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ANTI-LINEAR INVOLUTIONS ON A G-VECTOR BUNDLE

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ABSTRACT. We study the anti-linear involutions on a real algebraic vector bundle with a compact real algebraic group action.

1. Introduction

Let G be a compact real algebraic group and Ω be a real G-module. An *affine real algebraic G-variety* is a G-invariant set

$$X = \{x \in \Omega \mid p_1(x) = \cdots = p_m(x) = 0\}$$

for polynomials $p_1, \ldots, p_m : \Omega \to \mathbb{R}$, *i.e.*, X is a real algebraic variety with G-action.

DEFINITION. An algebraic G-vector bundle $E \to X$ is a vector bundle on X with an algebraic G-action such that the following holds:

1. (1) The total space E and the base space X are affine algebraic G-varieties.
2. (2) The projection $\pi : E \to X$ is a G-equivariant polynomial map.
3. (3) The G-action is linear on the fibers $E_x := \pi^{-1}(x)$, (*i.e.*, for every $g \in G$ and $x \in X$ the map $v \to gv : E_x \to E_{gx}$ is linear).

In this paper we are concerned with equivariant vector bundles in an algebraic category. In Section 2, we analyze anti-linear involutions on a vector bundle. But the arguments in Section 2 work in any category such

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as in topological and smooth ones. In Section 3, we prove some theorems which hold in an algebraic category but not in a smooth category. Anti-linear involutions come out when we study the relation between real G-vector bundles and complex G-vector bundles. First, take a real G-vector bundle and complexify with $\mathbb{C}$. Then we get a complex G-vector bundle with an anti-linear action. The fixed point set of the anti-linear action on complex G-vector bundle is an original real G-vector bundle. So it is useful to study anti-linear involutions on a G-vector bundle to understand the relation between real G-vector bundles and complex G-vector bundles.

2. Anti-linear involutions on a vector bundle

Let G be a compact real algebraic group and let X be an affine real algebraic G-variety. Let $\pi : E \to X$ be a real algebraic G-vector bundle. We denote by $\mathrm{Aut}_{\mathbb{R}}(E)^G$ the group of real G-vector bundle automorphisms of E.

REMARK. $\mathrm{Aut}_{\mathbb{R}}(E)^G$ depends on the category. For example, let E be a trivial vector bundle with trivial action, *i.e.*, $E = B \times \mathbb{R}^n$, where B is the base space. Then $\mathrm{Aut}_{\mathbb{R}}(E)^G$ can be viewed as the set of maps from B to $GL(n, \mathbb{R})$, where the maps are smooth in a smooth category or algebraic (*i.e.*, polynomial map) in an algebraic category. In general, there are many more smooth maps than polynomial maps from B to $GL(n, \mathbb{R})$. For example, if $B = \mathbb{R}$ and $n = 1$, then only the nonzero constant maps are polynomial maps from $B = \mathbb{R}$ to $GL(1, \mathbb{R}) = \mathbb{R}^*$, while there are many non-constant smooth maps from $\mathbb{R}$ to $\mathbb{R}^*$. One of them is the exponential map.

DEFINITION. We say that $\iota \in \mathrm{Aut}_{\mathbb{R}}(E)^G$ is a complex structure on E if $\iota^2 = -1$, and E with ι is called a complex G-vector bundle. (Note that each fiber over x in X admits a structure of complex G_x-module given by $(a + b\sqrt{-1})w = aw + b\iota(w)$ where $a, b \in \mathbb{R}$ and $w \in E$.)

In the following we assume that E is a complex G-vector bundle (with ι as the complex structure) and denote by $\mathrm{Aut}_{\mathbb{C}}(E)^G$ the group of complex G-vector bundle automorphisms of E, *i.e.*, the group consisting of elements in $\mathrm{Aut}_{\mathbb{R}}(E)^G$ which commute with the complex structure ι.

DEFINITION. We say that $\tau \in \mathrm{Aut}_{\mathbb{R}}(E)^G$ is *anti-linear* if $\tau\iota = -\iota\tau$ (in other words, if $\tau(cw) = \bar{c}\tau(w)$ for $c \in \mathbb{C}$ and $w \in E$) (see [2]). Furthermore, if $\tau^2 = 1$, then it is called an *anti-linear involution.* We denote by $\mathrm{AL}(E)^G$ the set of anti-linear involutions of E.

LEMMA 1. *If $\tau \in \mathrm{AL}(E)^G$, then the fixed part $\pi|E^\tau : E^\tau \to X$ is a real G-vector bundle of $\dim_{\mathbb{C}} E$ and $E = E^\tau \oplus \iota(E^\tau)$.*

PROOF. Any element $w \in E$ is expressed as

$$w = (w + \tau(w))/2 + (w - \tau(w))/2.$$

Since τ is an involution, the two terms on the right hand side are eigenvectors of τ with eigenvalues 1 and -1 respectively. The anti-linearity of τ implies that the eigenspaces with eigenvalues 1 and -1 are isomorphic via ι. This proves the lemma. □

DEFINITION. Two elements τ and τ' in $\mathrm{AL}(E)^G$ are said to be *conjugate* if there is $\Phi \in \mathrm{Aut}_{\mathbb{C}}(E)^G$ such that $\tau' = \Phi\tau\Phi^{-1}$.

LEMMA 2. *Two elements τ and τ' in $\mathrm{AL}(E)^G$ are conjugate if and only if E^τ and $E^{\tau'}$ are isomorphic.*

PROOF. If τ and τ' are conjugate, then there is a $\Phi \in \mathrm{Aut}_{\mathbb{C}}(E)^G$ such that $\tau'\Phi = \Phi\tau$. This means that Φ maps E^τ onto $E^{\tau'}$, and Φ is an isomorphism.

Conversely, suppose that E^τ and $E^{\tau'}$ are isomorphic and let $\phi : E^\tau \to E^{\tau'}$ be an isomorphism. By Lemma 1, ϕ extends to an element $\Phi \in \mathrm{Aut}_{\mathbb{C}}(E)^G$ by

$$\Phi(u + \iota(v)) = \phi(u) + \iota\phi(v)$$

where $u, v \in E^\tau$. Then one sees that $\tau' = \Phi\tau\Phi^{-1}$. □

Multiplication on fibers of E by a non-zero complex number gives an element of $\mathrm{Aut}_{\mathbb{C}}(E)^G$, so we regard $\mathbb{C}^* = \mathbb{C} - \{0\}$ as a subgroup of $\mathrm{Aut}_{\mathbb{C}}(E)^G$.

THEOREM 3. *If $\mathrm{Aut}_{\mathbb{C}}(E)^G = \mathbb{C}^*$, then E^τ and $E^{\tau'}$ are isomorphic for any $\tau, \tau' \in \mathrm{AL}(E)^G$.*

PROOF. Since $\tau'\tau$ is an element of $\mathrm{Aut}_{\mathbb{C}}(E)^G$, it follows from the hypothesis that $\tau' = \lambda\tau$ for some $\lambda \in \mathbb{C}^*$. Note that $|\lambda| = 1$ since both τ and τ' are anti-linear involutions. Let μ be a square root of λ. Then $\mu\tau\mu^{-1} = \lambda\tau$, which shows that τ' is conjugate to τ. The theorem then follows from Lemma 2. □

3. Equivariant vector bundles in an algebraic category

In this section, we are concerned with some properties of equivariant vector bundles which hold in an algebraic category but not in a smooth category. First, we shall give a case where $\mathrm{Aut}_{\mathbb{C}}(E)^G = \mathbb{C}^*$. Henceforth we assume that the base space X is a real G-module of finite dimension.

DEFINITION. Let H be a closed subgroup of G. A complex G-module of finite dimension is called *multiplicity free* with respect to H if when viewed as an H-module, each irreducible H-module occurs with multiplicity at most 1.

REMARK. Any complex G-module of dimension one is multiplicity free with respect to any closed subgroup H.

THEOREM 4. *If the fiber E_0 of E over the origin $0 \in X$, where X is a real G-module, is irreducible as a G-module and multiplicity free with respect to the principal isotropy group H of X, then* $\mathrm{Aut}_{\mathbb{C}}(E)^G = \mathbb{C}^*$.

PROOF. The restricted bundle $E|X^H$ decomposes into eigenbundles ([1], Th 1.6.2) as H-vector bundles:

$$E|X^H = \oplus_{\chi \in Irr(H)} E_\chi$$

where $Irr(H)$ denotes the isomorphism classes of complex irreducible H-modules and E_χ is the eigenbundle of type χ . Let A be an element of $\mathrm{Aut}_{\mathbb{C}}(E)^G$. The multiplicity free condition and the Schur's lemma ([4], Ch. 2, Prop. 4) imply that A gives multiplication by a non-zero complex number on each fiber of E_χ. Therefore $A|E_\chi$ can be viewed as a nowhere zero complex valued polynomial function on X^H. Since every non-constant polynomial has a complex root, this means that the polynomial is a constant a_χ. Note that the origin 0 lies in X^H and A

restricted to the G-module E_0 is an automorphism of E_0. Since E_0 is irreducible as a G-module by assumption, $A|E_0$ gives a scalar multiplication, again by the Schur's lemma. This means that constants a_χ are independent of χ. Thus A gives a scalar multiplication on $E|X^H$. It follows from the equivariance of A that A gives a scalar multiplication on E restricted to the G-orbit GX^H of X^H. Since H is the principal isotropy group of X, the closure of GX^H agrees with X; so A gives a scalar multiplication on the whole of E by continuity. □

REMARK. Theorem 4 does not hold in a smooth category. See "Remark" in Section 2.

COROLLARY 5. *If E is a complex G-line bundle over a real representation space X, then E^τ and $E^{\tau'}$ are isomorphic for any $\tau, \tau' \in \mathrm{AL}(E)^G$.*

PROOF. $\mathrm{Aut}_{\mathbb{C}}(E)^G = \mathbb{C}^*$ by Theorem 4 and the Remark above Theorem 4. So it follows from Theorem 3. □

REMARK. In a smooth category, any G-vector bundle over a representation space is trivial. But in algebraic category, there are non-trivial complex vector bundles over a representation space (see [3] for the real vector bundles and see [5] for the complex vector bundles).

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