THE GENERALIZED WITT ALGEBRAS
USING ADDITIVE MAPS I

Ki-Bong Nam

ABSTRACT. Kawamoto generalized the Witt algebra using $F[x_1^\pm 1, \ldots, x_n^\pm 1]$ instead of $F[x_1, \ldots, x_n]$. We construct the generalized Witt algebra $W_{g,h,n}$ by using additive mappings $g, h$ from a set of integers into a field $F$ of characteristic zero. We show that the Lie algebra $W_{g,h,n}$ is simple if $g$ and $h$ are injective, and also the Lie algebra $W_{g,h,n}$ has no ad-diagonalizable elements.

1. Introduction

Let $F$ be a field of characteristic zero contains the set of integers $Z$. Let $N$ be the set of non-negative integers. The Witt algebra is called the general algebra by Rudakov [8]. Kac [2] studied the generalized Witt algebra on the $F$-algebra in the formal power series $F[[x_1, \ldots, x_n]]$ for a fixed positive integer $n$. Nam [5] constructs the Lie algebra on the $F$-subalgebra $F[e^{\pm x_1}, \ldots, e^{\pm x_n}, x_1, \ldots, x_{n+1}, \ldots, x_{n+m}]$ in the formal power series $F[[x_1, \ldots, x_n]]$ for the given positive integers $n$ and $m$.

Consider the generalized Witt algebra $W(n, m)$ having a basis

$$B = \{ e^{a_1 x_1} \cdots e^{a_n x_n} x_1^{i_1} \cdots x_{n+m}^{i_{n+m}} \partial_k | a_1, \ldots, a_n, i_1, \ldots, i_{n+m} \in Z, \\
1 \leq k \leq n+m \}$$

Received February 9, 1998.
1991 Mathematics Subject Classification: Primary 17B40, 17B65; Secondary 17B56, 17B68.
Key words and phrases: simple Lie algebras, Lie derivation, Lie automorphism.
Ki-Bong Nam

with Lie bracket on basis elements given by

\[
[e^{a_1 x_1} \ldots e^{a_n x_n} x_1^{i_1} \ldots x_n^{i_{n+m}} \partial_{i_1} \ldots e^{b_n x_n} x_1^{j_1} \ldots x_n^{j_{n+m}} \partial_{j_1}] \\
= b_1 e^{a_1 x_1 + b_1 x_1} \ldots e^{a_n x_n + b_n x_n} x_1^{i_1 + j_1} \ldots x_n^{i_{n+m} + j_{n+m}} \partial_{i_1} \\
+ t_i e^{a_1 x_1 + b_1 x_1} \ldots e^{a_n x_n + b_n x_n} x_1^{i_1 + j_1} \ldots x_n^{i_{n+m} + j_{n+m}} x_{i_1}^{-1} \partial_{j_1} \\
- a_j e^{a_1 x_1 + b_1 x_1} \ldots e^{a_n x_n + b_n x_n} x_1^{i_1 + j_1} \ldots x_n^{i_{n+m} + j_{n+m}} \partial_{i_1} \\
- i_j e^{a_1 x_1 + b_1 x_1} \ldots e^{a_n x_n + b_n x_n} x_1^{i_1 + j_1} \ldots x_n^{i_{n+m} + j_{n+m}} x_{j_1}^{-1} \partial_{i_1}
\]

where \( b_i = 0 \) if \( n + 1 \leq i \leq n + m \), and \( a_j = 0 \) if \( n + 1 \leq j \leq n + m \). See [2], [5], [6].

In [3], it is noted that the Lie subalgebra \( W(0, m) \) of \( W(n, m) \) is the Witt algebra on \( F[x_1^{\pm 1}, \ldots, x_m^{\pm 1}] \).

Let \( g \) and \( h \) be additive maps from \( Z \) into \( F \), where \( Z \) is the set of integers. Let us define the Lie algebra \( W_{g,h,n} \) with basis

\[
\left\{ \begin{pmatrix} a_1 \\ i_1 \end{pmatrix} \ldots \begin{pmatrix} a_n \\ i_n \end{pmatrix} \mid a_1, \ldots, a_n, i_1, \ldots, i_n \in Z, 1 \leq k \leq n \right\}
\]

and a Lie bracket on basis elements given by

\[
\left[ \begin{pmatrix} a_1 \\ i_1 \end{pmatrix} \ldots \begin{pmatrix} a_n \\ i_n \end{pmatrix} , \begin{pmatrix} b_1 \\ j_1 \end{pmatrix} \ldots \begin{pmatrix} b_n \\ j_n \end{pmatrix} \right] \\
= g(b_k) \begin{pmatrix} a_1 + b_1 \\ i_1 + j_1 \end{pmatrix} \ldots \begin{pmatrix} a_n + b_n \\ i_n + j_n \end{pmatrix} \\
+ h(j_k) \begin{pmatrix} a_1 + b_1 \\ i_1 + j_1 \end{pmatrix} \ldots \begin{pmatrix} a_k + b_k \\ i_k + j_k - 1 \end{pmatrix} \begin{pmatrix} a_{k+1} + b_{k+1} \\ i_{k+1} + j_{k+1} \end{pmatrix} \ldots \begin{pmatrix} a_n + b_n \\ i_n + j_n \end{pmatrix} \\
- g(a_i) \begin{pmatrix} a_1 + b_1 \\ i_1 + j_1 \end{pmatrix} \ldots \begin{pmatrix} a_n + b_n \\ i_n + j_n \end{pmatrix} \\
- h(i_i) \begin{pmatrix} a_1 + b_1 \\ i_1 + j_1 \end{pmatrix} \ldots \begin{pmatrix} a_i + b_i \\ i_i + j_i - 1 \end{pmatrix} \begin{pmatrix} a_{i+1} + b_{i+1} \\ i_{i+1} + j_{i+1} \end{pmatrix} \ldots \begin{pmatrix} a_n + b_n \\ i_n + j_n \end{pmatrix}.
\]

We extend the above Lie bracket linearly to the given basis \( B \) (see [5], [6], [8]). Here it is not hard to show that the above bracket satisfies the Jacobi identity.

In this paper we will prove the following main theorem.

**Theorem.** The Lie algebra \( W_{g,h,n} \) is simple.
2. Main results

The Lie algebra $W_{g,h,n}$ has a $\mathbb{Z}^n$-gradation as follows [3]:

\[(1) \quad W_{g,h,n} = \bigoplus_{(a_1, \ldots, a_n) \in \mathbb{Z}^n} W(a_1, \ldots, a_n),\]

where $W(a_1, \ldots, a_n)$ is the subspace of $W_{g,h,n}$ with basis

\[B = \left\{ \left( \begin{array}{c} a_1 \\ i_1 \\ \vdots \\ a_n \\ i_n \end{array} \right) | a_1, \ldots, a_n, i_1, \ldots, i_n \in \mathbb{Z}, 1 \leq k \leq n \right\}.\]

Let $W(a_1, \ldots, a_n)$ denote the $(a_1, \ldots, a_n)$-homogeneous component of $W_{g,h,n}$ and call elements in $W(a_1, \ldots, a_n)$ the $(a_1, \ldots, a_n)$-homogeneous elements. Note that the $(0, \ldots, 0)$-homogeneous component is isomorphic to the Witt algebra $W(n)$ [8]. From now on let the $(0, \ldots, 0)$-homogeneous component denote the $0$-homogeneous component.

Let us define $H(l)$ to be the number of different homogeneous components for any $l \in W_{g,h,n}$.

For the simplicity of $W_{g,h,n}$, we assume the map $g$ and $h$ are injective maps. Now we introduce a lexicographic ordering on two basis elements of $W_{g,h,n}$ as follows:

For any two basis elements \( \left( \begin{array}{c} a_1 \\ i_1 \\ \vdots \\ a_n \\ i_n \end{array} \right)_l \) and \( \left( \begin{array}{c} b_1 \\ j_1 \\ \vdots \\ b_n \\ j_n \end{array} \right)_k \), we have

\( \left( \begin{array}{c} a_1 \\ i_1 \\ \vdots \\ a_n \\ i_n \end{array} \right)_l > \left( \begin{array}{c} b_1 \\ j_1 \\ \vdots \\ b_n \\ j_n \end{array} \right)_k \)

if

\( (b_1, \ldots, b_n, i_1, \ldots, j_n, k) \leq (a_1, \ldots, a_n, i_1, \ldots, i_n, l) \)

by the natural lexicographic ordering in $\mathbb{Z}^{2n} \times \mathbb{Z}$.

For any element $l \in W_{g,h,n}$, $l$ can be written as follows using the ordering and the gradation:

\[l = \sum_{i_1, \ldots, i_n, p} C(i_1, \ldots, i_n, p) \left( \begin{array}{c} a_{11} \\ i_1 \\ \vdots \\ a_{1n} \\ i_n \end{array} \right)_p + \cdots + \sum_{j_1, \ldots, j_n, q} C(j_1, \ldots, j_n, p) \left( \begin{array}{c} a_{t1} \\ i_1 \\ \vdots \\ a_{tn} \\ j_n \end{array} \right)_q \]

where $(a_{11}, \ldots, a_{1n}) > \cdots > (a_{t1}, \ldots, a_{tn})$ using the natural ordering on $\mathbb{Z}^n$.  

235
Next, define the string number $st(l) = t$ for $l$ (see [5], [6]), and $l_p(l)$ as
the max $\{i_1, \ldots, i_n, \ldots, j_1, \ldots, j_n\}$. For any basis element $\binom{a_1}{i_1} \cdots \binom{a_n}{i_n}_t$, in $B$, let us call $a_1, \ldots, a_n$ the upper indices and $i_1, \ldots, i_n$ the lower indices.

**Remark.** If $g$ and $h$ are inclusions, then we have $W_{g,h,n} = W(n,0)$
where $W(n,0)$ is the generalized Witt algebra studied by Kawamoto [3].

**Lemma 1.** If $l \in W_{g,h,n}$ is any non-zero element, then the ideal $\langle l \rangle$
generated by $l$ contains an element whose lower indices are positive.

**Proof.** Take an element $M = \binom{0}{j_1} \cdots \binom{0}{j_n}_t$ such that $j_1 >> \cdots >> j_n$ and $t$ such that either $a_t \neq 0$ or $i_t \neq 0$ of $l$, where $a >> b$ means $a$ is
sufficiently larger than $b$. Then $0 \neq [M, l]$ is the required element.

**Lemma 2.** If an ideal of $W_{g,h,n}$ contains $\binom{0}{0}_i \cdots \binom{0}{0}_t$, where $(1 \leq i \leq n)$,
then we have $I = W_{g,h,n}$.

**Proof.** The Witt algebra $W(n)$ is simple, so we know the ideal in the
lemma contains $W(n)$ [3].

For any basis element $\binom{a_1}{i_1} \cdots \binom{a_n}{i_n}_t$ of $W_{g,h,n}$, if $i_1 = 0$ and $a_1 \neq 0$, then we have

$$\begin{bmatrix}
\binom{0}{0}_t, \\
\binom{a_1}{i_1} \cdots \binom{a_n}{i_n}_t
\end{bmatrix} = g(a_1) \binom{a_1}{i_1} \cdots \binom{a_n}{i_n}_t \in I.$$

We can assume for any fixed $p \in N$, $\binom{a_1}{p} \cdots \binom{a_n}{i_n}_t \in I$, that we have

$$\begin{bmatrix}
\binom{0}{0}_t, \\
\binom{a_1}{p+1} \cdots \binom{a_n}{i_n}_t
\end{bmatrix} - h(p+1) \binom{a_1}{p} \cdots \binom{a_n}{i_n}_t
= g(a_1) \binom{a_1}{p+1} \cdots \binom{a_n}{i_n}_t \in I,$$

where $a_1 \neq 0$. We can assume for any fixed $p \in Z - N$ that

$$\begin{bmatrix}
\binom{0}{0}_t, \\
\binom{a_1}{p} \cdots \binom{a_n}{i_n}_t
\end{bmatrix} - g(a_1) \binom{a_1}{p} \cdots \binom{a_n}{i_n}_t
= h(p) \binom{a_1}{p-1} \cdots \binom{a_n}{i_n}_t \in I,$$

where $Z - N$ is the set of negative integers. Thus, we have $I = W_{g,h,n}$.
Therefore, we have proven the lemma.

**Theorem 1.** The Lie algebra $W_{g,h,n}$ is simple.
The generalized Witt algebras using additive maps I

Proof. Let $I$ be any non-zero ideal of $W_{g,h,n}$. Let us prove this theorem by induction on $H(l)$ for any non-zero element of $l \in I$.

Let $l$ be any nonzero element of $I$ whose lower indices are positive integers by lemma 1. If $H(l) = 1$ and $l \in W_{(0,\ldots,0)} \cong W(n)$, then we have proved the theorem by lemma 2.

Assume that we have proved the theorem for any element $l \in I$ such that $H(l) = p$. Consider the element $l \in I$ such that $H(l) = p + 1$. If $l$ contains a $(0,\cdots,0)$-homogeneous component, then we have

$$0 \neq l_1 = \begin{bmatrix} \left( \begin{array}{c} 0 \\ 0 \end{array} \right)_t, \left( \begin{array}{c} 0 \\ 0 \end{array} \right)_t, \cdots, \left( \begin{array}{c} 0 \\ 0 \end{array} \right)_t, l \end{bmatrix}$$

is the element which has no $(0,\cdots,0)$-homogeneous elements by taking appropriate $t$. Then we have $H(l_1) < p + 1$, and the theorem is proved.

If $l$ contains no $(0,\cdots,0)$-homogeneous component, then $l$ has an $(a_1,\cdots,a_n)$-homogeneous component. If we take an element $\left( \begin{array}{c} -a_1 \\ 0 \end{array} \right)_t \cdots \left( \begin{array}{c} -a_n \\ 0 \end{array} \right)_t$ and take an appropriate $t$, then we have an element $l_2 = \left[ \begin{array}{c} \left( \begin{array}{c} -a_1 \\ 0 \end{array} \right)_t \cdots \left( \begin{array}{c} -a_n \\ 0 \end{array} \right)_t, l \end{array} \right] \neq 0$ such that $H(l_2) < p + 1$. Therefore, we have proven the theorem by induction and Lemma 2.

\[ \square \]

Corollary 1. The Lie algebra $W(n,0)$ is simple.

Proof. If we take an additive embedding $g, h : Z \to F$, then we get the required result (see [5], [6]). \[ \square \]

It is interesting problem to find all the automorphisms of $W_{g,l,1}$, where $I : Z \to F$ is an embedding.

Theorem 2. For any automorphism $\theta \in Aut(W_{g,l,1})$,

$$\theta \left( \begin{array}{c} 0 \\ 0 \end{array} \right)_1 = \sum_j C_j \left( \begin{array}{c} 0 \\ j \end{array} \right)_1,$$

where $C_j \in F$.

Proof. It is not difficult to prove this theorem using the gradation (1) of $W_{g,l,1}$ and the action of $\left( \begin{array}{c} 0 \\ 0 \end{array} \right)_1$ on the 0-homogeneous component $W_0$ whose vector space basis is $\{ \left( \begin{array}{c} 0 \\ l \end{array} \right)_1 | l \in Z \}$ as an adjoint map. \[ \square \]

If the lower indices of $W_{g,l,1}$ are zero, then this is Block algebra [1], thus all the automorphisms of this Lie algebra are decided by Block in

237
Theorem 3 of [1]. The element \( l \in W_{g,h,n} \) is ad-diagonalizable element if \( [l, m] = \alpha(m)m \) for \( m \in B \) and \( \alpha(m) \in F \).

We have a following proposition.

**Proposition 1.** The Lie algebra \( W_{g,h,n} \) has no non-zero ad-diagonalizable elements with respect to the basis \( B \).

**Proof.** Since \( W_{g,h,n} \) is \( Z^n \)-graded Lie algebra, all the ad-diagonalizable elements are in the \( (0, \cdots, 0) \)-homogeneous component. The \( (0, \cdots, 0) \)-homogeneous component is isomorphic to the Witt algebra \( W(n) \) [8]. Thus all the ad-diagonalizable elements of \( W_{g,h,n} \) are of the form \( \sum_{i=1}^{n} C_i \binom{0}{i} \) where \( C_i \in F \). But \( \sum_{i=1}^{n} C_i \binom{0}{i}, \binom{a}{0} \neq \alpha \binom{a}{0} \) for any \( \alpha \in F \) where \( a \neq 0 \). Therefore, we have proved the proposition. \( \square \)

**Remark.** For the non-existence of ad-diagonalizable elements of \( W(n,0) \) see Corollary 1 of [5].

**Acknowledgement.** The author thanks the referee for valuable suggestions on this paper.

**References**


Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA

E-mail: nam@math.wisc.edu