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NONLINEAR INTEGRODIFFERENTIAL EQUATIONS OF SOBOLEV TYPE WITH NONLOCAL CONDITIONS IN BANACH SPACES

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Abstract. In this paper we prove the existence of solutions of a nonlinear integrodifferential equation of Sobolev type with nonlocal conditions. The results are obtained by using compact semigroups and the Schauder fixed point theorem.

1. Introduction

The problem of existence of solutions of evolution equations with nonlocal conditions in Banach spaces has been studied first by Byszewski [7]. In that paper he has established the existence and uniqueness of mild, strong and classical solutions of the following nonlocal Cauchy problem:

$$u'(t) + Au(t) = f(t, u(t)), \quad t \in (t_0, t_0 + a], \tag{1}$$

$$u(t_0) + g(t_1, t_2, \cdots, t_p, u(\cdot)) = u_0 \tag{2}$$

where $-A$ is the infinitesimal generator of a C_0-semigroup $T(t)$, on a Banach space X, $0 \leq t_0 < t_1 < t_2 < \cdots < t_p \leq t_0 + a, \quad a > 0, u_0 \in X$ and $f : [t_0, t_0 + a] \times X \to X$, $g : [t_0, t_0 + a]^p \times X \to X$ are given functions. Subsequently, several authors have investigated the same type of problem to a different class of abstract differential equations in Banach spaces [1-4, 8, 11-13]. Brill [6] and Showalter [15] established the existence of solutions of semilinear evolution equations of Sobolev type in Banach spaces. This type of equations arises in various applications such as in

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the flow of fluid through fissured rocks [5], thermodynamics [9] and shear in second order fluids [10,16]. The purpose of this paper is to prove the existence and uniqueness of mild and strong solutions for a semilinear integrodifferential equation of Sobolev type with nonlocal conditions of the form

$$(3)\qquad (Bu(t))' + Au(t) = f(t, u(t), \int_0^t k(t, s, u(s))ds), \quad t \in (0, a],$$

$$(4)\qquad u(0) + g(t_1, t_2, \cdots, t_p, u(t_1), u(t_2), \cdots u(t_p)) = u_0$$

where $f : I \times X \times X \to Y$, $k : I^2 \times X \to X$ and $g : I^p \times X^p \to Y$ and Y are Banach spaces, and $I = [0, a]$.

2. Preliminaries

Definition 1. [14]: A continuous solution u of the integral equation

$$\begin{aligned} u(t) =& B^{-1}T(t)Bu_0 - B^{-1}T(t)Bg(t_1, \cdots, t_p, u(t_1), \cdots, u(t_p)) \\ &+ \int_0^t B^{-1}T(t-s)f(s, u(s), \int_0^s k(s, \tau, u(\tau))d\tau)ds, t \in I \end{aligned}$$

is said to be a mild solution of problem (3)-(4) on I.

Definition 2 [14]. A function u is said to be a strong solution of problem (3)-(4) on I if u is differentiable almost everywhere on I, $u' \in L^1(I, X)$, $u(0) + g(t_1, t_2, \cdots, t_p, u(t_1), u(t_2), \cdots u(t_p)) = u_0$ and

$$(Bu(t))' + Au(t) = f(t, u(t), \int_0^t k(t, s, u(s))ds), a.e. \text{ on } I.$$

In order to prove our main theorem we assume certain conditions on the operators A and B. Let X and Y be Banach spaces with norm $|\cdot|$ and $\|\cdot\|$ respectively. The operators $A : D(A) \subset X \to Y$ and $B : D(B) \subset X \to Y$ satisfy the following hypotheses:

(M1) A and B are closed linear operators,

(M2) $D(B) \subset D(A)$ and B is bijective,

(M3) $B^{-1} : Y \to D(B)$ is continuous.

The hypothesis (M_1),(M_2) and the closed graph theorem imply the boundedness of the linear operator $AB^{-1} : Y \to Y$ and $-AB^{-1}$ generates a uniformly continuous semigroup $T(t)$, $t \geq 0$, of bounded linear operators from Y into Y.

(M4) For each $t \in I$ and for some $\lambda \in \rho(-AB^{-1})$, the resolvent set of $-AB^{-1}$, the resolvent $R(\lambda, -AB^{-1})$ is a compact operator.

LEMMA 1 [14]. *Let A be the infinitesimal generator of a uniformly continuous semigroup $T(t)$. If the resolvent set $R(\lambda, A)$ of A is compact for every $\lambda \in \rho(A)$, then $T(t)$ is a compact semigroup.*

From the above fact that $-AB^{-1}$ generates a compact semigroup $T(t), t \geq 0$ and so $\max_{t\in I} \|T(t)\|$ is finite. We denote $M = \max_{t\in I} \|T(t)\|$, $R = \|B\|$ and $R^* = \|B^{-1}\|$.

Further assume that,

(M5) $k : I^2 \times X \to X$ is continuous in t,

(M6) $f : I \times X^2 \to Y$ is continuous in t on I and there exist a constant $L > 0$ such that

$$\|f(t, u(t), \int_0^t k(t, s, u(s))ds \leq L \quad \text{for} \quad t \in I \quad \text{and} \quad u \in X,$$

(M7) $g : I^p \times X^p \to Y$ is continuous and there exists a constant $G > 0$ such that

$$G = \max_{u\in C(I,X)} \|g(t_1, t_2, \cdots, t_p, u(t_1), \cdots, u(t_p))\|.$$

3. Main results

THEOREM 1. *If the assumptions $M_1 \sim M_7$ hold, then the problem (3)-(4) has a mild solution on I.*

PROOF. Let $E = C(I, Y)$ and

$$Y_0 = \{u : u \in Y, u(0) + g(t_1, t_2, \cdots, t_p, u(t_1), \cdots, u(t_p)) = u_0, \\ \|u\| \leq r, \quad t \in I\},$$

where $r = R^*MR\|u_0\| + R^*MRG + R^*MLa$. Clearly, Y_0 is a bounded closed convex subset of Y. We define a mapping $F : E \to Y_0$ by

$$(Fu)(t) = B^{-1}T(t)Bu_0 - B^{-1}T(t)Bg(t_1, t_2, \cdots, t_p, u(t_1), \cdots, u(t_p)) + \int_0^t B^{-1}T(t-s)f(s, u(s), \int_0^s k(s, \tau, u(\tau))d\tau)ds, t \in I.$$

Obviously, F maps Y_0 into itself and it is also continuous.

Moreover, F maps Y_0 into a precompact subset of Y_0. We prove that the set $Y_0(t) = \{(Fu)(t) : u \in y_0\}$ is precompact in X for every fixed $t \in I$. Obviously, for $t = 0$, the set $Y_0(0) = \{u_0 - g\}$ is precompact. Let $t > 0$ be fixed. Define

$$\begin{aligned}(F_\epsilon)(t) &= B^{-1}T(t)Bu_0 - B^{-1}T(t)Bg(t_1, \cdots, t_p, u(t_1), \cdots, u(t_p)) \\ &\quad + \int_0^{t-\epsilon} B^{-1}T(t-s)f(s, u(s), \int_0^s k(s, \tau, u(\tau))d\tau)ds, \quad t \in I \\ &= B^{-1}T(t)Bu_0 - B^{-1}T(t)Bg(t_1, \cdots, t_p, u(t_1), \cdots, u(t_p)) \\ &\quad + T(\epsilon)\int_0^{t-\epsilon} B^{-1}T(t-\epsilon-s)f(s, u(s), \int_0^s k(s, \tau, u(\tau))d\tau)ds, \\ &\qquad t \in I.\end{aligned}$$

Since $T(t)$ is compact for every $t > 0$, the set $Y_0(t) = \{F_\epsilon(u)(t) : u \in Y_0\}$ is precompact in X for every ϵ , $0 < \epsilon < t$. Furthermore, for $u \in Y_0$, we have

$$\begin{aligned}&\|(Fu)(t) - (F_\epsilon u)(t)\| \\ &\leq \int_{t-\epsilon}^t \|B^{-1}\|\|T(t-s)\|\|f(s, u(s), \int_0^s k(s, \tau, u(\tau))d\tau\|ds \\ &\leq R^*ML_\epsilon\end{aligned}$$

which implies that $Y_0(t)$ is totally bounded. i.e., $Y_0(t)$ is precompact in X.

We shall show that $F(Y_0) = S = \{Fu : u \in Y_0\}$ is an equicontinuous family of functions.

For $0 < s < t$, we have

$$
\begin{aligned}
&\|(Fu)(t) - (Fu)(s)\| \\
&\leq \|B^{-1}(T(t) - T(s))Bu_0\| \\
&\quad + \|B^{-1}(T(t) - T(s))Bg(t_1, \cdots, t_p, u(t_1), \cdots, u(t_p))\| \\
&\quad + \int_0^t \|B^{-1}\| \|T(t-\eta) - T(s-\eta)\| \|f(\eta, u(\eta), \int_0^\eta k(\eta, \tau, u(\tau))d\tau)\| d\eta \\
&\quad + \int_s^t \|B^{-1}\| \|T(s-\eta)\| \|f(\eta, u(\eta), \int_0^\eta k(\eta, \tau, u(\tau))d\tau)\| d\eta \\
&\leq (R^* R\|u_0\| + R^* RG)\|T(t) - T(s)\| \\
&\quad + R^* L \int_0^t \|T(t-\eta) - T(s-\eta)\| d\eta + R^* ML|t-s|.
\end{aligned}
$$

The right hand side of the above inequality is independent of $u \in Y_0$ and tends to zero as $s \to t$ as a consequence of the continuity of $T(t)$ in the uniform operator topology for $t > 0$ which follows from the compactness of $T(t), t > 0$. It is also clear that S is bounded in Y. Thus by Arzela-Ascoli's theorem, S is precompact. Hence by the Schauder fixed point theorem, F has a fixed point in Y_0 and any fixed point of F is a mild solution of (3)-(4) on I such that $u(t) \in X$ for $t \in I$. □

Next we prove that the problem (3)-(4) has a strong solution.

THEOREM 2. *Assume that*

(i) *Conditions $M_1 \sim M_7$ hold,*
(ii) *Y is a reflexive Banach space with norm $\|\cdot\|$,*
(iii) *$f : I \times X^2 \to Y$ is continuous in t on I and there exist constants $L > 0$ and $L_1 > 0$ such that*

$$
\begin{aligned}
&\|f(t, u(t), \int_0^t k(t, s, u(s))ds)\| \leq L \quad \text{and} \\
&\|f(t, u_1, v_1) - f(s, u_2, v_2)\| \\
&\leq L_1 [\|t - s\| + \|u_1 - u_2\| + \|v_1 - v_2\|], \\
&\text{for } \ s, t \in I, \quad u_i, v_i \in X,
\end{aligned}
$$

(iv) $k : I^2 \times X \to Y$ *is continuous in* t *and there exist constants* $L_2 > 0$ and $L_3 > 0$ *such that*

$$\|k(t,s,u)\| \leq L_2,$$
$$\|k(t,\tau,u) - k(s,\tau,u)\| \leq L_3|t-s|, \quad \text{for } \tau, s, t \in I, \quad u \in X,$$

(v) $g(t_1, \cdots, t_p, u(t_1), \cdots, u(t_p)) \in D(-AB^{-1}), u_0 \in D(-AB^{-1})$.

Then u *is a strong solution of problem* (3)-(4) *on* I.

PROOF. Since all the assumptions of Theorem 1 are satisfied, then the problem (3)-(4) has a mild solution belonging to $C(I,X)$. Now we shall show that u is a strong solution of problem (3)-(4) on I.

For any $t \in I$, we have

$$\begin{aligned}
&\|u(t+h) - u(t)\| \\
&= B^{-1}[T(t+h) - T(t)]Bu_0 \\
&\quad - B^{-1}[T(t+h) - T(t)]Bg(t_1, \cdots, t_p, u(t_1, \cdots, u(t_p)) \\
&\quad + \int_0^h B^{-1}T(t+h-s)f(s, u(s), \int_0^s k(s,\tau,u(\tau))d\tau)ds \\
&\quad + \int_h^{t+h} B^{-1}T(t+h-s)f(s, u(s), \int_0^s k(s,\tau,u(\tau))d\tau)ds \\
&\quad - \int_0^t B^{-1}T(t-s)f(s, u(s), \int_0^s k(s,\tau,u(\tau))d\tau)ds \\
&= B^{-1}T(t)[T(h) - I]Bu_0 \\
&\quad - B^{-1}T(t)[T(h) - I]Bg(t_1, \cdots, t_p, u(t_1), \cdots, u(t_p)) \\
&\quad + \int_0^h B^{-1}T(t+h-s)f(s, u(s), \int_0^s k(s,\tau,u(\tau))d\tau)ds \\
&\quad + \int_0^t B^{-1}T(t-s)f(s+h, u(s+h), \int_0^{s+h} k(s+h,\tau,u(\tau))d\tau)ds \\
&\quad - \int_0^t B^{-1}T(t-s)f(s, u(s), \int_0^s k(s,\tau,u(\tau))d\tau)ds.
\end{aligned}$$

Using our assumptions we observe that

$$\begin{aligned}
&\|u(t+h)-u(t)\| \\
\leq& R^*MRh\|B^{-1}Au_0\| + RMR^*h\|B^{-1}Ag(t_1,\cdots,t_p,u(t_1),\cdots,u(t_p)) \\
&+hR^*ML + R^*M\int_0^t L_1\Big[h+\|u(s+h)-u(s)\| \\
&+\int_0^s \|k(s+h,\tau,u)-k(s,\tau,u)\|d\tau + \int_s^{s+h}\|k(s+h,\tau,u)\|d\tau\Big]ds \\
\leq& R^*MRh\|B^{-1}Au_0\| + RMR^*h\|B^{-1}Ag(t_1,\cdots,t_p,u(t_1),\cdots,u(t_p))\| \\
&+hR^*ML + R^*ML_1\int_0^t [h+\|u(s+h)-u(s)\| + L_2ha + L_3h]ds \\
\leq& Ph + R^*ML_1\int_0^t \|u(s+h)-u(s)\|ds,
\end{aligned}$$

where

$$\begin{aligned}
P =& R^*MR\|B^{-1}Au_0\| + RMR^*\|B^{-1}Ag(t_1,\cdots,t_p,u(t_1),\cdots u(t_p))\| \\
&+R^*ML + MR^*L_1a + R^*ML_1L_2a + R^*ML_1L_3.
\end{aligned}$$

By Gronwall's inequality

$$\|u(t+h)-u(t)\| \leq Ph\exp(R^*ML_1a), \quad \text{for} \quad t\in J.$$

Therefore, u is Lipschitz continuous on I.
The Lipschitz continuity of u on I combined with (iii) implies that

$$t \to f(t,u(t),\int_0^t k(t,s,u(s))ds)$$

is Lipschitz continuous on I.
Using a Theorem in [15] and the definition of strong solution we observe that the linear Cauchy problem:

$$\begin{aligned}
(Bv(t))' + Av(t) &= f\left(t,u(s),\int_0^t k(t,s,u(s))ds\right), t\in(0,a] \\
v(0) &= u_0 - g(t_1,\cdots,t_p,u(t_1),\cdots,u(t_p))
\end{aligned}$$

has a unique strong solution v satisfying the equation

$$\begin{aligned} v(t) &= B^{-1}T(t)Bu_0 - B^{-1}T(t)Bg(t_1, \cdots, t_p, u(t_1), \cdots, u(t_p)) \\ &\quad + \int_0^t B^{-1}T(t-s)f(s, u(s), \int_0^s k(s, \tau, u(\tau))d\tau)ds, t \in I \\ &= u(t). \end{aligned}$$

Consequently, u is a strong solution of problem (3)-(4) on I. □

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