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SURFACES OF 1-TYPE GAUSS MAP WITH FLAT NORMAL CONNECTION

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ABSTRACT. In this paper, we proved that the only surfaces of 1-type Gauss map with flat normal connection are spheres, products of two plane circles and helical cylinders.

1. Introduction and Preliminaries

The notion of submanifolds of finite type was introduced by B.-Y. Chen in the late seventies [2]. Since then many works were done to characterize or classify submanifolds in terms of finite type. The study of finite type submanifolds provided a natural way to combine spectral theory with the geometry of smooth maps, in particular, Gauss map. In [3] B.-Y. Chen and P. Piccinni gave a general study of submanifolds with finite-type Gauss map. In [1] C. Baikoussis, B.-Y. Chen and L. Verstraelen classified ruled surfaces and tubes with finite-type Gauss map. Also D.-S. Kim and S.-B. Kim proved that the only hyperquadrics with finite type Gauss map are hyperplanes, hyperspheres and spherical cylinders [8]. The classification problem for surfaces of 1-type Gauss map in Euclidean 3-space was solved by Y. H. Kim and the first author [7], [9]. In this article we continuously investigated surfaces with 1-type Gauss map in Euclidean n-space E^n and proved that the only surfaces of 1-type Gauss map with flat normal connection in E^n are spheres, a product of two plane circles and helical cylinders. (By a helical cylinder we mean the product of a straight line and a circular helix. If the torsion of the circular helix is zero, then the helical cylinder is nothing but an ordinary circular cylinder in E^3.) Since

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a surface in E^3 has spontaneously flat normal connection, this can be a generalization of Theorem in [7].

In general, a smooth map ϕ of a Riemannian manifold M into a Euclidean space is said to be of finite type if it is decomposed as a finite sum of E^m-valued eigenfunctions of the Laplacian Δ on M, that is,

$$\phi = \phi_0 + \phi_1 + \cdots + \phi_k,$$

where ϕ_0 is a constant function and $\phi_1, \cdots, \phi_k$ are nonconstant functions satisfying $\Delta\phi_i = \lambda_i\phi$, λ_i being constants, $i = 1, 2, \cdots, k$. In particular, if $\lambda_1, \lambda_2, \cdots, \lambda_k$ are mutually different, we say that ϕ is of k-type.

Let M^2 be a connected surface in E^n and let $e_1, e_2, \cdots, e_{n-1}, e_n$ be an oriented orthonormal local frame on M^2 such that e_1, e_2 are tangent to M^2, $e_3, \cdots, e_n$ are normal to M^2. From now on, the indices i, j, k run over the range $\{1, 2\}$ and the indices r, s over $\{3, \cdots, n\}$, unless stated otherwise. Let ∇ and ∇' be the Levi-Civita connections on M^2 and E^n, respectively. Denote by ω_B^A, $A, B = 1, 2, \cdots, n$, the connection forms. Then we have

$$(1.1)\qquad \begin{aligned} \nabla'_{e_i} e_j &= \nabla_{e_i} e_j + h(e_i, e_j), \\ \nabla_{e_i} e_j &= \sum_k \omega_j^k(e_i) e_k, \\ h(e_i, e_j) &= \sum_r h_{ij}^r e_r, \end{aligned}$$

$$(1.2)\qquad \nabla'_{e_i} e_r = D_{e_i} e_r - \sum_j h_{ij}^r e_j, \quad D_{e_i} e_r = \sum_s \omega_r^s(e_i) e_s.$$

where h is the second fundamental form, D is the normal connection and h_{ij}^r are the coefficients of the second fundamental form h. The Ricci equation of M^2 implies that

$$R^D(e_i, e_j; e_r, e_s) = \sum_k (h_{k2}^r h_{k1}^s - h_{k1}^r h_{k2}^s),$$

where R^D is the normal curvature tensor of M^2. Let G be the Gauss map of M^2 into $G(2, n)$ which is the Grassmannian manifold of the oriented 2-planes in E^n. Also, $G(2, n)$ can be identified with the decomposable 2-vectors of norm 1 in $\binom{n}{2}$-dimensional Euclidean space $\wedge^2 E^n = E^N$, where $N = \binom{n}{2}$. Then,

$$G : M^2 \longrightarrow G(2, n) \subset E^N$$

can given by $G(p) = (e_1 \wedge e_2)(p),\ p \in M^2$. Following [3] we can calculate ΔG, where Δ is the Laplacian on M^2 and $G = e_1 \wedge e_2$ is the Gauss map of M^2, as follows

$$\begin{aligned} \Delta G &= D_{e_1} H \wedge e_2 + e_1 \wedge D_{e_2} H \\ &\quad -2 \sum_{r<s} R^D(e_1, e_2; e_r, e_s) e_r \wedge e_s - \|h\|^2 e_1 \wedge e_2, \end{aligned} \tag{1.3}$$

where $H = \text{tr}\ h = \sum_i h(e_i, e_i)$ is the mean curvature vector of M^2 in E^n and $\|h\|^2 = \sum_{i,j,r} (h^r_{ij})^2$ is the square length of the second fundamental form h.

2. Surfaces of 1-type Gauss map with flat normal connection

Let M^2 be a connected surface in E^n and let its Gauss map G be of 1-type. Then there exist a constant λ and a constant vector c in $\wedge^2 E^n = E^N$ such that

$$\Delta G = \lambda(G - c). \tag{2.1}$$

Suppose that M^2 has flat normal connection, i.e., $R^D = 0$, and that $e_1, e_2, \cdots, e_n$ are orthonormal frame on M^2 such that e_1, e_2 are tangent to M^2, $e_3, \cdots, e_n$ are normal to M^2. Then without loss of generality we may assume that the coefficients h^r_{ij} of the second fundamental form h are given by

$$[h^r_{ij}] = \begin{bmatrix} x_r & 0 \\ 0 & y_r \end{bmatrix}, \quad r = 3, \cdots, n \tag{2.2}$$

and the normal frame $e_3, \cdots, e_n$ are parallel($De_r = 0,\ r = 3, \cdots, n$) [2]. Since (2.2) implies that $H = \sum_{r=3}^n (x_r + y_r) e_r$, from (1.3) and (2.1) it follows that

$$\begin{aligned} -\lambda c &= \left\{ \sum_{r=3}^n e_1(x_r + y_r) e_r \right\} \wedge e_2 + e_1 \wedge \left\{ \sum_{r=3}^n e_2(x_r + y_r) e_r \right\} \\ &\quad -(\|h\|^2 + \lambda) e_1 \wedge e_2. \end{aligned} \tag{2.3}$$

Let X be a tangent vector field on M^2. Taking the covariant differentiation of (2.3) in the direction X, we have

$$\begin{aligned}
&\left\{\left(\sum_{r=3}^{n} X(e_1(x_r+y_r))\right)e_r + \sum_{r=3}^{n} e_1(x_r+y_r)\nabla'_X e_r\right\}\wedge e_2\\
&+\left(\sum_{r=3}^{n} e_1(x_r+y_r)e_r\right)\wedge\nabla'_X e_2 + \nabla'_X e_1\wedge\left(\sum_{r=3}^{n} e_2(x_r+y_r)e_r\right)\\
&+e_1\wedge\left\{\sum_{r=3}^{n} X(e_2(x_r+y_r))e_r + \sum_{r=3}^{n} e_2(x_r+y_r)\nabla'_X e_r\right\}\\
&-X(\|h\|^2+\lambda)e_1\wedge e_2 - (\|h\|^2+\lambda)\nabla'_X e_1\wedge e_2 - (\|h\|^2+\lambda)e_1\wedge\nabla'_X e_2\\
&=0.
\end{aligned}$$

If we take X as e_1 and collect the coefficients of $e_A\wedge e_B (A<B)$, with the help of (1.1), (1.2), then we obtain the following equalities:

$$x_r e_2(x_s+y_s) - x_s e_2(x_r+y_r) = 0 \quad (r\neq s,\ r,s=3,\cdots,n), \tag{2.4}$$

$$\sum_{r=3}^{n} x_r e_1(x_r+y_r) + e_1(\|h\|^2+\lambda) = 0, \tag{2.5}$$

$$\begin{aligned}
e_1e_1(x_r+y_r) - \omega_1^2(e_1)e_2(x_r+y_r) &= x_r(\|h\|^2+\lambda) \quad (r=3,\cdots,n),\\
e_1e_2(x_r+y_r) - \omega_2^1(e_1)e_1(x_r+y_r) &= 0 \quad (r=3,\cdots,n).
\end{aligned} \tag{2.6}$$

Similarly, taking X as e_2, we get

$$y_r e_1(x_s+y_s) - y_s e_1(x_r+y_r) = 0 \quad (r\neq s,\ r,s=3,\cdots,n), \tag{2.7}$$

$$\sum_{r=3}^{n} y_r e_2(x_r+y_r) + e_2(\|h\|^2+\lambda) = 0, \tag{2.8}$$

$$\begin{aligned}
e_2e_2(x_r+y_r) - \omega_2^1(e_2)e_1(x_r+y_r) &= y_r(\|h\|^2+\lambda) \quad (r=3,\cdots,n),\\
e_2e_1(x_r+y_r) - \omega_1^2(e_2)e_2(x_r+y_r) &= 0 \quad (r=3,\cdots,n).
\end{aligned} \tag{2.9}$$

Note that the Coddazzi equations imply

$$e_1 y_r = \omega_2^1(e_2)(y_r - x_r), \tag{2.10}$$

$$e_2 x_r = \omega_1^2(e_1)(x_r - y_r)(r=3,\cdots,n). \tag{2.11}$$

And (2.3) yields

$$\sum_{r=3}^{n}\{e_1(x_r+y_r)\}^2+\sum_{r=3}^{n}\{e_2(x_r+y_r)\}^2+(\|h\|^2+\lambda)^2=d, \tag{2.12}$$

where $d=\langle\lambda c,\lambda c\rangle$ for the usual Euclidean metric $\langle\ ,\ \rangle$ of E^N. Let M_i $(i=0,1,2)$ be the set of points of M^2 at which the dimension of the first normal space $\mathrm{Im}\ h=\mathrm{Span}\{h(X,Y)|X$ and Y are tangent vectors on $M^2\}$ is i.

LEMMA 2.1. *Every component of M_2 is contained in a 4-dimensional affine subspace of E^n.*

PROOF. Let V be a component of M_2. Then without loss of generality we may assume that $x_3y_4-x_4y_3\neq 0$ on V. Thus there exist differentiable functions $a_r, b_r,\ r=5,\cdots n$ such that

$$\begin{bmatrix} x_r \\ y_r \end{bmatrix}=a_r\begin{bmatrix} x_3 \\ y_3 \end{bmatrix}+b_r\begin{bmatrix} x_4 \\ y_4 \end{bmatrix}. \tag{2.13}$$

By (2.4), we find

$$\begin{aligned} x_4e_2(x_r+y_r)-x_re_2(x_4+y_4) &= 0,\\ x_3e_2(x_r+y_r)-x_re_2(x_3+y_3) &= 0. \ (r=5,\cdots,n) \end{aligned}$$

Using (2.13) and $x_3e_2(x_4+y_4)-x_4e_2(x_3+y_3)=0$, from the above equation it follows that

$$\begin{aligned} x_4\{(x_3+y_3)e_2a_r+(x_4+y_4)e_2b_r\} &= 0,\\ x_3\{(x_3+y_3)e_2a_r+(x_4+y_4)e_2b_r\} &= 0. \end{aligned}$$

So we get

$$(x_3+y_3)e_2a_r+(x_4+y_4)e_2b_r=0. \tag{2.14}$$

From (2.11) and (2.13) we find

$$x_3e_2a_r+x_4e_2b_r=0. \tag{2.15}$$

Similarly, by (2.7), (2.10) and (2.13) we have

$$(x_3+y_3)e_1a_r+(x_4+y_4)e_1b_r = 0, \tag{2.16}$$

$$y_3e_1a_r+y_4e_1b_r = 0. \tag{2.17}$$

From (2.14), (2.15), (2.16) and (2.17) it follows that a_r, b_r are constants. Hence $e_r-a_re_3-b_re_4$ are constant normal vectors. Thus V is contained in a 4-dimensional affine subspace of E^n. □

LEMMA 2.2. *Let V be a component of the interior of M_1. Then V is contained in a 3-dimensional affine subspace of E^n or an open part of a helical cylinder.*

PROOF. We may assume that $(x_3, y_3) \neq (0,0)$ in V. Then there exist differentiable functions a_r such that

$$x_r = a_r x_3, \;\; y_r = a_r y_3, \;\; r = 4, \cdots, n.$$

From (2.10) and (2.11) we have $y_3 e_1 a_r = x_3 e_2 a_r = 0$. If $x_3 y_3 \neq 0$, then a_r are constants. In this case V is contained in 3-dimensional affine subspace of E^n. Otherwise we may assume that $y_3 = \cdots = y_n = 0$. Then (2.10) implies that $\omega_2^1(e_2) = 0$. Also we have $e_2 a_r = 0$ by (2.11). Thus from (2.8) we have $e_2 x_3 = 0$. This and (2.11) imply $\omega_2^1(e_1) = 0$. All these imply that e_2 is a constant tangent vector and $\nabla_{e_1} e_1 = 0$. Thus V is an open part of a cylinder $C \times E^1$, where C is a curve in E^{n-1}. Since C must have 1-type Gauss map, C is a circular helix. Consequently V is an open part of a helical cylinder. □

For the time being assume that M^2 is contained in 4-dimensional Euclidean space E^4, i.e., $n = 4$. Then we have the following equalities from (2.4), (2.5), (2.7), (2.8), (2.10) and (2.11)

$$(x_3 + 2y_3)e_1 y_3 + 3x_4 e_1 x_4 + (x_4 + 2y_4)e_1 y_4 = -3x_3 e_1 x_3, \tag{2.18}$$

$$y_4 e_1 y_3 - y_3 e_1 x_4 - y_3 e_1 y_4 = -y_4 e_1 x_3, \tag{2.19}$$

$$(y_4 - x_4)e_1 y_3 - (y_3 - x_3)e_1 y_4 = 0, \tag{2.20}$$

$$3y_3 e_2 y_3 + (y_4 + 2x_4)e_2 x_4 + 3y_4 e_2 y_4 = -(2x_3 + y_3)e_2 x_3, \tag{2.21}$$

$$x_4 e_2 y_3 + x_3 e_2 x_4 + x_3 e_2 y_4 = x_4 e_2 x_3, \tag{2.22}$$

$$(y_4 - x_4)e_2 x_3 - (y_3 - x_3)e_2 x_4 = 0. \tag{2.23}$$

Also the following sublemmas hold.

SUBLEMMA 2.3. *Under the assumption $n = 4$, the following equality holds.*

$$e_1(x_3 + y_3)e_2(x_4 + y_4) - e_1(x_4 + y_4)e_2(x_3 + y_3) = 0.$$

PROOF. Acting the laplacian Δ to the map $e_3 \wedge e_4$, we get

$$\begin{aligned} &\Delta(e_3 \wedge e_4) \\ &= e_1(x_3 + y_3)e_4 \wedge e_1 - e_2(x_3 + y_3)e_2 \wedge e_4 \\ &\quad - e_1(x_4 + y_4)e_3 \wedge e_1 - e_2(x_4 + y_4)e_3 \wedge e_2 - \|h\|^2 e_3 \wedge e_4. \end{aligned} \tag{2.24}$$

Since $\langle -\lambda c, e_3 \wedge e_4 \rangle = 0$ by (2.3), we obtain

$$\Delta\langle -\lambda c, e_3 \wedge e_4 \rangle = \langle -\lambda c, \Delta(e_3 \wedge e_4) \rangle = 0. \tag{2.25}$$

Thus we have $e_1(x_3+y_3)e_2(x_4+y_4) - e_1(x_4+y_4)e_2(x_3+y_3) = 0$ from (2.3), (2.24) and (2.25). □

SUBLEMMA 2.4. *If $x_3 = y_3$ or $x_4 = y_4$ in an open subset V in M^2, then x_3, y_3, x_4 and y_4 are all constants in V.*

PROOF. Suppose that $x_3 = y_3$. Then (2.10) and (2.11) imply that $e_1 y_3 = e_2 x_3 = 0$. Hence x_3 and y_3 are constants in V. Thus, from (2.18) and (2.19) the followings hold

$$-y_3 e_1 x_4 - y_3 e_1 y_4 = 0, \ 3x_4 e_1 x_4 + (x_4 + 2y_4) e_1 y_4 = 0.$$

If $y_3 \neq 0$, then the above equations imply that $e_1 x_4 = e_1 y_4 = 0$. We also find $e_2 x_4 = e_2 y_4 = 0$ in a similar way. Thus we can conclude that x_4 and y_4 are constants. If $y_3 = 0$, then V is contained in E^3. Then by Theorem in [7], x_4, y_4 are constants. In the case $x_4 = y_4$, the similar arguments lead to the same conclusion. □

From sublemmas we get the following lemma, which will be the crucial point in the proof of our main theorem.

LEMMA 2.5. *If M^2 is contained in E^4 and if $e_1 x_3 \neq 0$ or $e_1 x_4 \neq 0$ in a connected open subset V of M^2, then $y_3 = y_4 = 0$ in V. Similarly, if $e_2 y_3 \neq 0$ or $e_2 y_4 \neq 0$ in V, then $x_3 = x_4 = 0$ in V.*

PROOF. For notational simplicities, we will use the following abbreviations.

$$\begin{aligned} P &= (x_3 - y_3)^2 + (x_4 - y_4)^2, \\ Q &= x_3 y_3 + x_4 y_4, \\ R &= \|h\|^2 = x_3^2 + y_3^2 + x_4^2 + y_4^2, \\ f &= 2y_3 P + 3(y_3 - x_3) Q, \ g = 3(x_3 - y_3) Q. \end{aligned}$$

Assume that $e_1 x_3 \neq 0$ in V. It's enough to consider this case because the other cases can be dealt in similar fashions. We will work in V. Now suppose that $f \neq 0$. Then (2.18), (2.19) and (2.20) imply that

$$e_1y_3 = \frac{g}{f}e_1x_3, \tag{2.26}$$

$$e_1x_4 = \frac{1}{f}\{2y_4P + 3(y_4 - x_4)Q\}e_1x_3, \tag{2.27}$$

$$e_1y_4 = \frac{1}{f}3(x_4 - y_4)Qe_1x_3. \tag{2.28}$$

If $Q = 0$, then we have $e_1y_3 = e_1y_4 = 0$ from (2.26) and (2.28). Hence we get $y_3e_1x_3 + y_4e_1x_4 = 0$ by differentiating $Q = 0$ in the direction e_1. From this and (2.19), $e_1x_3 \neq 0$ implies $y_4 = y_3 = 0$, which contradicts $f \neq 0$. Thus we may assume that $g \neq 0$ by Sublemma 2.4. From (2.21), (2.22) and (2.23) we obtain

$$e_2y_3 = \frac{1}{g}\{-2x_3P + 3(y_3 - x_3)Q\}e_2x_3, \tag{2.29}$$

$$e_2x_4 = \frac{1}{g}3(x_4 - y_4)Qe_2x_3, \tag{2.30}$$

$$e_2y_4 = \frac{1}{g}\{-2x_4P + 3(y_4 - x_4)Q\}e_2x_3. \tag{2.31}$$

From (2.26) $\sim$ (2.31) and Sublemma 2.3 it follows that

$$(x_3y_4 - y_3x_4)Pe_1x_3e_2x_3 = 0.$$

If $x_3y_4 - y_3x_4 = 0$, then from (2.27), (2.30) and $y_4 = \frac{y_3x_4}{x_3}$ it follows that $e_1x_4 = \frac{x_4}{x_3}e_1x_3$ and $e_2x_4 = \frac{x_4}{x_3}e_2x_3$. This implies $e_1(\frac{x_4}{x_3}) = e_2(\frac{x_4}{x_3}) = 0$. Hence $x_4 = ax_3$ for a constant a, from which $x_3y_4 - y_3x_4 = 0$ implies that $y_4 = ay_3$. Therefore $ae_3 - e_4$ is a constant normal vector field. So V is contained in a 3-dimensional affine subspace of E^4. Then by Theorem in [7] x_3 and y_3 are constants in V, which will contradict the assumption $e_1x_3 \neq 0$. Hence we must have $e_2x_3 = 0$. Therefore we get $e_2y_3 = e_2x_4 = e_2y_4 = 0$ from (2.29), (2.30) and (2.31). This and (2.9) imply that

$$-\omega_2^1(e_2)e_1(x_3 + y_3) - y_3(\|h\|^2 + \lambda) = 0,$$

or

$$\frac{e_1y_3}{x_3 - y_3}e_1(x_3 + y_3) - y_3(\|h\|^2 + \lambda) = 0.$$

This and (2.26) mean

$$(e_1x_3)^2 = \frac{f^2(\|h\|^2+\lambda)}{6PQ}. \tag{2.32}$$

Since $e_1(x_3+y_3) = \frac{2y_3P}{f}e_1x_3$ and $e_1(x_4+y_4) = \frac{2y_4P}{f}e_1x_3$, (2.3) and (2.6) imply

$$x_4e_1(y_3U) = x_3e_1(y_4U),$$

where $U = \frac{2P}{f}e_1x_3$. Thus we find

$$(x_4y_3 - x_3y_4)e_1U = (x_3e_1y_4 - x_4e_1y_3)U.$$

From this, using (2.26), (2.28) and (2.32) we obtain

$$e_1U = \|h\|^2 + \lambda. \tag{2.33}$$

From (2.5), (2.26), (2.27) and (2.28) the followings hold

$$e_1R = \frac{2Q(2Q-R)}{f}e_1x_3,\ e_1Q = \frac{(R-2Q)(2S+3Q)}{f}e_1x_3, \tag{2.34}$$

where $S = y_3^2 + y_4^2$. Differentiating $U^2 = \frac{2}{3}\frac{(R+\lambda)(R-2Q)}{Q}$ in the direction e_1 and multiplying by $3Q^2$ we have

$$3Ue_1UQ^2 = e_1\{(R+\lambda)(R-2Q)\}Q - (R+\lambda)(R-2Q)e_1Q. \tag{2.35}$$

(2.34) implies

$$e_1\{(R+\lambda)(R-2Q)\} = \frac{e_1x_3}{f}(4Q-2R)\{(4Q+2S)(R+\lambda)+(R-2Q)Q\}.$$

Substituting this into (2.35) and using (2.32), (2.33) we find

$$\begin{aligned} 6(R+\lambda)Q^2 &= -2Q\{2(2Q+S)(R+\lambda)+(R-2Q)Q\} \\ &\quad -(R+\lambda)(R-2Q)(2S+3Q). \end{aligned} \tag{2.36}$$

By (2.26), (2.27), (2.28) and (2.32), the equation (2.12) becomes

$$2(R-2Q)(R+\lambda)S + 3Q(R+\lambda)^2 = 3dQ.$$

Thus we find

$$S = \frac{3Q}{2(R-2Q)(R+\lambda)}\{d-(R+\lambda)^2\} \tag{2.37}$$

Substituting (2.37) into (2.36), the following holds,

$$(4Q-3\lambda)R^2 + (3d-3\lambda^2+2\lambda Q-24Q^2)R - 4Q^2(4\lambda-2Q) = 0. \tag{2.38}$$

Differentiating (2.38) in the direction e_1, we have

$$\begin{aligned}&\{2(4Q-3\lambda)R+(3d-24Q^2-3\lambda^2+2\lambda Q)\}e_1R\\&+\{4R^2+(-48Q+2\lambda)R+(24Q^2-32Q\lambda)\}e_1Q=0.\end{aligned}$$

From this, using (2.34) and (2.37), we find

$$\begin{aligned}&2\{2(4Q-3\lambda)R+(3d-24Q^2-3\lambda^2+2\lambda Q)\}\\&-\{\frac{3d-3(R+\lambda)^2}{(R+\lambda)(R-2Q)}+3\}\{4R^2+(-48Q+2\lambda)R+(24Q^2-32Q\lambda)\}\\&=0.\end{aligned}$$

After some computations the above equation becomes

$$\begin{aligned}&(20Q)R^3-(184Q^2+32\lambda Q+3d)R^2\\&+\{120Q^3-248\lambda Q^2-94\lambda^2Q+66dQ\}R\\&+120\lambda Q^3-64\lambda^2Q^2-36dQ^2-42\lambda^3Q+42d\lambda Q=0.\end{aligned}\tag{2.39}$$

Consider the following two polynomials in a polynomial ring $\mathbf{R}[u,v]$ over real field $\mathbf{R}$.

$$\begin{aligned}F_1(u,v)&=(4u-3\lambda)v^2+(3d-3\lambda^2+2\lambda u-24u^2)v-4u^2(4\lambda-2u).\\F_2(u,v)&=(20u)v^3-(184u^2+32\lambda u+3d)v^2\\&\quad+\{120u^3-248\lambda u^2-94\lambda^2u+66du\}v\\&\quad+120\lambda u^3-64\lambda^2u^2-36du^2-42\lambda^3u+42d\lambda u.\end{aligned}$$

It is easy to see that $F_1(u,v)$ is irreducible in $\mathbf{R}[u,v]$. And $F_1(u,v)$ cannot divide $F_2(u,v)$ in $\mathbf{R}[u,v]$. Thus $F_1(u,v)$ and $F_2(u,v)$ are relatively prime in $\mathbf{R}[u,v]$. Therefore there exist only finitely many solutions satisfying $F_1(u,v)=0$ and $F_2(u,v)=0$ [5]. Since Q,R satisfy $F_1(Q,R)=0$ and $F_2(Q,R)=0$ by (2.38) and (2.39), Q,R are constants. Then (2.34) implies that $x_3=y_3$ and $x_4=y_4=0$. This with Sublemma 2.4 contradicts the assumption $e_1x_3\neq 0$. Therefore we can conclude $f=0$. Then from (2.18), (2.19) and (2.20) we know $(y_4-x_4)(x_4y_4+x_3y_3)=0$. This and Sublemma 2.4 mean that $x_4y_4+x_3y_3=0$. Subsequently this and $f=0$ imply that $y_3=0$. And (2.19) implies that $y_4=0$. □

Now we will state the main theorem and will prove it.

THEOREM 2.6. *Let M^2 be a connected surface of 1-type Gauss map with $R^D=0$ in E^n. Then M^2 is one of the followings:*

1) *an open part of a sphere,*

2) *an open part of a product of two plane circles,*
3) *an open part of a helical cylinder.*

PROOF. Assume that $e_1, e_2, \cdots, e_n$ are orthonormal frame on M^2 such that e_1, e_2 are tangent to M^2, $e_3, \cdots, e_n$ are normal to M^2 and let M_i denote the set points of M^2 at which the dimension of the first normal space $Im\ h$ is i. Also suppose that the coefficients $[h_{ij}^r]$ of the second fundamental form h are given by (2.2) and that the normal frame $e_3, \cdots, e_n$ are parallel in the normal bundle. Let V be a component of M_2. Then V is contained in a 4-dimensional Euclidean space by Lemma 2.1 and thus we may assume that $n = 4$. Lemma 2.5 implies that if $e_1 x_3 \neq 0$ or $e_1 x_4 \neq 0$, then $\dim Im\ h \leq 1$. Thus we can conclude that $e_1 x_3 = e_1 x_4 = 0$ in V. Similarly we have $e_2 y_3 = e_2 y_4 = 0$. These and (2.18) $\sim$ (2.23) imply that x_3, y_3, x_4, y_4 are all constants in V, which implies that $c = 0$ in (2.3). If the interior of M_1 is nonempty, then every component of the interior of M_1 is contained in E^3 or an open part of a helical cylinder by Lemma 2.2. If a component of M_1 is contained in E^3, then it is easy to see that $c = 0$ in (2.3) by theorem in [7]. Subsequently we can conclude that $c = 0$ in (2.3) or $M^2 = M_1 \cup M_0$ (In this case the interior of M_1 consists of open parts of helical cylinders fully contained in 4-dimensional Euclidean spaces.). In the latter case since every component of the interior of M_1 has nonzero constant mean curvature $|H|$ by Lemma 2.2, $M^2 = M_1$ or $M^2 = M_0$ by continuity. But the case $M^2 = M_0$ can't occur. This implies that if $c \neq 0$ in (2.3), then M^2 is is an open part of a helical cylinder fully contained in 4-dimensional Euclidean space. If $c = 0$ in (2.3), then M^2 has parallel mean curvature vector, constant square length of second fundamental form and flat normal connection. Thus M^2 is possibly an open part of a sphere or a circular cylinder or a product of two plane circles by Lemma 2.5 in [4, page 108] and Theorem 3.1 in [6]. Conversely, it is easy to see that these surfaces have 1-type Gauss map. □

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