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A CERTAIN CLASS OF ANALYTIC FUNCTIONS DEFINED BY RUSCHEWEYH DERIVATIVES

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ABSTRACT. This paper deals with a class $H_\lambda^b(A, B, \alpha)$ of analytic functions and we obtain coefficient estimates and related results.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0 \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$ and normalized such that $f(0) = 0, f'(0) = 1$. Also let S be the subclass of $\mathcal{A}$ consisting of analytic and univalent functions of the form (1.1). We denote by $S^*(\alpha)$ and $K(\alpha)$ the subclasses of S consisting of all functions which are, respectively starlike and convex of order α in $U(0 \leq \alpha < 1)$, that is,

$$S^*(\alpha) = \{f \in S : Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \ 0 \leq \alpha < 1, \ z \in U\} \tag{1.2}$$

and

$$K(\alpha) = \{f \in S : Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \ 0 \leq \alpha < 1, z \in U\}. \tag{1.3}$$

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We use Ω to denote that class of analytic functions $w(z)$ in U, satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$. In [8] Ruscheweyh introduced the class K_λ as follows. Denote $D^\lambda : \mathcal{A} \to \mathcal{A}$ the operator defined by

$$D^\lambda f = \frac{z}{(1-z)^{\lambda+1}} * f, \ \ \lambda \geq -1, \ \ z \in U$$

where the operation " $*$ " is the usual Hadamard product of the series. We note that $D^0 f = f$, $D^1 f = zf'$, $D^2 f = (f' + zf''/2)$ and

$$K_\lambda := \{f \in \mathcal{A} : Re\left(\frac{D^{\lambda+1} f}{D^\lambda f}\right) > \frac{1}{2}, \ \ z \in U\}. \tag{1.4}$$

Let R_α be the class of pre-starlike functions of order α, that is, $f \in R_\alpha$ if and only if

$$f * \frac{z}{(1-z)^{2-2\alpha}} \in S^*(\alpha), \ \ \alpha < 1, \qquad Re\left(\frac{f(z)}{z}\right) > \frac{1}{2}, \alpha = 1, \ \ z \in U.$$

Note that

$$S_\alpha(z) = \frac{z}{(1-z)^{2-a\alpha}}$$

is a well known extremal function in $S^*(\alpha)$. Further, Ruscheweyh [9] observed that

$$f \in R_\alpha \text{ if and only if } Re\left(\frac{D^{2-2\alpha} f}{D^{1-2\alpha} f}\right) > \frac{1}{2}, \ \ \alpha \leq 1, \ \ z \in U, \tag{1.5}$$

and in [12] Silverman and Silvia proved that $R_\alpha \subseteq S$ if and only if $\alpha \leq \frac{1}{2}$.

In [6] Aouf and Nasr introduced the class S_b of starlike functions of complex order $b (b \neq 0)$, that $f \in S_b$ if and only if

$$Re\left(1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right) > 0, \ \ z \in U. \tag{1.6}$$

The class K_λ and various other subclasses have been rather extensively studied by Sheil-Small et al., [11], Schild and Silverman [10], Al-Amiri [4, 5], Ahuja and Silverman [2, 3], Ahuja [1]. Motivated especially by the work of Ahuja [1], we aim at presenting here a systematic study of the class $H_\lambda^b(A, B; \alpha)$, which provides a unified approach for the classes considered by Ruscheweyh [8], Aouf and Nasr [6], Ahuja [1], and others [4, 5, 12, 13].

DEFINITION. A function f of $\mathcal{A}$ belongs to the class $H_\lambda^b(A,B;\alpha)$ if

$$(1.7)\qquad 1+\frac{\lambda+1}{b}\left(\frac{D^{\lambda+1}f}{D^\lambda f}-1\right)\prec\frac{1+\big(B+(A-B)(1-\alpha)\big)z}{1+Bz},$$

where $z\in U$, $-1\leq B<A\leq 1$, $0\leq\alpha<1$, and " $\prec$ " denotes subordination.

Equivalently, a function f of $\mathcal{A}$ belongs to $H_\lambda^b(A,B;\alpha)$ if and only if there exists $w(z)\in\Omega$ such that for $z\in U$,

$$(1.8)\qquad 1+\frac{\lambda+1}{b}\left(\frac{D^{\lambda+1}f}{D^\lambda f}-1\right)=\frac{1+\big(B+(A-B)(1-\alpha)\big)w(z)}{1+Bw(z)}.$$

2. Main theorems

Our first result provides coefficient estimates for a function f to be in $H_\lambda^b(A,B;\,\alpha)$, but before that we state a following result which we are going to use.

LEMMA 1 [7]. *Let*

$$g(z)=\sum_{p=q}^{\infty}d_pz^p,\ \ G(z)=\sum_{p=q}^{\infty}D_pz^p,\ \ q\geq 0.$$

If $g(z)=w(z)G(z)$, where $w(0)=0$ and $|w(z)|<1$, $z\in U$, then $d_q=0$ and

$$\sum_{p=q+1}^{k}|d_p|^2\leq\sum_{p=q}^{k-1}|D_p|^2,\ \ k=q+1,q+2,\cdots.$$

THEOREM 1. *For a fixed integer $m(m\geq 3)$, let*

$$M_j=\frac{|(A-B)b(1-\alpha)-(j-2)B|^2}{(\lambda+j-1)^2},\ \ j=2,3,\cdots m,$$

and

$$C(\lambda,p)=\frac{(\lambda+1)_{p-1}}{(p-1)!}=\frac{(\lambda+1)(\lambda+2)\cdots(\lambda+p-1)}{(p-1)!},\ \ p=2,3,\cdots.$$

Then

(2.1)

$$\left(\frac{1}{(m-1)C(\lambda,m)}\right)^2 \Bigg\{(A-B)^2|b|^2(1-\alpha)^2 + \sum_{p=2}^{m-1}\left(|(A-B)b(1-\alpha)-(p-1)B|^2-(p-1)^2\right)(C(\lambda,p))^2\prod_{j=2}^{p}M_j\Bigg\} = \prod_{j=2}^{m}M_j.$$

Proof. We prove (2.1) by the method of mathematical induction on m. The equality (2.1) holds for $m=3$. Assume that (2.1) is true for $m=4,5,\cdots,t-1$. Then for $m=t$, the left side of (2.1) gives

$$\begin{aligned}
&\left(\frac{1}{(t-1)C(\lambda,t)}\right)^2 \Bigg\{(A-B)^2|b|^2(1-\alpha)^2 \\
&\quad+\sum_{p=2}^{t-2}\left(|(A-B)b(1-\alpha)-(p-1)B|^2-(p-1)^2\right)(C(\lambda,p))^2\prod_{j=2}^{p}M_j \\
&\quad+\left(|(A-B)b(1-\alpha)-(t-2)B|^2-(t-2)^2\right)(C(\lambda,t-1))^2\prod_{j=2}^{t-1}M_j\Bigg\} \\
&=\left(\frac{1}{(t-1)C(\lambda,t)}\right)^2 \Bigg\{((t-2)C(\lambda,t-1))^2\prod_{j=2}^{t-1}M_j \\
&\quad+\left((\lambda+t-1)^2M_t-(t-2)^2\right)(C(\lambda,t-1))^2\prod_{j=1}^{t-1}M_j\Bigg\} \\
&=\prod_{j=2}^{t}M_j.
\end{aligned}$$

Hence the theorem is proved. □

THEOREM 2. *Let* $f \in H_\lambda^b(A,B;\alpha)$, *and* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, $z \in U$. *Then*

$$|a_2| \leq \frac{(A-B)|b|(1-\alpha)}{\lambda+1}, \tag{2.2}$$

and if $|(A-B)b(1-\alpha) - B| \leq 1$ *and* $k \geq 3$, *then*

$$|a_k| \leq \frac{(A-B)|b|(1-\alpha)(k-2)!}{(\lambda+1)_{k-1}}. \tag{2.3}$$

Furthermore, if $|(A-B)b(1-\alpha) - (k-2)B| > (k-2)$, $k \geq 3$, *let*

$$M = \left[\frac{|(A-B)b(1-\alpha) - (k-2)B|}{k-2}\right] \tag{2.4}$$

be the largest integer less than or equal to the expression within the square bracket. Then

$$|a_k| \leq \frac{1}{(\lambda+1)_{k-1}} \prod_{j=2}^{k} |(A-B)b(1-\alpha) - (j-2)B| \tag{2.5}$$

for $k = 3, 4, \cdots, m+2$ *and*

$$|a_k| \leq \frac{(k-2)!}{(m+1)!(\lambda+1)_{k-1}} \prod_{j=2}^{M+3} |(A-B)b(1-\alpha) - (j-2)B| \tag{2.6}$$

for $k > M+2$.

The bounds in (2.2), (2.3) *and* (2.5) *are sharp for all admissible values of* b, λ, A *and* B, *and for each* k.

PROOF. Using (1.7), we have

$$1 + \frac{\lambda+1}{b}\left(\frac{D^{\lambda+1}f}{D^\lambda f} - 1\right) = \frac{1 + \big(B + (A-B)(1-\alpha)\big)w(z)}{1 + Bw(z)} \tag{2.7}$$

$z \in U$, $w \in \Omega$. Simplifying (2.7) we have

$$\begin{aligned}&(\lambda+1)(1+Bw)D^{\lambda+1}f\\&\quad= \{(\lambda+1) + \big((A-B)b(1-\alpha) + (\lambda+1)B\big)w\}D^\lambda f\end{aligned}$$

that is,

(2.8)

$$(\lambda+1)(D^{\lambda+1}f - D^{\lambda}f)$$
$$= \Big\{\big((A-B)b(1-\alpha) + (\lambda+1)B\big)D^{\lambda}f - B(\lambda+1)D^{\lambda+1}f\Big\}w(z).$$

As

$$D^{\lambda}f = z + \sum_{p=2}^{\infty} C(\lambda,p)a_p z^p$$

and

$$C(\lambda+1,p) = \frac{\lambda+p}{\lambda+1}C(\lambda,p),$$

it follows that (2.8) is equivalent to

$$\sum_{p=2}^{\infty}(p-1)C(\lambda,p)a_p z^p$$
$$=\Big\{\sum_{p=1}^{\infty}\big((A-B)b(1-\alpha) - (p-1)B\big)C(\lambda,p)a_p z^p\Big\}w(z)$$

where $a_1 = 1$. Making use of Lemma 1, we have

$$\sum_{p=2}^{\infty}(p-1)^2(C(\lambda,p))^2|a_p|^2$$
$$\leq \sum_{p=1}^{k-1}\big(|(A-B)b(1-\alpha) - (p-1)B|^2\big)(C(\lambda,p))^2|a_p|^2$$

which leads to

(2.9)

$$|a_k|^2 \leq \left(\frac{1}{(k-1)C(\lambda,k)}\right)^2 \Big\{(A-B)^2|b|^2(1-\alpha)^2$$
$$+ \sum_{p=2}^{k-1}\big(|(A-B)b(1-\alpha)B|^2 - (p-1)^2\big)(C(\lambda,p))^2|a_p|^2\Big\}$$

for every $k = 2, 3, \cdots$. For $k = 2$, we get

$$|a_k|^2 \leq \left(\frac{(A-B)|b|(1-\alpha)}{\lambda+1}\right)^2,$$

which implies (2.2). Let

$$|(A-B)b(1-\alpha) - B| \leq 1, \;\; k \geq 3.$$

Then it follows that

$$|(A-B)b(1-\alpha) - (k-2)B| \leq k-2, \;\; k \geq 3.$$

Since all the terms under summation in (2.9) are non-positive, we get

$$|a_k| \leq \frac{(A-B)|b|(1-\alpha)}{(k-1)C(\lambda,k)}, \;\; k \geq 3$$

which establishes (2.3). However, if

$$|(A-B)b(1-\alpha) - (k-2)B| > k-2, \;\; k \geq 3,$$

then all terms under summation in (2.9) are positive. We prove (2.5) for integer $k \geq 3$ and $k \leq M+2$, from (2.9), by mathematical induction. For $k = 3$, (2.9) contributes,

$$|a_3|^2 \leq \frac{(A-B)|b|(1-\alpha)|(A-B)b(1-\alpha)-B|^2}{(\lambda+1)(\lambda+2)}$$

which proves (2.5) for $k = 3$.

Suppose that (2.5) holds for $k = 4, 5, \cdots, m-1$. Then for $k = m$, (2.9) yields

$$
\begin{aligned}
|a_m|^2 & \\
\leq & \left(\frac{1}{(m-1)C(\lambda,m)}\right)^2 \Big\{(A-B)^2|b|^2(1-\alpha)^2 \\
& + \sum_{p=2}^{m-1}\left(|(A-B)b(1-\alpha)-(p-1)B|^2-(p-1)^2\right)(C(\lambda,p))^2|a_p|^2\Big\} \\
\leq & \left(\frac{1}{(m-1)C(\lambda,m)}\right)^2 \Big\{(A-B)^2|b|^2(1-\alpha)^2 \\
& + \sum_{p=2}^{m-1}\left(|(A-B)b(1-\alpha)-(p-1)B|^2-(p-1)^2\right) \\
& (C(\lambda,p))^2\prod_{j=2}^{p}\frac{|(A-B)b(1-\alpha)-(j-2)B|^2}{(\lambda+j-2)^2}\Big\} \\
= & \prod_{j=2}^{m}\frac{|(A-B)b(1-\alpha)-(j-2)B|^2}{(\lambda+j-2)^2},
\end{aligned}
$$

by Theorem 1, and it is very easy to prove (2.5) holds for $k \leq M+2$.

Lastly, suppose that $k > M+2$. Then we can write (2.9) as

$$
\begin{aligned}
|a_k|^2 & \\
\leq & \left(\frac{1}{(k-1)C(\lambda,k)}\right)^2 \Big\{(A-B)^2|b|^2(1-\alpha)^2 \\
& + \sum_{p=2}^{m+2}\left(|(A-B)b(1-\alpha)-(p-1)B|^2-(p-1)^2\right)(C(\lambda,p))^2|a_p|^2 \\
& + \sum_{p=M+3}^{\infty}\left(|(A-B)b(1-\alpha)-(p-1)B|^2-(p-1)^2\right)(C(\lambda,p))^2|a_p|^2\Big\} \\
\leq & \left(\frac{1}{(k-1)C(\lambda,k)}\right)^2 \Big\{(A-B)^2|b|^2(1-\alpha)^2 \\
& + \sum_{p=2}^{m+2}\left(|(A-B)b(1-\alpha)-(p-1)B|^2-(p-1)^2\right)(C(\lambda,p))^2|a_p|^2\Big\}
\end{aligned}
$$

$$\leq \left(\frac{1}{(k-1)C(\lambda,k)}\right)^2 \Bigg\{(A-B)^2|b|^2(1-\alpha)^2$$

$$+\sum_{p=2}^{m+2}\left(|(A-B)b(1-\alpha)-(p-1)B|^2-(p-1)^2\right)$$

$$(C(\lambda,p))^2\prod_{j=2}^{p}\frac{|(A-B)b(1-\alpha)-(j-2)B|^2}{(\lambda+j-2)^2}\Bigg\}$$

$$=\left(\frac{(M+2)C(\lambda,M+3)}{(k-1)C(\lambda,k)}\right)^2\prod_{j=2}^{m+3}\frac{|(A-B)b(1-\alpha)-(j-2)|}{(\lambda+j-2)^2},$$

$$=\left(\frac{(k-2)!}{(M+1)!(\lambda+1)\cdots(\lambda+k-1)}\right)^2\prod_{j=2}^{m+3}|(A-B)b(1-\alpha)-(j-2)B|^2$$

by using Theorem 1, and thus (2.6) follows immediately from the above expression.

The equalities (2.2), (2.3) are sharp and are given by

$$f_k * \frac{z}{(1-z)^{\lambda+1}} = \begin{cases} z(1+Bz^{k-1})^{\frac{(A-B)b(1-\alpha)}{B(k-1)}}, & B\neq 0\\ zexp(\frac{Ab(1-\alpha)z^{k-1}}{k-1}), & B=0\end{cases}$$

where $|(A-B)b(1-\alpha)-B|\leq 1$.

The inequality (2.5) is sharp for

$$f * \frac{z}{(1-z)^{\lambda+1}} = \begin{cases} z(1+Bz)^{\frac{(A-B)b(1-\alpha)}{B}}, & B\neq 0\\ zexp(Ab(1-\alpha)z), & B=0\end{cases}$$

if $|(A-B)b(1-\alpha)-B(k-2)|>k-2,\ k\geq 3.$ □

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