u|^{p-2}u + g(x,u)$$

(5) $$\text{with } a(x) \in L^\infty(\Omega),$$

(6) $$g(x,u) = o(u^{p-1}) \text{ as } u \to 0^+, \text{ uniformly in } x,$$

(7) $$g(x,u) = o(u^{p^*-1}) \text{ as } u \to +\infty, \text{ uniformly in } x;$$

the operator $-\triangle_p u - a(x)|u|^{p-2}u$ has its smallest positive eigenvalue, that is,

(8) $$\int |\bigtriangledown \phi|^p - a(x)\phi^p \ge \alpha \int \phi^p \text{ for all } \phi \in W_0^{1,p}(\Omega), \alpha > 0$$

or equivalently

(9) $$\int |\bigtriangledown \phi|^p - a(x)\phi^p \ge \alpha' \int |\bigtriangledown \phi|^p \text{ for all } \phi \in W_0^{1,p}(\Omega), \alpha' > 0.$$

Since the value of $f(x,u)$ for $u < 0$ is irrelevant, we may define

(10) $$f(x,u) = 0 \text{ for all } x \in \Omega,\ u \le 0.$$

Set

(11) $$F(x,u) = \int_0^u f(x,t)dt \quad \text{for all } x \in \Omega, u \in R$$

and

(12)

$$\psi(u) = \int \frac{1}{p}|\bigtriangledown u|^p - \int \frac{Q(x)}{p^*}|u|^{p^*} - \int F(x,u) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

In fact, the solutions of (1) correspond to the critical point of $\psi(u)$. Since p^* is the critical Sobolev exponent corresponding to the non-compact embedding of $W_0^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$, the functional ψ does not, in general, satisfy the Palais-Smale condition and it is not possible to obtain critical points of ψ via simple variational arguments. Thus we rely on a variant of the mountain pass theorem of Ambrosetti and Rabinowiz without the (PS) condition [3]. The main result in this paper is the following:

MAIN THEOREM. *Assume that the conditions* (2)-(8) *be satisfied and suppose that there exists some* $v_0 \in W_0^{1,p}(\Omega)$, $v_0 \geq 0$ *on* Ω, $v_0 \not\equiv 0$ *such that*

$$\sup_{t\geq 0} \psi(tv_0) < \frac{1}{N} \frac{S^{\frac{N}{p}}}{(\max \quad Q(x))^{\frac{N-p}{p}}}. \tag{13}$$

Then problem (1) *possesses a solution.*

Here S is the best constant for the Sobolev imbedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$.

2. Proof of main theorem

Using (4)-(7) we may fix a constant $\mu \geq 0$ so large that

$$-f(x,u) \leq \mu u^{p-1} + u^{p^*-1} \tag{14}$$

for almost all $x \in \Omega$ and for all $u \geq 0$. In case $f(x,u) \geq 0$ for all $u \geq 0$, we may choose $\mu = 0$. On $E = W_0^{1,p}(\Omega)$ we define

$$\Phi(u) = \int \left\{ \frac{1}{p} |\nabla u|^p + \frac{\mu}{p}|u|^p - \frac{Q(x)}{p^*}(u^+)^{p^*} - F(x,u^+) - \frac{1}{p}\mu(u^+)^p \right\}.$$

Then Φ is C^1 on E. First, we must prove that Φ satisfies the conditions of the mountain pass theorem of Ambrosetti and Rabinowitz without the (PS) condition [3]. It is stated that for C^1 function Φ on a Banach space E, if Φ satisfies the following two conditions:

(CON1) there exist a neighborhood U of 0 in E and a constant ρ such that $\Phi(u) \geq \rho$ for every u in the boundary of U,

(CON2) $\Phi(0) < \rho$ and $\Phi(v) < \rho$ for some $v \notin U$.

Set

$$C = \operatorname*{Inf}_{P\in\mathcal{P}} \max_{w\in P} \Phi(w) \geq \rho$$

where $\mathcal{P}$ denotes the class of continuous paths joining 0 to v. Then there is a sequence(u_j) in E such that

$$\Phi(u_j) \to C \quad \text{and} \quad \Phi'(u_j) \to 0 \quad \text{in} \quad E^*.$$

PROOF OF (CON1). It follows from condition (6) that for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$g(x,u) \le \varepsilon u^{p-1} \text{ for almost all } x \in \Omega \text{ and for all } 0 \le u \le \delta.$$

Thus from (7), we obtain

$$g(x,u) \le \varepsilon u^{p-1} + Cu^{p^*-1} \text{ for almost all } x \in \Omega \text{ and for all } u \ge 0.$$

and for some constant C (depending on ε). Therefore we have

$$F(x,u) \le \frac{1}{p}a(x)u^p + \frac{\varepsilon}{p}u^p + \frac{C}{p^*}u^{p^*} \tag{15}$$

for almost all $x \in \Omega$ and for all $u \ge 0$. Hence we can easily see that for all $u \in W_0^{1,p}(\Omega)$,

$$\Phi(u) \ge \int \left\{ \frac{1}{p}|\nabla u|^p - \frac{a(x)}{p}(u^+)^p - \frac{\varepsilon}{p}(u^+)^p - \frac{C+1}{p^*}Q(x)(u^+)^{p^*} \right\}$$

Using (9) and the fact that $\int|\nabla u|^p = \int|\nabla u^+|^p + \int|\nabla u^-|^p$ we conclude that (with ε small enough) there exist constants ρ and a neighborhood U of 0 in $W_0^{1,p}(\Omega)$ such that

$$\begin{aligned}\Phi(u) &\ge \int \frac{1}{p}|\nabla u^+|^p + \frac{1}{p}|\nabla u^-|^p - \frac{a(x)}{p}(u^+)^p - \frac{C+1}{p^*}Q(x)(u^+)^{p^*} \\ &\ge K\|u\|^p_{W_0^{1,p}} - C'\|u\|^{p^*}_{W_0^{1,p}} > \rho \\ &\text{for every } u \text{ in the boundary of } U. \end{aligned}$$

□

PROOF OF (CON2). For any $u \in W_0^{1,p}(\Omega)$, $u \ge 0$, $u \not\equiv 0$, we have by (7) $\lim_{t\to\infty}\Phi(tu) = -\infty$. Thus there are many $v's$ satisfying $\Phi(0) < \rho$ and $\Phi(v) < \rho$ for some $v \notin U$. However, it will be important for later purposes to use Theorem [3] with a special v, namely $v = t_0v_0$ where v_0 is given by (13) and $t_0 > 0$ is chosen so large that $v \notin U$ and $\Phi(v) \le 0$. It follows from (13) that

$$\sup_{t\ge 0}\Phi(tv) < \frac{1}{N}\frac{S^{\frac{N}{p}}}{(\max \quad Q(x))^{\frac{N-p}{p}}}$$

and therefore we have

$$C < \frac{1}{N} \frac{S^{\frac{N}{p}}}{(\max \quad Q(x))^{\frac{N-p}{p}}}. \tag{16}$$

Applying Theorem [3], we obtain a sequence (u_j) in $W_0^{1,p}(\Omega)$ such that $\Phi(u_j) \to C$ and $\Phi'(u_j) \to 0$ in $W^{-1,p'}(\Omega)$ $(\frac{1}{p} + \frac{1}{p'} = 1)$, that is,

$$\begin{aligned} &\int \left\{ \frac{1}{p} | \nabla u_j|^p + \frac{\mu}{p}|u_j|^p - \frac{1}{p^*} Q(x)(u_j^+)^{p^*} - F(x, u_j^+) - \frac{\mu}{p}(u_j^+)^p \right\} \\ &= C + O(1) \end{aligned} \tag{17}$$

and

$$\begin{aligned} &- \triangle_p u_j + \mu |u_j|^{p-2} u_j - Q(x)(u_j^+)^{p^*-1} - f(x, u_j^+) - \mu(u_j^+)^{p-1} \\ &= \xi_j \quad \text{with} \quad \xi_j \to 0 \quad \text{in} \quad W^{-1,p'}(\Omega). \end{aligned} \tag{18}$$

We claim that

$$\|u_j\|_{W_0^{1,p}} \le C. \tag{19}$$

Indeed, multiply (18) by u_j, we obtain

$$\int \{ |\nabla u_j|^p + \mu |u_j|^p - Q(x)(u_j^+)^{p^*} - f(x, u_j^+)u_j^+ - \mu (u_j^+)^p \} = \langle \xi_j, u_j \rangle. \tag{20}$$

Taking - $\frac{1}{p}\times$(20) + (17), we obtain

$$\begin{aligned} \frac{1}{N} \int Q(x)(u_j^+)^{p^*} \le &\left\{ \int \{ F(x, u_j^+) - \frac{1}{p} f(x, u_j^+) u_j^+ \right\} + C + O(1) \\ &+ \|\xi_j\|_{W^{-1,p'}} \|u_j\|_{W_0^{1,p}}. \end{aligned} \tag{21}$$

On the other hand, from (7) we have for all $\varepsilon > 0$, there exists a constant C such that

$$|f(x, u)| \le \varepsilon u^{p^*-1} + C \tag{22}$$

for almost all $x \in \Omega$ and for all $u \geq 0$. So

$$|F(x,u)| \leq \frac{\varepsilon}{p^*} u^{p^*} + Cu \tag{23}$$

for almost all $x \in \Omega$ and for all $u \geq 0$. We deduce from (21)-(23) (with ε small enough)that

$$\int Q(x)(u_j^+)^{p^*} \leq C + C\|u_j\|_{W_0^{1,p}}. \tag{24}$$

Combining (17) and (24), we obtain (19).

Extract a subsequence, still denoted by u_j, so that

$$\begin{aligned}
&u_j \rightharpoonup u \ \text{ weakly in } W_0^{1,p}(\Omega),\\
&u_j \to u \ \text{ strongly in } L^q \ \text{ for all } 1 < q < p^*,\\
&u_j \to u \ \text{ a.e on } \Omega,\\
&(u_j^+)^{p^*-1} \rightharpoonup (u^+)^{p^*-1} \ \text{ weakly in } (L^{p^*})' = L^{\frac{p^*}{p^*-1}},\\
&f(x,u_j^+) \rightharpoonup f(x,u_j^+) \ \text{ weakly in } (L^{p^*})' = L^{\frac{p^*}{p^*-1}},\\
&-\triangle_p u_j \rightharpoonup -\triangle_p u \ \text{ weakly in } W^{-1,p'}.
\end{aligned}$$

Passing to the limit in (18), we obtain
(25)

$$-\triangle_p u + \mu|u|^{p-2}u = Q(x)(u^+)^{p^*-1} + f(x,u^+) + \mu(u^+)^{p-1} \ \text{ in } W^{-1,p'}(\Omega).$$

We deduce from (14) and (25) in which the right-hand side is greater than or equal to 0 and from the Vazquez maximum principle [13], that $u \geq 0$ on Ω and u satisfies

$$-\triangle_p u = Q(x)|u|^{p^*-2}u + f(x,u).$$

We shall now verify that $u \not\equiv 0$ (and consequently $u > 0$ on Ω by the strict Vazquez maximum principle). Indeed, suppose that $u \equiv 0$ we claim that

$$\int f(x,u_j^+)u_j^+ \to 0, \tag{26}$$

$$\int F(x, u_j^+) \to 0. \tag{27}$$

From (22), (23), we deduce that

$$\left|\int f(x, u_j^+)u_j^+\right| \le \varepsilon \int (u_j^+)^{p^*} + C \int u_j^+$$

$$\left|\int F(x, u_j^+)\right| \le \frac{\varepsilon}{p^*} \int (u_j^+)^{p^*} + C \int u_j^+.$$

Since u_j remains bounded in L^{p^*} and $u_j \to 0$ in L^2, we obtain (26) and (27). Extracting still another sequence, we may assume that

$$\int |\nabla u_j|^p \to \ell \tag{28}$$

for some constant $\ell \ge 0$. Passing to the limit in (20) and then in (17), we obtain

$$\int Q(x)(u_j^+)^{p^*} \to \ell \tag{29}$$

and

$$\frac{1}{N}\ell = C. \tag{30}$$

On the other hand, from (29), since $\max_{\Omega} Q(x) \int (u_j^+)^{p^*} \ge \ell$ for sufficiently large j, we have

$$\| \nabla u_j\|_{L^p}^p \ge S\|u_j\|_{L^{p^*}}^p \ge S\|u_j^+\|_{L^{p^*}}^p.$$

Hence, using (28) and (29) we find in the limit

$$\ell \ge \frac{S\ell^{\frac{p}{p^*}}}{(\max \quad Q(x))^{\frac{p}{p^*}}}. \tag{31}$$

From (30) and (31) we deduce that

$$C \ge \frac{1}{N}\frac{S^{\frac{N}{p}}}{(\max \quad Q(x))^{\frac{N-p}{p}}}.$$

This contradicts (16). Thus $u \not\equiv 0$. □

Lemma below furnishes the assumption under which the crucial condition (13) of main theorem holds.

LEMMA. *Assume that $f(x,u)$ satisfies the conditions (2)-(8) and $Q(x)$ satisfies property (P). Also suppose that there is a function $f(u)$ such that*

$$f(x,u) \geq f(u) \geq 0 \text{ for almost all } x \in w \text{ and for all } u \geq 0 \tag{32}$$

where w is some nonempty open set in Ω and the primitive $F(u) = \int_0^u f(t)dt$ satisfies

$$\lim_{\varepsilon \to 0} \varepsilon^{\frac{pN-2N+p}{p}} \int_0^{\varepsilon^{-\frac{p-1}{p}}} F\left[\left(\frac{\varepsilon^{\frac{1-p}{p}}}{1+s^{\frac{p}{p-1}}}\right)^{\frac{N-p}{p}}\right] s^{N-1} ds = \infty. \tag{33}$$

Then the condition (13) holds.

To prove the Lemma, we need the following estimates (39)-(43). Without loss of generality, we can assume that $0 \in \Omega$ and that $\max_{\Omega} Q(x) = Q(0)$. Let us define for $\lambda \in R$

$$S_\lambda = \inf\left\{\int_\Omega (|Du|^p - \lambda |u|^p)dx : \ u \in W_0^{1,p}(\Omega), \ \int_\Omega |u|^{p^*} dx = 1\right\}. \tag{34}$$

Then

$$S_0 = S = \inf\left\{\int_\Omega |Du|^p dx : u \in W_0^{1,p}(\Omega), \ \int_\Omega |u|^{p^*} dx = 1\right\} \tag{35}$$

is the best constant for the Sobolev imbedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$. Note that S is independent of Ω and is never achieved for a bounded domain of R^N. When Ω is replaced by R^N, then S is achieved by the function

$$U_a(x) = \left(Na\left(\frac{N-p}{p-1}\right)^{p-1}\right)^{\frac{N-p}{p^2}} \left(a + |x|^{\frac{p}{p-1}}\right)^{\frac{p-N}{p}} \tag{36}$$

for some $a > 0$. Assume $0 \in w$ and fix a function $\phi \in C_0^\infty(\Omega)$, $0 \leq \phi \leq 1$ and $\phi(x) \equiv 1$ in some neighborhood of 0.

Set

$$u_\varepsilon(x) = \frac{\phi(x)}{(\varepsilon + |x|^{\frac{p}{p-1}})^{N-p/p}} \tag{37}$$

and

$$v_\varepsilon(x) = \frac{u_\varepsilon(x)}{\|Q(x)^{\frac{1}{p^*}} u_\varepsilon\|_{L^{p^*}}}. \tag{38}$$

We claim that v_ε satisfies the condition (13) for $\varepsilon > 0$ sufficiently small. To show this, we need to estimate the following: As $\varepsilon \to 0$,

$$\| \nabla u_\varepsilon\|_{L^p}^p = K_1 \varepsilon^{\frac{p-N}{p}} + O(1), \quad K_1 = \left(\frac{N-p}{p-1}\right)^p \| \nabla u_1(x)\|_{L^p}^p, \tag{39}$$

$$\int_\Omega Q(x)|u_\varepsilon|^{p^*} dx = o\left(\varepsilon^{\frac{p^*-p-N}{p}}\right) + K_2 Q(0)\varepsilon^{-\frac{N}{p}} + O(1), \quad K_2 = \|u_1\|_{L^{p^*}}^{p^*}, \tag{40}$$

$$\| \nabla v_\varepsilon\|_{L^p}^N = Q(0)^{-\frac{N}{p^*}} S^{\frac{N}{p}} + o\left(\varepsilon^{\frac{N-p}{p}}\right), \tag{41}$$

$$\|v_\varepsilon\|_{L^p}^p = \frac{K_3}{K_2}\varepsilon^{p-1} + o\left(\varepsilon^{\frac{np-p-n}{n}}\right), \quad K_3 = \|u_1\|_{L^{p^*}}^p \text{ if } 1 < p^2 < N, \tag{42}$$

$$\|v_\varepsilon\|_{L^p}^p = \frac{K_4}{K_2}\varepsilon^{p-1}|\log\, \varepsilon| + O\left(\varepsilon^{\frac{p^2-p-1}{p}}\right)|\log\varepsilon| \text{ if } p^2 = N. \tag{43}$$

PROOF OF (39).

$$\nabla u_\varepsilon(x) = \frac{\nabla\phi(x)}{\left(\varepsilon + |x|^{\frac{p}{p-1}}\right)^{N-p/p}} + \frac{p-N}{p-1}\frac{x\phi(x)}{\left(\varepsilon + |x|^{\frac{p}{p-1}}\right)^{N/p}|x|^{\frac{p-2}{p-1}}}.$$

As $\phi \equiv 1$ is in some neighborhood of 0,

$$\int_\Omega |\nabla u_\varepsilon(x)|^p dx = \left(\frac{p-N}{p-1}\right)^p \int_\Omega \frac{|x|^{\frac{p}{p-1}}\phi(x)^p}{\left(\varepsilon + |x|^{\frac{p}{p-1}}\right)^N} dx + O(1).$$

Writing $\phi(x)^p = 1 + \phi(x)^p - 1$, we obtain

$$\int_\Omega |\nabla u_\varepsilon(x)|^p dx = \left(\frac{N-p}{p-1}\right)^p \int_\Omega \frac{|x|^{\frac{p}{p-1}}}{\left(\varepsilon + |x|^{\frac{p}{p-1}}\right)^N} dx + O(1)$$

$$(\text{ put } \; x = \varepsilon^{\frac{p-1}{p}} t, \; dx = \varepsilon^{\frac{(p-1)N}{p}} dt)$$

$$= \varepsilon^{\frac{p-N}{p}} \left(\frac{N-p}{p-1}\right)^p \int_{R^N} \frac{|t|^{\frac{p}{p-1}}}{\left(1 + |t|^{\frac{p}{p-1}}\right)^N} dt + O(1)$$

$$= K_1 \varepsilon^{\frac{p-N}{p}} + O(1)$$

with $K_1 = \left(\frac{N-p}{p-1}\right)^p \int_{R^N} \frac{|t|^{\frac{p}{p-1}}}{\left(1 + |t|^{\frac{p}{p-1}}\right)^N} dt = L(N,p) \| \nabla u_1(x) \|^p_{L^p}.$ □

PROOF OF (40).

$$\int_\Omega Q(x) u_\varepsilon^{p^*}(x) dx$$

$$= \int_\Omega (Q(x) - Q(0)) \frac{\phi(x)^{p^*}}{\left(\varepsilon + |x|^{\frac{p}{p-1}}\right)^N} dx + \int_\Omega Q(0) \frac{\phi(x)^{p^*}}{\left(\varepsilon + |x|^{\frac{p}{p-1}}\right)^N} dx$$

$$= o\left(\varepsilon^{\frac{p^*-p-N}{p}}\right) + \int_\Omega Q(0) \frac{\phi^{p^*}}{\left(\varepsilon + |x|^{\frac{p}{p-1}}\right)^N} dx$$

$$(\text{ put } \; x = \varepsilon^{\frac{p-1}{p}} t \;\; dx = \varepsilon^{\frac{p-1}{p} \cdot N} dt)$$

$$= o\left(\varepsilon^{\frac{p^*-p-N}{p}}\right) + Q(0) \varepsilon^{\frac{-N}{p}} \int_{R^N} \frac{dt}{\left(1 + |t|^{\frac{p}{p-1}}\right)^N}.$$

$$= o\left(\varepsilon^{\frac{p^*-p-N}{p}}\right) + K_2 Q(0) \varepsilon^{\frac{-N}{p}} + 0(1)$$

with $K_2 = \|u_1\|^{p^*}_{L^{p^*}}$ □

PROOF OF (41). From (40), (41),

$$\| \nabla v_\varepsilon\|_{L^p}^N = \frac{(\int_\Omega | \nabla u_\varepsilon(x)|^p dx)^{\frac{N}{p}}}{(\int_\Omega Q(x)u_\varepsilon(x)^{p^*} dx)^{\frac{N}{p^*}}} = Q(0)^{\frac{-N}{p^*}} S^{\frac{N}{p}} + o\left(\varepsilon^{\frac{N-p}{p}}\right). \quad \square$$

PROOF OF (42). For $1 < p^2 < N$,

$$\begin{aligned}
\int_\Omega u_\varepsilon^p dx &= \int_\Omega \frac{dx}{\left(\varepsilon + |x|^{\frac{p}{p-1}}\right)^{N-p}} + \int_\Omega \frac{\phi^p - 1}{\left(\varepsilon + |x|^{\frac{p}{p-1}}\right)^{N-p}} dx \\
&= \int_{R^N} \frac{dx}{\left(\varepsilon + |x|^{\frac{p}{p-1}}\right)^{N-p}} + O(1) \\
&\text{(put } x = \varepsilon^{\frac{p-1}{p}} t,\ dx = \varepsilon^{\frac{p-1}{p}\cdot N} dt) \\
&= \varepsilon^{\frac{p^2-N}{p}} \int_{R^N} \frac{dt}{\left(1 + |t|^{\frac{p}{p-1}}\right)^{N-p}} + O(1) \\
&= K_3 \varepsilon^{\frac{p^2-N}{p}} + O(1) \quad \text{for } 1 < p^2 < N, \\
&\text{with } K_3 = \int_{R^N} \frac{dt}{\left(1 + |t|^{\frac{p}{p-1}}\right)^{N-p}} = \|u_1\|_{L^{p^*}}^p.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\|u_\varepsilon\|_{L^p}^p}{\|Q(x)^{\frac{1}{p^*}} u_\varepsilon\|_{L^{p^*}}^p} = \|v_\varepsilon\|_{L^p}^p &= \frac{K_3 \varepsilon^{\frac{p^2-N}{p}} + O(1)}{\left(o\left(\varepsilon^{\frac{p^*-p-N}{p}}\right) + K_2 Q(0) \varepsilon^{-\frac{N}{p}} + O(1)\right)^{\frac{p}{p^*}}} \\
&= \frac{K_3}{K_2} \varepsilon^{p-1} + o(\varepsilon^{\frac{Np-p-N}{N}}) \quad \text{if } 1 < p^2 < N. \quad \square
\end{aligned}$$

PROOF OF (43). For $N = p^2$,

$$\int_\Omega u_\varepsilon^p dx = O(1) + \int_\Omega \frac{dx}{\left(\varepsilon + |x|^{\frac{p}{p-1}}\right)^{p(p-1)}} = O(1) + I(\varepsilon)$$

and there exists $0 < R_1 < R_2$ such that

$$\int_{|x|\le R_1} \frac{dx}{\left(\varepsilon + |x|^{\frac{p}{p-1}}\right)^{p(p-1)}} \le I(\varepsilon) \le \int_{|x|\le R_2} \frac{dx}{\left(\varepsilon + |x|^{\frac{p}{p-1}}\right)^{p(p-1)}}$$

and it is clear that for a fixed constant $R > 0$, we have

$$\begin{aligned}\int_{|x|\le R} \frac{dx}{\left(\varepsilon + |x|^{\frac{p}{p-1}}\right)^{p(p-1)}} &= |S^{N-1}| \int_0^R \frac{r^{N-1}dr}{\left(\varepsilon + r^{\frac{p}{p-1}}\right)^{p(p-1)}} \\ &= |S^{N-1}| \int_0^{R\varepsilon^{-\frac{p-1}{p}}} \frac{s^{p^2-1}}{\left(1 + s^{\frac{p}{p-1}}\right)^{p(p-1)}} ds \\ &= K_4 \log\left(\frac{1}{\varepsilon}\right) + O(1) \quad \text{as } \varepsilon \to 0 \text{ if } p^2 = N\end{aligned}$$

where $|S^{N-1}|$ is the measure of the unit sphere in R^N. Hence

$$\begin{aligned}\frac{\|u_\varepsilon\|^p_{L^p}}{\|u_\varepsilon\|^p_{L^{p^*}}} = \|v_\varepsilon\|^p_{L^p} &= \frac{K_4 \log(\frac{1}{\varepsilon}) + O(1)}{\left(o\left(\varepsilon^{\frac{p^*-p-N}{p}}\right) + K_2 Q(0)\varepsilon^{-\frac{N}{p}} + O(1)\right)^{\frac{p}{p^*}}} \\ &= \frac{K_4}{K_2} \log\left(\frac{1}{\varepsilon}\right) \varepsilon^{p-1} + o\left(\varepsilon^{\frac{p^2-p-1}{p}}\right) |\log \varepsilon| \quad \text{if } p^2 = N.\end{aligned}$$

□

PROOF OF LEMMA. We set $X_\varepsilon = \| \nabla v_\varepsilon\|^p_{L^p}$ and so we have

$$\begin{aligned}\psi(tv_\varepsilon) &= \left\{\int \frac{1}{p} t^p |\nabla v_\varepsilon|^p - \int \frac{Q(x)}{p^*} t^{p^*} |v_\varepsilon|^{p^*} - \int F(x, tv_\varepsilon)\right\} \\ &= \frac{1}{p} t^p X_\varepsilon - \frac{t^{p^*}}{p^*} - \int F(x, tv_\varepsilon).\end{aligned}$$

Note that $\psi(tv_\varepsilon) \le \frac{1}{p} t^p X_\varepsilon - \frac{t^{p^*}}{p^*}$ and fix ε,

$$\lim_{t\to\infty} \psi(tv_\varepsilon) = -\infty.$$

Therefore $\sup_{t\geq 0}\psi(tv_\varepsilon)$ is achieved at some $t_\varepsilon > 0$ (if $t_\varepsilon = 0$, then $\sup_{t\geq 0}\psi(tv_\varepsilon) = 0$ and there is nothing to prove). Since the derivative of the function $t \mapsto \psi(tv_\varepsilon)$ vanishes at $t = t_\varepsilon$, we have

$$t_\varepsilon^{p-1}X_\varepsilon - t_\varepsilon^{p^*-1} - \int f(x, t_\varepsilon v_\varepsilon)v_\varepsilon = 0 \tag{44}$$

and therefore

$$t_\varepsilon^{p-1}X_\varepsilon - t_\varepsilon^{p^*-1} \geq 0 \tag{45}$$

that is,

$$X_\varepsilon \geq t_\varepsilon^{p^*-p}$$
$$t_\varepsilon \leq X_\varepsilon^{\frac{1}{p^*-p}}.$$

Set $Y_\varepsilon = \sup\limits_{t\geq 0}\psi(tv_\varepsilon) = \psi(t_\varepsilon v_\varepsilon)$.

Since the function $t \mapsto (\frac{1}{p}t^pX_\varepsilon - \frac{t^{p^*}}{p^*})$ is increasing on the interval $[0, X_\varepsilon^{\frac{1}{p^*-p}}]$. It follows from (45) that

$$\begin{aligned}
Y_\varepsilon &= \frac{1}{p}t_\varepsilon^pX_\varepsilon - \frac{t_\varepsilon^{p^*}}{p^*} - \int F(x, t_\varepsilon v_\varepsilon)\\
&\leq \frac{1}{p}\left(X_\varepsilon^{\frac{1}{p^*-p}}\right)^p X_\varepsilon - \frac{1}{p^*}\left(X_\varepsilon^{\frac{1}{p^*-p}}\right)^{p^*} - \int F(x, t_\varepsilon v_\varepsilon)\\
&= \frac{1}{p}X_\varepsilon^{\frac{p}{p^*-p}} - \frac{1}{p^*}X_\varepsilon^{\frac{p^*}{p^*-p}} - \int F(x, t_\varepsilon v_\varepsilon)\\
&= \frac{1}{N}X_\varepsilon^{\frac{p^*}{p^*-p}} - \int F(x, t_\varepsilon v_\varepsilon).
\end{aligned}$$

Using (39), we obtain

$$\begin{aligned}
Y_\varepsilon &\leq \frac{1}{N}\left(Q(0)^{-\frac{N}{p^*}}S^{\frac{N}{p}} + o\left(\varepsilon^{\frac{N-p}{p}}\right)\right) - \int F(x, t_\varepsilon v_\varepsilon)\\
&= \frac{1}{N}\frac{S^{\frac{N}{p}}}{Q(0)^{\frac{N}{p^*}}} + o(\varepsilon^{\frac{N-p}{p}}) - \int F(x, t_\varepsilon v_\varepsilon).
\end{aligned} \tag{46}$$

On the other hand, we claim that

$$t_\varepsilon \to (Q(0)^{-\frac{p}{p^*}} S)^{\frac{1}{p^*-p}} \quad \text{as } \varepsilon \to 0. \tag{47}$$

Indeed, by (44) we have

$$X_\varepsilon - t_\varepsilon^{p^*-p} - \int \frac{f(x, t_\varepsilon v_\varepsilon) v_\varepsilon}{t_\varepsilon^{p-1}} = 0. \tag{48}$$

Thus it suffices to verify that

$$\int \frac{f(x, t_\varepsilon v_\varepsilon) v_\varepsilon}{t_\varepsilon^{p-1}} \to 0 \quad \text{as } \varepsilon \to 0. \tag{49}$$

Using (4)-(7), we see that for all $\delta > 0$, there exists a constant $C > 0$ such that $|f(x,u)| \le \delta u^{p^*-1} + C u^{p-1}$ for almost all $x \in \Omega$ and for all $u \ge 0$. Therefore we have

$$\left| \int \frac{f(x, t_\varepsilon v_\varepsilon) v_\varepsilon}{t_\varepsilon^{p-1}} \right| \le \left| \int \frac{[\delta (t_\varepsilon v_\varepsilon)^{p^*-1} + C (t_\varepsilon v_\varepsilon)^{p-1}] v_\varepsilon}{t_\varepsilon^{p-1}} \right|$$
$$= \left| \int \delta t_\varepsilon^{p^*-p} v_\varepsilon^{p^*} + C v_\varepsilon^p \right| \le \delta t_\varepsilon^{p^*-p} \|v_\varepsilon\|_{L^{p^*}}^{p^*} + C \|v_\varepsilon\|_{L^p}^p.$$

From (48), $t_\varepsilon^{p^*-p} \le \| \nabla v_\varepsilon\|_{L^p}^p = Q(0)^{-\frac{p}{p^*}} S^{\frac{N}{p}} + o(\varepsilon^{\frac{N-p}{p}})$ as $\varepsilon \to 0$. For $\varepsilon > 0$ small, we get

$$t_\varepsilon^{p^*-p} < Q(0)^{-\frac{p}{p^*}} S^{\frac{N}{p}} + 1.$$

Thus $\delta t_\varepsilon^{p^*-p} \|v_\varepsilon\|_{L^{p^*}}^{p^*} + C\|v_\varepsilon\|_{L^p}^p < \delta (Q(0)^{-\frac{p}{p^*}} S^{\frac{N}{p}} + 1) \|v_\varepsilon\|_{L^{p^*}}^{p^*} + C\|v_\varepsilon\|_{L^p}^p \to 0$ as $\varepsilon \to 0$ which implies (49) and thereby (47).

It follows from (47), (37), (38) and (39) that, for $\varepsilon > 0$ sufficiently small,

$$\begin{aligned} \int F(x, t_\varepsilon v_\varepsilon) &\ge \int_{|x|<R} F(x, t_\varepsilon v_\varepsilon) \\ &\ge \int_{|x|<R} F\left[\frac{A \varepsilon^{\frac{N-p}{p^2}}}{(\varepsilon + |x|^{p/p-1})^{N-p/p}} \right] dx \end{aligned} \tag{50}$$

for some constant $A (0 < A \le t_\varepsilon)$. From (44) and (48), we deduce that

$$(51)\quad Y_\varepsilon \le \frac{1}{N}\frac{S^{\frac{N}{p}}}{Q(0)^{\frac{N}{p^*}}} + o(\varepsilon^{\frac{N-p}{p}}) - \int_{|x|<R} F\left[\frac{A\varepsilon^{\frac{N-p}{p^2}}}{(\varepsilon + |x|^{p/p-1})^{N-p/p}}\right] dx.$$

Finally, we claim that

$$(52)\qquad \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{\frac{N-p}{p}}}\int_{|x|<R} F\left[\frac{A\varepsilon^{\frac{N-p}{p^2}}}{(\varepsilon + |x|^{p/p-1})^{N-p/p}}\right] dx = \infty$$

which implies, together with (51), that $Y_\varepsilon < \frac{1}{N}\frac{S^{\frac{N}{p}}}{Q(0)^{\frac{N}{p^*}}}$ for $\varepsilon > 0$ sufficiently small. □

VERIFICATION OF (52). We have

$$\frac{1}{\varepsilon^{N-p/p}}\int_{|x|<R} F\left[\frac{A\varepsilon^{\frac{N-p}{p^2}}}{(\varepsilon + |x|^{p/p-1})^{N-p/p}}\right] dx$$

$$= \frac{|S^{N-1}|}{\varepsilon^{N-p/p}}\int_0^R F\left[\frac{A\varepsilon^{\frac{N-p}{p^2}}}{(\varepsilon + r^{p/p-1})^{N-p/p}}\right] r^{N-1}dr$$

(put $r = \varepsilon^{\frac{p-1}{p}} s \quad dr = \varepsilon^{\frac{p-1}{p}} ds.$)

$$= |S^{N-1}|\varepsilon^{\frac{pN-2N+p}{p}}\int_0^{R\varepsilon^{\frac{-p-1}{p}}} F\left[A\left(\frac{\varepsilon^{\frac{p-1}{p}}}{1+s^{\frac{p}{p-1}}}\right)^{\frac{N-p}{p}}\right] s^{N-1}ds.$$

After rescaling ε we see that (52) is equivalent to

$$(53)\qquad \lim_{\varepsilon\to 0}\varepsilon^{\frac{pN-2N+p}{p}}\int_0^{R'\varepsilon^{-\frac{p-1}{p}}} F\left[\left(\frac{\varepsilon^{\frac{p-1}{p}}}{1+s^{\frac{p}{p-1}}}\right)^{\frac{N-p}{p}}\right] s^{N-1}ds = \infty$$

for some constant $R' > 0$. When $R' \ge 1$, (53) implies (13). Otherwise, when $R' < 1$, consider

$$z_\varepsilon = \varepsilon^{\frac{pN-2N+p}{p}}\int_{R'\varepsilon^{-\frac{p-1}{p}}}^{\varepsilon^{-\frac{p-1}{p}}} F\left[\left(\frac{\varepsilon^{\frac{p-1}{p}}}{1+s^{\frac{p}{p-1}}}\right)^{\frac{N-p}{p}}\right] s^{N-1}ds$$

and note that (for some constant C')

$$
\begin{aligned}
|z_\varepsilon| &\le C'\varepsilon^{\frac{pN-2N+p}{p}} F\left(C'\varepsilon^{\frac{N-p}{p^2}}\right)\varepsilon^{-\frac{p-1}{p}\cdot N} \\
&\le C'\varepsilon^{\frac{p-N}{p}}\left(\frac{a(x)}{p}(C')^p\varepsilon^{\frac{N-p}{p}} + \frac{\varepsilon}{p}(C')^p\varepsilon^{\frac{N-p}{p}} + \frac{C}{p^*}(C')^{p^*}\cdot\varepsilon^{\frac{N-p}{p^2}\cdot\frac{pN}{N-p}}\right) \\
&= \left((C')^{p+1}\frac{a(x)}{p} + (C')^{p+1}\frac{\varepsilon}{p} + \frac{C}{p^*}(C')^{p^*+1}\right)\varepsilon
\end{aligned}
$$

which is bounded as $\varepsilon \to 0$ and thus (52) implies (13). Thus the proof of Lemma is complete. □

EXAMPLE. All the assumptions of Lemma are satisfied if $f(x,u) = f(u) = \mu u^q$ with $\mu > 0$ and $p-1 < q < p^*-1$, $1 < p < N$. And $F(x,u) = \int_0^u f(t)\,dt$ satisfy (33). Thus (1) possess a solution. Now we will show that $F(x,u)$ satisfy (33).

We have $F(u) = \frac{\mu}{q+1}u^{q+1} \ge \beta$ for all $u \ge B$ for some constant $\beta > 0$ and $B > 0$.

Then $F\left[\left(\frac{\epsilon^{\frac{1-p}{p}}}{1+s^{\frac{p}{p-1}}}\right)^{\frac{N-p}{p}}\right] \ge \beta$ for all s such that $\frac{\epsilon^{\frac{1-p}{p}}}{1+s^{\frac{p}{p-1}}} \ge B^{\frac{p}{N-p}}$ and this holds for all $s \le C\epsilon^{-\frac{(p-1)^2}{p^2}}$ where C is some constant and ϵ is small.

Thus we have for ϵ small,

$$
\begin{aligned}
&\epsilon^{\frac{pN-2N+p}{p}}\int_0^{-\frac{p-1}{p}} F\left[\left(\frac{\epsilon^{\frac{1-p}{p}}}{1+s^{\frac{p}{p-1}}}\right)^{\frac{N-p}{p}}\right] s^{N-1}\,ds \\
&\ge \beta\epsilon^{\frac{pN-2N+p}{p}}\int_0^{C\epsilon^{-\frac{(p-1)^2}{p^2}}} s^{N-1}\,ds \\
&= \frac{\beta}{N}\epsilon^{\frac{pN-2N+p}{p}}\cdot\epsilon^{-\frac{(p-1)^2N}{p^2}} \\
&= \frac{\beta}{N}\epsilon^{p^2-N^2} \to \infty \text{ as } \epsilon \to 0
\end{aligned}
$$

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