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WEYL'S THEOREM FOR ISOLOID AND REGULOID OPERATORS

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ABSTRACT. In this paper we find some classes of operators for which Weyl's theorem holds. The main result is as follows. If $T \in \mathcal{L}(\mathcal{H})$ satisfies the following:

(i) Either T or T^* is reduced by each of its eigenspaces;
(ii) Weyl's theorem holds for T;
(iii) T is isoloid,

then for every polynomial p, Weyl's theorem holds for $p(T)$.

1. Introduction

Throughout this paper let $\mathcal{X}$ denote an infinite dimensional Banach space and let $\mathcal{H}$ denote an infinite dimensional Hilbert space. Let $\mathcal{L}(\mathcal{X})$ denote the set of bounded linear operators on $\mathcal{X}$. If $T \in \mathcal{L}(\mathcal{X})$, write $N(T)$ and $R(T)$ for the null space and range of T; $\rho(T)$ for the resolvent set of T ; $\sigma(T)$ for the spectrum of T; $\pi_0(T)$ for the set of eigenvalues of T; $\pi_{0f}(T)$ for the eigenvalues of finite multiplicity; $\pi_{0i}(T)$ for the eigenvalues of infinite multiplicity; $\pi_{00}(T)$ for the isolated eigenvalues of finite multiplicity. Recall ([6],[7]) that $T \in \mathcal{L}(\mathcal{X})$ is called *regular* if there is an operator $T' \in \mathcal{L}(\mathcal{X})$ for which $T = TT'T$. It is familiar that if T is regular then T has closed range and that its converse is also true in the Hilbert space setting. An operator $T \in \mathcal{L}(\mathcal{X})$ is called *left-Fredholm* if it has closed range with finite dimensional null space and *right-Fredholm* if it has closed range with its range of finite co-dimension. If T is either left- or right-Fredholm, we call it *semi-Fredholm* and if T is both left-

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and right-Fredholm, we call it *Fredholm*. The *index* of a semi-Fredholm operator $T \in \mathcal{L}(\mathcal{X})$ is given by

$$\operatorname{ind}(T) = \dim N(T) - \dim \mathcal{X}/R(T).$$

An operator $T \in \mathcal{L}(\mathcal{X})$ is called *Weyl* if it is Fredholm of index zero. An operator $T \in \mathcal{L}(\mathcal{X})$ is called *Browder* if T is Fredholm and $T - \lambda I$ is invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\omega(T)$ and the Browder spectrum $\sigma_b(T)$ of $T \in \mathcal{L}(\mathcal{X})$ are defined by

$$\begin{aligned}\sigma_e(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\};\\ \omega(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\};\\ \sigma_b(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\} :\end{aligned}$$

evidently

$$\sigma_e(T) \subseteq \omega(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \operatorname{acc}\sigma(T), \tag{0.1}$$

where we write $\operatorname{acc}\mathbf{K}$ for the accumulation points of $\mathbf{K} \in \mathbb{C}$. If we write $\operatorname{iso}\mathbf{K} = \mathbf{K} \setminus \operatorname{acc}\mathbf{K}$ and

$$p_{00}(T) := \sigma(T) \setminus \sigma_b(T) \tag{0.2}$$

for the *Riesz points* of T, then

$$\operatorname{iso}\sigma(T) \setminus \sigma_e(T) = \operatorname{iso}\sigma(T) \setminus \omega(T) = p_{00}(T) \subseteq \pi_{00}(T). \tag{0.3}$$

We say that *Weyl's theorem holds for* $T \in \mathcal{L}(\mathcal{X})$ if there is equality

$$\sigma(T) \setminus \omega(T) = \pi_{00}(T). \tag{0.4}$$

H. Weyl ([17]) showed that the equality (0.4) holds for hermitian operators. Weyl's theorem has been extended from hermitian operators to hyponormal operators and to Toeplitz operators by L. Coburn ([4]), to several classes of operators including seminormal operators by S. Berberian ([2],[3]), and to a few classes of Banach space operators by many authors ([10],[11],[13]). In this paper we try to find some classes of operators for which Weyl's theorem holds.

1. Reguloid operators

If $T \in \mathcal{L}(\mathcal{X})$, write $r(T)$ for the spectral radius of T. It is familiar that $r(T) \leq ||T||$. An operator $T \in \mathcal{L}(\mathcal{X})$ is called *normaloid* if $r(T) = ||T||$ and *isoloid* if $\text{iso}\,\sigma(T) \subseteq \pi_0(T)$. An operator $T \in \mathcal{L}(\mathcal{X})$ is said to satisfy condition (G_1) if $(T - \lambda I)^{-1}$ is normaloid for all $\lambda \notin \sigma(T)$. An operator $T \in \mathcal{L}(\mathcal{X})$ is called *reguloid* ([9, Definition 13]) if $T - \lambda I$ is regular for each $\lambda \in \text{iso}\,\sigma(T)$.

We begin with:

LEMMA 1.1. *If $T \in \mathcal{L}(\mathcal{X})$ then*

(1.1.1) *T satisfies (G_1) $\Longrightarrow$ T is reguloid $\Longrightarrow$ T is isoloid.*

PROOF. The first implication is known from [9, Theorem 14]. For the second implication, suppose $T \in \mathcal{L}(\mathcal{X})$ is reguloid and $\lambda \in \text{iso}\,\sigma(T)$. Assume to the contrary that $T - \lambda I$ is one-one. Then since by assumption $T - \lambda I$ is left invertible (cf. [7, (3.8.3.12)]), so that λ cannot lie on the boundary of $\sigma(T)$. This contradicts to the fact that $\lambda \in \text{iso}\,\sigma(T)$. □

We introduce the conditions that an operator T may satisfy (cf. [2]):

(β) Each point of $\sigma(T)$ is a *bare point* of $\sigma(T)$, in the sense that it lies on the circumference of some closed disc that contains $\sigma(T)$.

(β') Each point of $\pi_{0f}(T)$ is a *semibare point* of $\sigma(T)$, in the sense that it lies on the circumference of some closed disc that contains no other points of $\sigma(T)$.

We now add a condition to the above.

(β'') Each point of $\pi_{0f}(T)$ is called a *pseudo-bare point* of $\sigma(T)$, in the sense that it lies on a circle that contains no other points of $\sigma(T)$.

Evidently, $(\beta) \Longrightarrow (\beta') \Longrightarrow (\beta'')$. In general the reverse implications are not true. For example if

$$T = \begin{pmatrix} 0_1 & 0 & 0 \\ 0 & U - I & 0 \\ 0 & 0 & U + I \end{pmatrix}$$

(where 0_1 is the one-dimensional zero operator and U is the unilateral shift on ℓ_2) then

$$\sigma(T) = \{z : |z - 1| \leq 1\} \cup \{z : |z + 1| \leq 1\}.$$

Thus 0 cannot lie on the circumference of any closed disc that contains no other points of $\sigma(T)$. However, evidently, 0 lies in the circle $\{z : |z-2| = 2\}$, which intersects with $\sigma(T)$ at only 0. Thus 0 is a pseudo-bare point of $\sigma(T)$, but not a semibare point of $\sigma(T)$.

S. Berberian ([2, Corollary 2]) has shown that if every direct summand of T satisfies the property (G_1) and if T satisfies (β'), then Weyl's theorem holds for T. We can prove more:

THEOREM 1.2. *If $T \in \mathcal{L}(\mathcal{X})$ is reguloid and satisfies (β'') then Weyl's theorem holds for T.*

PROOF. Suppose $\lambda \in \pi_{00}(T)$. Since T is reguloid, we have that $T-\lambda I$ is left-Fredholm. Write $\partial\mathbf{K}$ for the boundary of $\mathbf{K} \subseteq \mathbb{C}$. Since $\lambda \in \partial\sigma(T)$, it follows from the continuity of the (semi-Fredholm) index that $\operatorname{ind}(T-\lambda I) = 0$. Therefore $\pi_{00}(T) \subseteq \sigma(T) \setminus \omega(T)$. For the reverse inclusion assume that $\lambda \in \sigma(T) \setminus \omega(T)$. Then $0 < \dim (T-\lambda I)^{-1}(0) < \infty$. But since by assumption, λ is a pseudo-bare point, it follows that $\lambda \in \partial\sigma(T)$. Thus by the punctured neighbourhood theorem (cf. [7],[8]), we must have that $\lambda \in \operatorname{iso}\sigma(T)$ and therefore $\lambda \in \pi_{00}(T)$. □

REMARK. If in Theorem 1.2, the "reguloid" condition is replaced by the "isoloid" condition, then Theorem 1.2 may fail. For example, if $T : \ell_2 \to \ell_2$ is defined by

$$T(x_1, x_2, \cdots) = (\frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \cdots), \tag{1.2.1}$$

then $0 \in \pi_{00}(T) = \pi_{0f}(T)$ is a pseudo-bare point of $\sigma(T)$, while Weyl's theorem does not hold for T. Note that T is isoloid, while T is not reguloid.

THEOREM 1.3. *If $T \in \mathcal{L}(\mathcal{X})$ is reguloid and $\pi_{0f}(T) \subseteq \operatorname{iso}\sigma(T)$, then*

$$\omega(p(T)) = p(\omega(T)) \quad \textit{for every polynomial } p. \tag{1.3.1}$$

Thus in particular, for every polynomial p, Weyl's theorem holds for $p(T)$.

PROOF. If $\pi_{0f}(T) \subseteq \operatorname{iso}\sigma(T)$, then evidently, each point of $\pi_{0f}(T)$ is a pseudo-bare point of $\sigma(T)$. Thus by Theorem 1.2, Weyl's theorem holds for T. For (1.3.1) suppose $\mu \notin \omega(p(T))$. Writing $p(\lambda) - \mu = a_0(\lambda - \lambda_1)\cdots(\lambda - \lambda_n)$, we have that

$$p(T) - \mu I = a_0(T - \lambda_1 I)\cdots(T - \lambda_n I)$$

is Weyl. Thus $T - \lambda_i I$ is Fredholm for each $i = 1, \cdots, n$ and

$$\sum_{i=1}^{n} \operatorname{ind}(T - \lambda_i I) = 0. \tag{1.3.2}$$

We now claim that

$$\operatorname{ind}(T - \lambda_i I) \leq 0 \qquad \text{for each } i = 1, \cdots, n. \tag{1.3.3}$$

To the contrary we assume that $\operatorname{ind}(T - \lambda_i I) > 0$ for some i, so that $0 < \dim N(T - \lambda_i I) < \infty$. Thus we have that $\lambda_i \in \pi_{0f}(T)$ and hence by assumption, $\lambda_i \in \operatorname{iso}\sigma(T)$. But then it follows from the continuity of the index that $\operatorname{ind}(T - \lambda_i I) = 0$, a contradiction. Therefore by (1.3.2) and (1.3.3), $T - \lambda_i I$ is Weyl for each $i = 1, \cdots, n$, so that $\mu \notin p(\omega(T))$. This gives that $p(\omega(T)) \subseteq \omega(p(T))$. The reverse inclusion holds for any operator $T \in \mathcal{L}(\mathcal{X})$. This completes the proof of (1.3.1). The second assertion follows from the first assertion and Theorem 1 of K. Oberai ([14]) which states that if T is isoloid and Weyl's theorem holds for T then Weyl's theorem holds for $p(T)$ if and only if $\omega(p(T)) = p(\omega(T))$. $\square$

Note that the "reguloid" condition cannot also be relaxed to the "isoloid" condition in Theorem 1.3: for example consider the operator in (1.2.1).

An operator $T \in \mathcal{L}(\mathcal{X})$ is said to be *polynomially compact* if there exists a nonzero complex polynomial p such that $p(T)$ is compact. It is familiar that every polynomially compact operator has a nontrivial invariant subspace. Also the structure of polynomially compact operators has been described by F. Gilfeather ([5]) and C. Olsen ([15]). Recall ([1],[16]) that an operator $T \in \mathcal{L}(\mathcal{X})$ is said to satisfy the condition (C_2) if $\lambda \in \pi_{00}(T)$ implies that $R(T - \lambda I)$ is closed. We now have:

THEOREM 1.4. *If $T \in \mathcal{L}(\mathcal{X})$ is polynomially compact, and satisfies (C_2) then Weyl's theorem holds for T.*

PROOF. Suppose $p(T)$ is compact for some nonzero polynomial p. Since the essential spectrum of any compact operator is contained in $\{0\}$, it follows that $p(\sigma_e(T)) = \sigma_e(p(T)) \subseteq \{0\}$, which says that $\sigma_e(T)$ is finite, say $\sigma_e(T) = \{\lambda_1, \cdots, \lambda_n\}$. We now claim that $\operatorname{acc}\sigma(T) \subseteq \{\lambda_1, \cdots, \lambda_n\}$. Indeed, if $T - \lambda I$ is left-Fredholm but not invertible then since $\lambda \in \operatorname{int}\sigma(T)$ implies that $\sigma_e(T)$ is infinite, it follows that $\lambda \in \partial\sigma(T)$. Then by the punctured neighborhood theorem, we should have that $\lambda \in \operatorname{iso}\sigma(T)$. Therefore $\sigma(T) \setminus \omega(T) \subseteq \pi_{00}(T)$. The reverse inclusion follows from the observation that if $\lambda \in \pi_{00}(T)$ then by our assumption $T - \lambda I$ is left-Fredholm, and hence by the continuity of the (semi-Fredholm) index we have that $\lambda \notin \omega(T)$. Therefore Weyl's theorem holds for T. □

2. Operators reduced by each of its eigenspaces

Suppose that $T \in \mathcal{L}(\mathcal{H})$ is reduced by each of its eigenspaces. If

$$\mathfrak{M} := \bigvee_{\lambda \in \pi_0(T)} N(T - \lambda I) \quad \text{(where } \bigvee(\cdot) \text{ denotes the closed linear span)},$$

then $\mathfrak{M}$ reduces to T. Write

$$T_1 := T|\mathfrak{M} \quad \text{and} \quad T_2 := T|\mathfrak{M}^{\perp}. \tag{2.0.1}$$

Then we have ([3, Proposition 4.1])

(i) T_1 is a normal operator with pure point spectrum;
(ii) $\pi_0(T_1) = \pi_0(T)$;
(iii) $\sigma(T_1) = \operatorname{cl}\pi_0(T)$;
(iv) $\pi_0(T_2) = \emptyset$.

The Weyl spectrum and the Browder spectrum coincide when the operator is reduced by each of its eigenspaces.

LEMMA 2.1. *If $T \in \mathcal{L}(\mathcal{H})$ is reduced by each of its eigenspaces then*

$$\omega(T) = \sigma_b(T). \tag{2.1.1}$$

PROOF. Since $\omega(T) \subseteq \sigma_b(T)$ in general, it suffices to show

$$\sigma_b(T) \subseteq \omega(T). \tag{2.1.2}$$

For (2.1.2) suppose $\lambda \in \sigma(T) \setminus \omega(T)$. Then $T - \lambda I$ is Weyl but not invertible. Observe that if $\omega(T_1) = \sigma_e(T_1)$ (e.g., T_1 is normal), then

$$\omega(T) = \omega(T_1) \cup \omega(T_2). \tag{2.1.3}$$

But since $\pi_0(T_2) = \emptyset$, we should have that $T_2 - \lambda I$ is one-one and hence invertible. On the other hand since T_1 is normal it follows from Weyl's theorem that $\lambda \in \pi_{00}(T_1)$ and therefore $\lambda \in \operatorname{iso}\sigma(T)$, which implies that $\lambda \notin \sigma_b(T)$. This proves (2.1.1). □

We now have:

THEOREM 2.2. *If $T \in \mathcal{L}(\mathcal{H})$ satisfies the following:*

(i) *Either T or T^* is reduced by each of its eigenspaces;*
(ii) *Weyl's theorem holds for T;*
(iii) *T is isoloid,*

then for every polynomial p, Weyl's theorem holds for $p(T)$.

PROOF. If Weyl's theorem holds for an isoloid operator T then by Theorem 1 of [14], for any polynomial p Weyl's theorem holds for $p(T)$ if and only if $p(\omega(T)) = \omega(p(T))$. Thus it will suffice to show that $p(\omega(T)) = \omega(p(T))$ for any polynomial p. If T is reduced by each of its eigenspaces, write $T = T_1 \oplus T_2$ as in (2.0.1). Then $p(T) = p(T_1) \oplus p(T_2)$ shows that $p(T_1)$ is normal and hence $\omega(p(T_1)) = \sigma_b(p(T_1))$, and $\pi_0(p(T_2)) = p(\pi_0(T_2)) = p(\emptyset) = \emptyset$ and hence $\omega(p(T_2)) = \sigma(p(T_2))$. Therefore by (2.1.3) we have

$$\sigma_b(p(T)) = \sigma_b(p(T_1)) \cup \sigma_b(p(T_2)) = \omega(p(T_1)) \cup \omega(p(T_2)) = \omega(p(T)), \tag{2.2.1}$$

where the first equality follows from the same argument as (2.1.3) with σ_b in place of ω. But since the Browder spectrum obeies the spectral mapping theorem it follows from (2.1.1) and (2.2.1) that

$$p(\omega(T)) = p(\sigma_b(T)) = \sigma_b(p(T)) = \omega(p(T)).$$

If instead T^* is reduced by each of its eigenspaces, write $T^* = S_1 \oplus S_2$ as in (2.0.1), where S_1 is normal with pure point spectrum and $\pi_0(S_2) = \emptyset$. Then $T = S_1^* \oplus S_2^*$, where S_1^* is normal and $\Gamma(S_2^*) = \emptyset$, where $\Gamma(S_2^*)$ denotes the compression spectrum of S_2^*, i.e., $\Gamma(S_2^*) = \pi_0(S_2)^-$. Therefore $p(T) = p(S_1^*) \oplus p(S_2^*)$ shows that $p(S_1^*)$ is normal and hence $\omega(p(S_1^*)) = \sigma_b(p(S_1^*))$, and $\Gamma(p(S_2^*)) = p(\Gamma(S_2^*)) = p(\pi_0(S_2)^-) = p(\emptyset) = \emptyset$ and hence $\omega(p(S_2^*)) = \sigma(p(S_2^*))$. Therefore by the same argument as the above, we have that $\omega(p(T)) = p(\omega(T))$. This completes the proof.□

COROLLARY 2.3 ([12]). *If $T \in \mathcal{L}(\mathcal{H})$ is seminormal then for every polynomial p, Weyl's theorem holds for $p(T)$.*

PROOF. This follows from the fact that seminormal operators satisfy assumptions of Theorem 2.2. □

The following example shows that Theorem 2.2 may fail if any one of three conditions is dropped:

EXAMPLE 2.4. (1) If U is the unilateral shift on ℓ_2, define T on $\ell_2 \oplus \ell_2$ by

$$T = (U + I) \oplus (U^* - I).$$

Then we have

$$\sigma(T) = \omega(T) = \{\lambda \in \mathbb{C} : |1 - \lambda| \leq 1\} \bigcup \{\lambda \in \mathbb{C} : |1 + \lambda| \leq 1\}$$

$$\text{and} \quad \pi_{00}(T) = \emptyset.$$

Thus T is isoloid and Weyl's theorem holds for T. But T is not reduced by each of its eigenspaces. To see this let $\mathfrak{M}$ be the eigenspace of T corresponding to the eigenvalue -1, so that

$$\mathfrak{M} = \{0\} \oplus N(U^*), \quad \text{and hence} \quad \mathfrak{M}^\perp = \ell_2 \oplus N(U^*)^\perp.$$

But for some $x \oplus y \in \mathfrak{M}^\perp$,

$$T(x \oplus y) = (U + I)x \oplus (U^* - I)y \notin \mathfrak{M}^\perp$$

because $(U^* - I)y \notin N(U^*)^\perp$ for $y = (0, 1, 0, \cdots) \in N(U^*)^\perp$, which says that T is not reduced by its eigenspaces. Also the same argument as the above gives that T^* is not reduced by its eigenspaces. On the other hand we observe that $\sigma(T^2)$ is the Cardioid $\{re^{i\theta} : r \le 2(1+\cos\theta)\}$ (use the spectral mapping theorem). We however have

$$T^2 - I = (T+I)(T-I) = \big((U+2I)\oplus U^*\big)\,\big(U\oplus(U^*-2I)\big),$$

so that $\operatorname{ind}(T^2-I) = \operatorname{ind}(U^*) + \operatorname{ind}(U) = 1 + (-1) = 0$, which says that $1 \notin \omega(T^2)$. Thus we have that $1 \in \sigma(T^2)$ and $1 \notin \omega(T^2)\cup\pi_{00}(T^2)$, which implies that Weyl's theorem does not hold for T^2.

(2) To show that the "isoloid" condition is essential in Theorem 2.2, we may borrow an example due to K. Oberai ([14]). Let on ℓ_2

$$T_1(x_1, x_2, \cdots) = (x_1, 0, \frac{x_2}{2}, \frac{x_3}{2}, \cdots)$$

$$\text{and}\quad T_2(x_1, x_2, \cdots) = (0, \frac{x_1}{2}, \frac{x_2}{3}, \frac{x_3}{4}, \cdots).$$

Define T on $\ell_2\oplus\ell_2$ by $T = T_1\oplus(T_2-I)$. Then T is not isoloid and Weyl's theorem holds for T but fails for T^2 (see [14, Example 1]). We need to show that T is reduced by each of its eigenspaces. Since $\pi_0(T_1) = \{1\}$ and $\pi_0(T_2-I) = \emptyset$, it follows that $\pi_0(T) = \{1\}$. Let $\mathfrak{M}$ be the eigenspace of T corresponding to the eigenvalue 1, so that $\mathfrak{M} = N(U^*)\oplus\{0\}$. Then evidently $T(\mathfrak{M}) \subseteq \mathfrak{M}$ and $\mathfrak{M}^\perp = N(U^*)^\perp\oplus\ell_2$. Also since $T_1\big(N(U^*)^\perp\big) \subseteq N(U^*)^\perp$ and hence $T(\mathfrak{M}^\perp) \subseteq \mathfrak{M}^\perp$, it follows that T is reduced by each of its eigenspaces.

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