Comm. Korean Math. Soc. **14** (1999), No. 1, pp. 201–210

FINSLER METRICS COMPATIBLE WITH $f(5,1)$-STRUCTURE

HONG-SUH PARK AND HA-YONG PARK

ABSTRACT. We introduce the notion of the Finsler metrics compatible with $f(5,1)$-structure and investigate the properties of Finsler space with such metrics.

1. Introduction

In a manifold admitting a Finsler metric and an almost complex structure, if the manifold satisfies the Rizza condition, that is, the Finsler metric is compatible with an almost complex structure, then the manifold is called an almost Hermitian Finsler manifold or simply a Rizza manifold. The notion of the Rizza manifold has been studied by G. B. Rizza [8], Y. Ichijyō [4] and M. Fukui [1].

On the other hand, the f-structure in a Riemannian manifold was, for the first time, introduced and studied by K. Yano [9]. Recently, in [5], Y. Ichijyō introduced the notion of the Finsler metrics compatible with f-structure and found the equivalent conditions for the Finsler metric to be compatible with f-structure. The $\varphi(4,2)$-structure in a Riemannian manifold was introduced and studied by K. Yano, C. S. Houh and B. Y. Chen [10]. The Finsler metrics compatible with a $\varphi(4,2)$-structure were introduced and studied by the present authors [7]. In [2] F. Gouli-Andreou introduced and studied the $f(5,1)$-structure in a Riemannian manifold.

In the present paper, we consider the Finsler metrics compatible with $f(5,1)$-structure in analogous manner of [5] and [7]. We shall find the

Received June 6, 1998. Revised July 20, 1998.

1991 Mathematics Subject Classification: 53B40.

Key words and phrases: Finsler metric, $f(5,1)$-structure, complex Banach space, $(f(5,1),L)$-manifold, (a,b,f)-metric.

equivalent conditions for the Finsler metric to be compatible with $f(5,1)$-structure. An example of the Finsler metric compatible with $f(5,1)$-structure is shown in the last section.

The authors would like to express their gratitude to Professor Y. Ichijyō for his valuable suggestions and encouragement.

2. Preliminaries

In an n-dimensional differentiable manifold M^n, let f be an $f(5,1)$-structure [2] of rank r, that is, a tensor field $f(\neq 0)$ of type (1,1) and class C^∞ such that

$$f^5 + f = 0, \qquad rank\ f = r. \tag{2.1}$$

If we put

$$\ell = -f^4, \qquad m = f^4 + E, \tag{2.2}$$

where E is the identity operator, we have

$$\begin{aligned} &\ell + m = E, \quad \ell^2 = \ell, \quad \ell m = m\ell = 0, \\ &m^2 = m, \quad f\ell = \ell f = f, \quad fm = mf = 0, \\ &f^4\ell = -\ell, \quad f^4 m = 0, \quad f^2 m = mf^2 = 0. \end{aligned} \tag{2.3}$$

Hence, ℓ and m are complementary projection operators on the tangent space at each point of M^n. Let $\mathfrak{L}$ and $\mathfrak{M}$ be the distributions corresponding to ℓ and m respectively. Since the rank of f is r, the dimensions of $\mathfrak{L}$ and $\mathfrak{M}$ are r and $n-r$ respectively.

If we put $F = f^2$, then the tensor F of type (1,1) satisfies

$$F^3 + F = 0 \tag{2.4}$$

and from (2.2) and (2.3) we have

$$\begin{aligned} &\ell = -F^2, \quad m = F^2 + E, \quad F\ell = \ell F = F, \\ &F^2\ell = -\ell, \quad Fm = mF = 0, \quad F^2 m = 0. \end{aligned} \tag{2.5}$$

That is, F acts on $\mathfrak{L}$ as an almost complex structure operator and on $\mathfrak{M}$ as a null operator. It is well known [9] that, in a manifold with the structure F satisfying (2.4), there exists a positive definite Riemannian metric a_{ij} with respect to which the distributions $\mathfrak{L}$ and $\mathfrak{M}$ are orthogonal and such that

$$a_{ij} = a_{pq}F^p{}_iF^q{}_j + a_{ip}m^p{}_j, \quad F_{ij} = -F_{ji}, \tag{2.6}$$

where $F_{ij} = a_{ip}F^p{}_j$.

3. An $(f(5,1),\ L)$-manifold

Let us assume that M^n admits a Finsler metric $L(x,\ y)$. As a matter of course, $L(x,\ y)$ satisfies

$$\text{(3.1)} \qquad \begin{aligned} &1)\quad L(x,\ ky) = kL(x,\ y) \quad \text{for any} \quad k > 0, \\ &2)\quad g_{ij}(x,\ y)\xi^i\xi^j \quad \text{is positive definite,} \end{aligned}$$

where $g_{ij}(x,\ y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_jL^2(x,\ y), \quad \dot{\partial}_i = \partial/\partial y.$

For any point $x \in M^n$, we define the norm of y in the tangent space $T_x(M)$ as

$$\|y\|_x = L(x,\ y). \tag{3.2}$$

Then $T_x(M)$ can be regarded as an n-dimensional normed linear space by the properties of Finsler metric $L(x,y)$.

Now, we define the scalar product of a complex number $\tilde{c} = |\tilde{c}|(\cos\theta + i\sin\theta)$ and any vector ℓy on the subspace $\mathfrak{L}$ in $T_x(M)$ as follows

$$\tilde{c}\ell y = |\tilde{c}|(\delta^i_j\cos\theta + F^i{}_j\sin\theta)\ell^j{}_ky^k. \tag{3.3}$$

Then, we have $\|\ell y\| = L(x,\ell y) \geq 0$ and

$$\|\ell y_1 + \ell y_2\| = L(x,\ell y_1 + \ell y_2) \leq L(x,\ell y_1) + L(x,\ell y_2) = \|\ell y_1\| + \|\ell y_2\|.$$

for any $\ell y_1, \ell y_2 \in \mathfrak{L}$. Since the dimension of $T_x(M)$ is finite, $\mathfrak{L}$ is complete. Thus the tangent subspace $\mathfrak{L}$ of $T_x(M)$ is a complex Banach space with respect to F if and only if

$$\|\tilde{c}\ell y\| = |\tilde{c}|\ \|\ell y\|. \tag{3.4}$$

If we put $\varphi_\theta = E\cos\theta + F\sin\theta$, we have from (3.3)

$$\|\tilde{c}\ell y\| = |\tilde{c}|L(x, \varphi_\theta \ell y). \tag{3.5}$$

Therefore (3.4) is equivalent to

$$L(x,\ \varphi_\theta \ell y) = L(x,\ \ell y) \tag{3.6}$$

for any θ. If (3.6) holds, the Finsler metric $L(x,\ y)$ is said to be *compatible with an* $f(5,1)$*-structure* and the manifold with the Finsler metric satisfying (3.6) is called an $(f(5,1),\ L)$*-manifold.*

From the definition of the Finsler metric tensor, we have

$$\dot{\partial}_i\dot{\partial}_j L^2(x,\ \ell y) = 2g_{pq}(x,\ \ell y)\ell^p{}_i\ell^q{}_j,$$
$$\dot{\partial}_i\dot{\partial}_j L^2(x,\ \varphi_\theta\ell y) = 2g_{pq}(x,\ \varphi_\theta\ell y)\varphi_\theta{}^p{}_r\varphi_\theta{}^q{}_s\ell^r{}_i\ell^s{}_j,$$

where $\varphi_\theta{}^i{}_j = \delta^i_j\ \cos\theta + f^i{}_r f^r{}_j\ \sin\theta$.

Therefore, if (3.6) holds good, then

$$g_{pq}(x,\ \varphi_\theta\ell y)\varphi_\theta{}^p{}_r\varphi_\theta{}^q{}_s\ell^r{}_i\ell^s{}_j = g_{pq}(x,\ \ell y)\ell^p{}_i\ell^q{}_j. \tag{3.7}$$

Conversely, by the homogeneity of $L(x,\ y)$ in y, we get easily (3.6), so (3.6) is equivalent to (3.7).

Thus we have

THEOREM 3.1. *Let M^n admit a Finsler metric $L(x,\ y)$ and an $f(5,1)$-structure. The tangent subspace $\mathfrak{L}$ at each point x of M^n is a complex Banach space, if and only if* (3.6) *or* (3.7) *is satisfied.*

Next, differentiating (3.7) with respect to θ, we have

$$\begin{aligned} &2C_{pqh}(x,\ \varphi_\theta\ell y)\frac{d\varphi_\theta{}^h{}_k}{d\theta}\ell^k{}_t y^t\varphi_\theta{}^p{}_r\varphi_\theta{}^q{}_s\ell^r{}_i\ell^s{}_j \\ &+ g_{pq}(x,\ \varphi_\theta\ell y)\frac{d\varphi_\theta{}^p{}_r}{d\theta}\varphi_\theta{}^q{}_s\ell^r{}_i\ell^s{}_j \\ &+ g_{pq}(x,\ \varphi_\theta\ell y)\varphi_\theta{}^p{}_r\frac{d\varphi_\theta{}^q{}_s}{d\theta}\ell^r{}_i\ell^s{}_j = 0, \end{aligned} \tag{3.8}$$

where $2C_{pqh}(x,y) = \dot{\partial}_h g_{pq}(x,y)$.

Using the following equations:

$$\left[\frac{d{\varphi_\theta}^i{}_j}{d\theta}\right]_{\theta=0} = F^i{}_j, \qquad \left[{\varphi_\theta}^i{}_j\right]_{\theta=0} = \delta^i_j,$$

the equation (3.8) is reduced to

$$(3.9)\quad g_{pq}(x,\ \ell y)\lambda^p{}_i\ell^q{}_j + g_{pq}(x,\ \ell y)\lambda^q{}_j\ell^p{}_i + 2C_{pqr}(x,\ \ell y)\lambda^r{}_t y^t \ell^p{}_i \ell^q{}_j = 0,$$

where we put $\lambda^p{}_i = F^p{}_r\ell^r{}_i$.

Now we assume that (3.9) holds good. Differentiating the left side of (3.7) with respect to θ, we get

$$\begin{aligned}
&\frac{d}{d\theta}\left\{g_{pq}(x,\ \varphi_\theta\ell y){\varphi_\theta}^p{}_r{\varphi_\theta}^q{}_s\ell^r{}_i\ell^s{}_j\right\}\\
(3.10)\quad &= \left\{2C_{pqm}(x,\ \varphi_\theta\ell y)\frac{d{\varphi_\theta}^m{}_k}{d\theta}\ell^k{}_t y^t {\varphi_\theta}^p{}_r{\varphi_\theta}^q{}_s\right.\\
&\left. + g_{pq}(x,\ \varphi_\theta\ell y)\frac{d{\varphi_\theta}^p{}_r}{d\theta}{\varphi_\theta}^q{}_s + g_{pq}(x,\ \varphi_\theta\ell y){\varphi_\theta}^p{}_r\frac{d{\varphi_\theta}^q{}_s}{d\theta}\right\}\ell^r{}_i\ell^s{}_j.
\end{aligned}$$

From (2.5) we have

$$(3.11)\qquad \frac{d{\varphi_\theta}^r{}_k}{d\theta}\ell^k{}_t = f^r{}_m f^m{}_k{\varphi_\theta}^k{}_h\ell^h{}_t, \qquad {\varphi_\theta}^q{}_m\ell^m{}_i = \ell^q{}_r{\varphi_\theta}^r{}_i.$$

Substituting (3.11) in the right side of (3.10), we obtain

$$\begin{aligned}
&\frac{d}{d\theta}\left\{g_{pq}(x,\ \varphi_\theta\ell y){\varphi_\theta}^p{}_r{\varphi_\theta}^q{}_s\ell^r{}_i\ell^s{}_j\right\}\\
&= {\varphi_\theta}^r{}_i{\varphi_\theta}^s{}_j\left\{2C_{pqm}(x,\ \ell\varphi_\theta y)\lambda^m{}_h{\varphi_\theta}^h{}_t y^t\ell^p{}_r\ell^q{}_s\right.\\
&\left. + g_{pq}(x,\ \ell\varphi_\theta y)\lambda^p{}_r\ell^q{}_s + g_{pq}(x,\ \ell\varphi_\theta y)\lambda^q{}_s\ell^p{}_r\right\} = 0
\end{aligned}$$

by virtue of (3.9). Therefore $g_{pq}(x,\ \varphi_\theta\ell y){\varphi_\theta}^p{}_r{\varphi_\theta}^q{}_s\ell^r{}_i\ell^s{}_j$ is independent of θ. That is, (3.7) holds good. Thus (3.7) is equivalent to (3.9).

Next, transvecting (3.9) by $y^i y^j$, we get

$$g_{pq}(x,\ \ell y)\lambda^p{}_i y^i \ell^q{}_j y^j + g_{pq}(x,\ \ell y)\lambda^q{}_j \ell^p{}_i y^i y^j = 0,$$

that is,

$$g_{pq}(x,\ \ell y)\lambda^p{}_i y^i \ell^q{}_j y^j = 0. \tag{3.12}$$

Differentiating (3.12) with respect to y^h, we have

$$\begin{aligned}&2C_{pqr}(x,\ \ell y)\ell^r{}_h \lambda^p{}_i y^i \ell^q{}_j y^j \\ &\quad + g_{pq}(x,\ \ell y)\lambda^p{}_h \ell^q{}_j y^j + g_{pq}(x,\ \ell y)\lambda^p{}_i y^i \ell^q{}_h = 0,\end{aligned}$$

from which

$$\{g_{pq}(x,\ \ell y)\lambda^p{}_k \ell^q{}_j + g_{pq}(x,\ \ell y)\lambda^p{}_j \ell^q{}_k\} y^j = 0. \tag{3.13}$$

Conversely we assume that (3.13) holds good. Differentiating (3.13) with respect to y^i, we easily get (3.9). Thus (3.9) is equivalent to (3.13).

Consequently we have

THEOREM 3.2. *In order that a Finsler metric be compatible with an $f(5,1)$-structure, it is necessary and sufficient that one of the following four equations be satisfied:*

(1) $g_{pq}(x,\ \varphi_\theta \ell y)\varphi_\theta{}^p{}_r \varphi_\theta{}^q{}_s \ell^r{}_i \ell^s{}_j = g_{pq}(x, \ell y)\ell^p{}_i \ell^q{}_j,$

(2) $g_{pq}(x,\ \ell y)\lambda^p{}_i \ell^q{}_j + g_{pq}(x,\ \ell y)\lambda^q{}_j \ell^p{}_i + 2C_{rsh}(x,\ \ell y)\lambda^h{}_t y^t \ell^r{}_i \ell^s{}_j = 0,$

(3) $g_{pq}(x, \ell y)\lambda^p{}_i y^i \ell^q{}_j y^j = 0,$

(4) $\{g_{pq}(x, \ell y)\lambda^p{}_k \ell^q{}_j + g_{pq}(x,\ \ell y)\lambda^p{}_j \ell^q{}_k\} y^j = 0,$

where $\lambda^i{}_j = f^i{}_r f^r{}_s \ell^s{}_j$.

THEOREM 3.3. *If an $(f(5,1),\ L)$-manifold satisfies*

$$g_{ij}(x,\ \ell y) = g_{pq}(x,\ \ell y)F^p{}_i F^q{}_j, \tag{3.14}$$

then $g_{ij}(x,\ \ell y)$ is a Riemannian metric.

PROOF. Differentiating (3.14) with respect to y^k, we have

$$C_{ijr}(x,\ \ell y)\ell^r{}_k = C_{pqr}(x,\ \ell y)F^p{}_iF^q{}_j\ell^r{}_k. \tag{3.15}$$

Transvecting (3.15) by $\ell^j{}_h$ and using (2.3), we have

$$C_{ijr}(x,\ \ell y)\ell^r{}_k\ell^j{}_h = C_{pqr}(x,\ \ell y)F^p{}_iF^q{}_h\ell^r{}_k. \tag{3.16}$$

Since C_{ijr} is symmetric in the all indices, we get

$$C_{pqr}(x,\ \ell y)F^p{}_iF^q{}_h\ell^r{}_k = C_{pqr}(x,\ \ell y)F^p{}_iF^q{}_k\ell^r{}_h. \tag{3.17}$$

From (3.15) and (3.17), we have

$$C_{ijr}(x,\ \ell y)\ell^r{}_k = C_{pqr}(x,\ \ell y)F^p{}_iF^q{}_k\ell^r{}_j. \tag{3.18}$$

Using (2.3), (3.18) is rewritten as

$$\begin{aligned} &C_{ijk}(x,\ \ell y) - C_{ijr}(x,\ \ell y)m^r{}_k \\ &\quad = C_{pqj}(x,\ \ell y)F^p{}_iF^q{}_k - C_{pqr}(x,\ \ell y)F^p{}_iF^q{}_km^r{}_j. \end{aligned} \tag{3.19}$$

Transvecting (3.19) by $F^i{}_mF^j{}_s\ell^k{}_h$ and using (2.5), we get

$$C_{ijk}(x,\ \ell y)F^i{}_mF^j{}_s\ell^k{}_h = -C_{pqj}(x,\ \ell y)\ell^p{}_mF^q{}_hF^j{}_s.$$

That is,

$$C_{ijk}(x,\ \ell y)F^i{}_mF^j{}_s\ell^k{}_h = 0 \tag{3.20}$$

by virtue of (3.17). Therefore, from (3.15) and (3.20), we have

$$2C_{ijr}(x,\ \ell y)\ell^r{}_k = \dot{\partial}_k\{g_{ij}(x,\ \ell y)\} = 0,$$

that is, $g_{ij}(x,\ \ell y)$ is a Riemannian metric. □

4. An example of $(f(5,1),\ L)$-manifold

In an n-dimensional manifold M^n, let $a_{ij}(x)$ be the positive definite Riemannian metric, b_i be a nonvanishing covariant vector field and $J^i{}_j(x)$ be an almost Hermitian structure. A Finsler metric $L(x,y)$ defined by

$$L(x,\ y) = \sqrt{a_{ij}y^iy^j} + \sqrt{(b_my^m)^2 + (b_mJ^m{}_ry^r)^2}.$$

is called an (a,b,f)-metric [3].

Now we consider a Finsler space with an analogous (a,b,f)-metric defined by

$$L(x,y) = \sqrt{a_{ij}y^iy^j} + \sqrt{(b_my^m)^2 + (b_mf^m{}_sf^s{}_ry^r)^2}. \tag{4.1}$$

Since $\varphi_\theta{}^i{}_p = \delta^i_p\ \cos\theta + F^i{}_p\ \sin\theta$, we have

$$\begin{aligned} &a_{ij}\varphi_\theta{}^i{}_p\varphi_\theta{}^j{}_q\ell^p{}_r\ell^q{}_sy^ry^s \\ &\quad = \{\cos^2\theta\ a_{ij}\ell^i{}_r\ell^j{}_s + \sin\theta\cos\theta\ (a_{ij}\ell^i{}_rF^j{}_s + a_{ij}F^i{}_r\ell^j{}_s) \\ &\qquad + \sin^2\theta\ a_{ij}F^i{}_rF^j{}_s\}y^ry^s. \end{aligned} \tag{4.2}$$

Now, from (2.5) and (2.6) we get

$$\begin{aligned} a_{ij}\ell^i{}_r\ell^j{}_s &= a_{ij}F^i{}_pF^p{}_rF^j{}_qF^q{}_s \\ &= (a_{pq} - a_{pt}m^t{}_q)F^p{}_rF^q{}_s = a_{pq}F^p{}_rF^q{}_s, \end{aligned} \tag{4.3}$$

$$\begin{aligned} (a_{ij}\ell^i{}_rF^j{}_s + a_{ij}F^i{}_r\ell^j{}_s)y^ry^s &= (F_{is}\ell^i{}_r + F_{jr}\ell^j{}_s)y^ry^s \\ &= 2F_{is}\ell^i{}_ry^sy^r = 0 \end{aligned} \tag{4.4}$$

by virtue of $F_{is}\ell^i{}_r = -F_{sr}$ and (2.6).

Substituting (4.3) and (4.4) in (4.2), we have

$$a_{ij}\varphi_\theta{}^i{}_p\varphi_\theta{}^j{}_q\ell^p{}_r\ell^q{}_sy^ry^s = a_{ij}\ell^i{}_p\ell^j{}_qy^py^q. \tag{4.5}$$

Next, from (2.3) and (2.5)

$$\begin{aligned} F^m{}_p \varphi_\theta{}^p{}_r \ell^r{}_s &= F^m{}_p(\ell^p{}_s \cos\theta + F^p{}_s \sin\theta) \\ &= F^m{}_p \ell^p{}_s \cos\theta - \ell^m{}_s \sin\theta. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &(b_m \varphi_\theta{}^m{}_r \ell^r{}_p y^p)^2 + (b_m F^m{}_p \varphi_\theta{}^p{}_r \ell^r{}_s y^s)^2 \\ &\quad = (\cos\theta\, b_m \ell^m{}_r y^r + \sin\theta\, b_m F^m{}_r \ell^r{}_s y^s)^2 \\ &\qquad + (\cos\theta b_m F^m{}_r \ell^r{}_s y^s - \sin\theta\, b_m \ell^m{}_r y^r)^2 \\ &\quad = (b_m \ell^m{}_r y^r)^2 + (b_m F^m{}_r \ell^r{}_s y^s)^2. \end{aligned} \tag{4.6}$$

From (4.1), (4.5) and (4.6) we have

$$\begin{aligned} &L(x,\ \varphi_\theta \ell y) \\ &= \sqrt{a_{ij} \varphi_\theta{}^i{}_p \varphi_\theta{}^j{}_q \ell^p{}_r \ell^q{}_s y^r y^s} + \sqrt{(b_m \varphi_\theta{}^m{}_r \ell^r{}_p y^p)^2 + (b_m F^m{}_p \varphi_\theta{}^p{}_r \ell^r{}_s y^s)^2} \\ &= \sqrt{a_{ij} \ell^i{}_p \ell^j{}_q y^p y^q} + \sqrt{(b_m \ell^m{}_r y^r)^2 + (b_m f^m{}_s f^s{}_r \ell^r{}_t y^t)^2} = L(x,\ \ell y), \end{aligned}$$

that is, the Finsler metric $L(x,\ y)$ defined by (4.1) satisfies (3.6).

Thus we have

THEOREM 4.1. *A Finsler space with a metric* (4.1) *is an* $(f(5,1), L)$-*manifold.*

References

[1] M. Fukui, *Complex Finsler manifolds*, J. Math. Kyoto Univ. **29** (1989), 609-624.

[2] F. Gouli-Andreou, *On a structure defined by a tensor field of type* $(1,1)$ *satisfying* $f^5 + f = 0$, Tensor, N. S. **36** (1982), 79-84.

[3] Y. Ichijyō, *Almost Hermitian Finsler manifolds*, Tensor, N. S. **37** (1982), 279-284.

[4] ———, *Finsler metrics on almost complex manifolds*, Riv, Mat. Univ. Parma **14** (1988), 1-28.

[5] ———, *On Finsler metrics compatible with f-structures*, Proceedings of the 30th symposium on Finsler Geometry (Nagasaki), Japan (1995), 18-20.

[6] M. Matsumoto, *Foundations of Finsler geometry and special Finsler spaces*, Kaiseisha press, Ōtsu, Japan, 1986.

[7] H. S. Park and H. Y. Park, *On Finsler metrics compatible with $\varphi(4,2)$-structure*, Tensor, N. S. **58** (1997), 67-70.
[8] G. B. Rizza, *Strutture di Finsler di tipo quasi Hermitiano*, Riv. Mat. Univ. Parma **4** (1963), 83-106.
[9] K. Yano, *On a structure defined by a tensor field f of type $(1,1)$ satisfying* $f^3 + f = 0$, Tensor, N. S. **14** (1963), 99-109.
[10] K. Yano, C. S. Houh and B. Y. Chen, *Structures defined by a tensor field φ of type $(1,1)$ satisfying* $\varphi^4 \pm \varphi^2 = 0$, Tensor, N. S. **23** (1972), 81-87.

Hong-Suh Park
Department of Mathematics
Yeungnam University
Gyongsan 712-749, Korea

Ha-Yong Park
Department of Mathematics
Kyungil University
Gyongsan 712-701, Korea