ON NON-PROPER PSEUDO-EINSTEIN RULED REAL HYPERSURFACES IN COMPLEX SPACE FORMS

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ABSTRACT. In the paper [12] we have introduced the new kind of pseudo-Einstein ruled real hypersurfaces in complex space forms $M_n(c)$, $c \neq 0$, which are foliated by pseudo-Einstein leaves. The purpose of this paper is to give a geometric condition for non-proper pseudo-Einstein ruled real hypersurfaces to be totally geodesic in the sense of Kimura [8] for $c > 0$ and Ahn, Lee and the present author [1] for $c < 0$.

1. Introduction

A complex $n(\geq 2)$-dimensional Kaehlerian manifold of constant holomorphic sectional curvature $c$ is called a complex space form, which is denoted by $M_n(c)$. A complete and simply connected complex space form is a complex projective space $P_n(\mathbb{C})$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $H_n(\mathbb{C})$, according as $c > 0$, $c = 0$ or $c < 0$. The induced almost contact metric structure of a real hypersurface $M$ of $M_n(c)$ is denoted by $(\phi, \xi, \eta, g)$.

There exist many studies about real hypersurfaces of $M_n(c)$. One of the first research is the classification of homogeneous real hypersurfaces in a complex projective space $P_n(\mathbb{C})$ by Takagi [14], who showed that these hypersurfaces of $P_n(\mathbb{C})$ could be divided into six types which are said to be of type $A_1, A_2, B, C, D$, and $E$, and in Cecil-Ryan [4] and Kimura [7] proved that they are realized as the tubes of constant curvature.
radius over Kaehlerian submanifolds if the structure vector field $\xi$ is principal. Also Berndt [2,3] showed recently that all real hypersurfaces with constant principal curvatures of a complex hyperbolic space $H_n(C)$ are realized as the tubes of constant radius over certain submanifolds when the structure vector field $\xi$ is principal. Nowadays in $H_n(C)$ they are said to be of type $A_0, A_1, A_2,$ and $B$.

When the structure vector field $\xi$ is not principal, Kimura [8] and Ahn, Lee and the present author [1] have constructed an example of ruled real hypersurfaces foliated by totally geodesic leaves, which are integrable submanifolds of the distribution $T_0$ defined by the subspace $T_0(x) = \{X \in T_x M | X \perp \xi, x \in M\}$, along the direction of $\xi$ and Einstein complex hypersurfaces in $P_n(C)$ and $H_n(C)$ respectively. The expression of the Weingarten map is given by

$$A\xi = \alpha\xi + \beta U, \ AU = \beta \xi \text{ and } AX = 0,$$

where we have defined a unit vector $U$ orthogonal to $\xi$ in such a way that $\beta U = A\xi - \alpha \xi$ and $\beta$ denotes the length of a vector field $A\xi - \alpha \xi$ and $\beta(x) \neq 0$ for any point $x$ in $M$, and for any $X$ in the distribution $T_0$ and orthogonal to $\xi$. Recently, several characterizations of such kind of ruled real hypersurfaces have been studied by the papers ([1], [3], [8], [9] and [13]). Moreover, among them there are so many ruled real hypersurfaces, which are foliated in parallel by the leaves of the distribution $T_0 = \{X \in T_x M | X \perp \xi\}$ along the integral curve of the structure vector $\xi$. Then in such a situation the vector field $U$ defined in above is always parallel along the direction of $\xi$.

Now as a general extension of this fact we introduce a new kind of ruled real hypersurfaces in $M_n(c)$ foliated by pseudo-Einstein leaves, which are integrable submanifolds of the distribution $T_0$ defined by the subspace $\{X \in T_x M | X \perp \xi\}$, along the direction of $\xi$ and pseudo-Einstein complex hypersurfaces in $M_n(c)$. Then such kind of ruled real hypersurfaces are said to be pseudo-Einstein, because its Ricci tensor of the integral submanifold $M(t)$ is given by

$$S^t = (\frac{n}{2}c - \mu)I + (\mu - \lambda)\{U \otimes U^* + \phi U \otimes (\phi U)^*\}.$$
Moreover, its expression of the Weingarten map is given by
\[ AU = \beta \xi + \gamma U + \delta \phi U \quad \text{and} \quad A\phi U = \delta U - \gamma \phi U. \]

In Lemma 3.1 we know that the function \( \lambda \) in above is given by \( \lambda = 2(\gamma^2 + \delta^2) \). When \( \lambda = \mu \), ruled real hypersurfaces foliated by such kind of leaves are said to be \textit{Einstein}. In particular, \( \lambda = \mu = 0 \), this kind of Einstein ruled real hypersurfaces are congruent to ruled real hypersurfaces in \( M_n(c) \) foliated by totally geodesic Einstein leaves \( M_{n-1}(c) \), which are said to be \textit{totally geodesic} ruled real hypersurfaces in the sense of Kimura [8] for \( c > 0 \) and Ahn, Lee and the present author [1] for \( c < 0 \). In such a situation the function \( \gamma \) and \( \delta \) both vanish identically.

When the function \( \mu = 0 \) and at least one of the functions \( \gamma \) and \( \delta \) vanishes identically, this kind of \textit{pseudo-Einstein} ruled ones are said to be \textit{non-proper}. Of course, totally geodesic ruled ones in the sense of Kimura [8] and Ahn, Lee and the present author [1] are contained in the class of \textit{non-proper} pseudo-Einstein ruled real hypersurfaces.

Then it naturally rises to the question that “Whether this kind of non-proper \textit{pseudo-Einstein} ruled real hypersurfaces in \( M_n(c) \) except totally geodesic ruled ones can be existed or not?” Or otherwise, “What kind of geometric condition can be imposed for non-proper pseudo Einstein ruled ones to be congruent to one of geodesic ruled ones?” From this point of view we answer this problem affirmatively and assert the following:

\textbf{Theorem.} Let \( M \) be a non-proper pseudo-Einstein ruled real hypersurface in \( M_n(c) \), \( c \neq 0 \), \( n \geq 2 \). If the vector \( U \) is parallel along the direction of \( \xi \), then \( M \) is locally congruent to one of ruled real hypersurfaces with each leaves totally geodesics and parallel along the direction of the structure vector field \( \xi \).

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\textbf{2. Preliminaries}

First of all, we recall fundamental properties of real hypersurfaces
of a complex space form. Let $M$ be a real hypersurface of a complex $n$-dimensional complex space form $M_n(c)$ of constant holomorphic sectional curvature $c(\neq 0)$ and let $C$ be a unit normal vector field on a neighborhood of a point $x$ in $M$. We denote by $J$ an almost complex structure of $M_n(c)$. For a local vector field $X$ on a neighborhood of $x$ in $M$, the transformation of $X$ and $C$ under $J$ can be represented as

$$JX = \phi X + \eta(X)C, \quad JC = -\xi,$$

where $\phi$ defines a skew-symmetric transformation on the tangent bundle $TM$ of $M$, while $\eta$ and $\xi$ denote a 1-form and a vector field on a neighborhood of $x$ in $M$, respectively. Moreover, it is seen that $g(\xi, X) = \eta(X)$, where $g$ denotes the induced Riemannian metric on $M$. By properties of the almost complex structure $J$, the set $(\phi, \xi, \eta, g)$ of tensors satisfies

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi \xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1,$$

where $I$ denotes the identity transformation. Accordingly, the set is so called an almost contact metric structure. Furthermore the covariant derivative of the structure tensors are given by

$$(\nabla_X \phi) Y = \eta(Y)AX - g(A X, Y)\xi, \quad \nabla_X \xi = \phi AX,$$

where $\nabla$ is the Riemannian connection of $g$ and $A$ denotes the shape operator with respect to the unit normal $C$ on $M$.

Since the ambient space is of constant holomorphic sectional curvature $c$, the equation of Gauss and Codazzi are respectively given as follows

$$(\nabla_X \phi) Y = \eta(Y)AX - g(A X, Y)\xi, \quad \nabla_X \xi = \phi AX,$$

$$R(Y, Z)X = \frac{c}{4} \{g(Z, X)Y - g(Y, X)Z + g(\phi Z, X)\phi Y - g(\phi Y, X)\phi Z - 2g(\phi Y, Z)\phi X\} + g(A Z, X)AY - g(A Y, X)A Z,$$

$$\nabla_X A - \nabla_Y A + \frac{c}{4} \{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\},$$

where $R$ denotes the Riemannian curvature tensor of $M$ and $\nabla_X A$ denotes the covariant derivative of the shape operator $A$ with respect to $X$. Now let us suppose that the structure vector $\xi$ is a principal vector.
with principal curvature $\alpha$, that is, $A\xi = \alpha\xi$. Then, differentiating this, we have

$$\tag{2.4} (\nabla_X A)\xi = (X\alpha)\xi + \alpha\phi AX - A\phi AX,$$

where we have used (2.1). Then it follows

$$\tag{2.5} g((\nabla_X A)Y, \xi) = (X\alpha)\eta(Y) + \alpha g(Y, \phi AX) - g(Y, A\phi AX)$$

for any tangent vector fields $X$ and $Y$ on $M$. By the equation of Codazzi (2.3), we have

$$\tag{2.6} 2A\phi AX - \frac{c}{2}\phi X = \alpha (\phi A + A\phi) X.$$

Now in order to get our results, we introduce a lemma, which was derived from the formulas (2.4), (2.5) and (2.6), in the paper [5] due to Ki and the present author as follows:

**Lemma 2.1.** Let $M$ be a real hypersurface in a complex space form $M_n(c), n \geq 2$. If it satisfies

$$\tag{2.7} A\phi + \phi A = 0,$$

then we have $c = 0$.

3. Pseudo-Einstein ruled real hypersurface

This section is concerned with the necessary properties about *pseudo-Einstein ruled* real hypersurfaces. Before going to give the notion of pseudo-Einstein ruled ones, we recall a ruled real hypersurface $M$ of $M_n(c), c \neq 0$ which is defined in Kimura [7]. Let us denote by $D$ a $J$-invariant integrable $(2n - 2)$-dimensional distribution defined on $M_n(c)$ whose integral manifolds are holomorphic planes normal to the plane spanned by unit normals $C$ and $JC$ and let $\gamma : I \rightarrow M_n(c)$ be an integral curve for the vector $\xi = -JC$.

For any $t \in I$ let $M_{n-1}^{(t)}(c)$ be a totally geodesic complex hypersurface through the point $\gamma(t)$ of $M_n(c)$ which is orthogonal to a holomorphic
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plane spanned by $\gamma'(t)$ and $J\gamma'(t)$. Set $M = \{x \in M_{n-1}^{(t)}(c) : t \in I\}$. Then the construction of $M$ asserts that $M$ is a real hypersurface of $M_n(c)$, which is called a ruled real hypersurface. This means that there exists a ruled real hypersurfaces of $M_n(c)$ with the given distribution $D$. This kind of ruled real hypersurface is foliated by leaves, which are totally geodesic complex hypersurfaces $M_{n-1}^{(t)}(c)$. Then from its construction it can be easily seen that the expression of the Weingarten map is given by

$$A_\xi = \alpha \xi + \beta U, \quad AU = \beta \xi \text{ and } AX = 0,$$

where $U$ is a unit vector orthogonal to $\xi$ and $\alpha$ and $\beta$ ($\beta \neq 0$) denote certain differentiable functions defined on $M$ and for any $X$ in $D$ orthogonal to $U$. Moreover, it can be easily seen that the Ricci tensor $S^t$ of the complex hypersurface $M(t)$ in $M_n(c)$ is proportional to its Riemannian metric such that $S^t = \frac{\nu c}{2} g$. That is, all of its leaves are Einstein complex hypersurfaces in $M_n(c)$. So such a ruled real hypersurface is naturally said to be Einstein ruled.

Now let us consider more generalized notion than the above ones. We want to consider a generalized ruled real hypersurface $M$, which is foliated by pseudo-Einstein leaves. Here, the meaning of pseudo-Einstein leaves are integrable submanifolds of the distribution $D$ which are pseudo-Einstein complex hypersurfaces in $M_n(c)$. Then in this case, this kind of generalized ruled real hypersurface is said to be pseudo-Einstein ruled real hypersurfaces.

For the construction of this, let us also consider a regular curve $\gamma : I \to M_n(c)$. Then for any $t \in I$, let $\Gamma_{n-1}^{(t)}$ be a pseudo-Einstein complex hypersurface through the point $\gamma(t)$ of $M_n(c)$ which is orthogonal to a holomorphic plane spanned by $\gamma'(t)$ and $J\gamma'(t)$. Set $M = \{x \in \Gamma_{n-1}^{(t)} : t \in I\}$. Then this construction gives us many pseudo-Einstein ruled real hypersurface.

Now, let us consider two shape operators $A_C$ and $A_\xi$ of any integral submanifold $M(t) = \Gamma_{n-1}^{(t)}$ of the distribution $D = \{X \in T_xM | X \perp \xi\}$ in $M_n(c)$ in the direction of $C$ and $\xi$. For any unit vector field $V$ along $D$, let $V^*$ be the corresponding 1-form defined by $V^*(V) = g(V, V) = 1$. 320
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If they satisfy

\[ A_{\xi}^2 + A_C^2 = \mu I + \lambda (V \otimes V^* + \phi V \otimes (\phi V)^*) \]

for a certain vector field \( V \), where \( \lambda \) and \( \mu \) are smooth function on \( M \), then the real hypersurface \( M \) with the given distribution \( D \) of \( M_n(c) \) is said to be \textit{pseudo-Einstein ruled}. In particular, if \( \lambda = \mu \), then it is said to be \textit{Einstein ruled} and if \( \lambda = \mu = 0 \), then it is said to be \textit{totally geodesic} and \textit{Einstein ruled}, and is the ruled real hypersurface as discussed in above. Accordingly, we say that the real hypersurface \( M \) is \textit{pseudo-Einstein ruled}, \textit{Einstein ruled} or \textit{totally geodesic ruled}, then it is easily seen that any integral submanifold of \( D \), which is a submanifold of real codimension 2 in \( M_n(c) \), is \textit{pseudo-Einstein}, \textit{Einstein} or \textit{totally geodesic}, respectively.

Since \( T_0(=D) \) is integrable, we know that

\[
(3.1) \quad g((A\phi + \phi A)X, Y) = 0
\]

for any vector fields \( X \) and \( Y \) in \( T_0 \) (See Kimura [8], Kimura and Maeda [9]).

Now we are going to give a Ricci tensor of the integral submanifold \( M(t) \) of the distribution \( D \), which is a \textit{pseudo-Einstein} submanifold of real codimension 2 in \( M_n(c) \). Since \( M(t) \) is a submanifold of codimension 2, \( \xi \) and \( C \) are orthonormal vector fields on its leaf in \( M_n(c) \). So by the equation of Gauss, we have

\[
\tilde{\nabla}_XY = \nabla XY + g(AX, Y)C \\
= \nabla^t_XY + g(A_{\xi}X, Y)\xi + g(A_CCX, Y)C,
\]

where \( \tilde{\nabla} \) and \( \nabla^t \) are the covariant derivatives in the ambient space \( M_n(c) \) and in the submanifold \( M(t) \), respectively and moreover \( A_C \) and \( A_{\xi} \) are the shape operators in the direction of \( C \) and \( \xi \), respectively. Then we have

\[
g(\nabla XY, \xi) = g(\nabla X\xi, Y) = -g(\nabla X\xi, Y) = g(A_{\xi}X, Y),
\]

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for any \( X, Y \in T_0 \), from which it implies that

\[ (3.2) \quad A_\xi X = -\phi AX, \quad X \in T_0. \]

On the other hand, by the equation of Gauss, we have

\[ g(AX, Y) = g(A_C X, Y), \quad X, Y \in T_0 \]

and therefore

\[ (3.3) \quad A_C X = AX - \beta g(X, U)\xi, \quad X \in T_0. \]

By (3.1) we have a formula

\[ (3.4) \quad A\phi X = -\phi AX - \beta g(X, \phi U)\xi, \quad X \in T_0. \]

From this it can be easily seen that the traces of these two shape operators \( A_\xi \) and \( A_C \) are both equal to zero. Now by using (2.2), the curvature tensor of the integral submanifold \( M(t) \) is given by

\[
g(R^t(X, Y)Z, W) = \frac{c}{4} \left\{ g(Y, Z)g(X, W) - g(X, Z)g(Y, W) + g(\phi Y, Z)g(\phi X, W) - g(\phi X, Z)g(\phi Y, W) - 2g(\phi X, Y)g(\phi Z, W) \right\} + g(A_\xi Y, Z)g(A_\xi X, W) + g(A_C Y, Z)g(A_C X, W) - g(A_\xi X, Z)g(A_\xi Y, W) - g(A_C X, Z)g(A_C Y, W)\]

for any vector fields \( X, Y, Z \) and \( W \) in \( D \). Since the traces of the above two shape operators \( A_\xi \) and \( A_C \) are both equal to zero, its Ricci tensor \( S^t \) of \( M(t) \) in \( M_n(c) \) is given by

\[ (3.5) \quad g(S^t Y, Z) = \sum_{i=1}^{2n-2} g(R^t(e_i, Y)Z, e_i) = \frac{n}{2} cg(Y, Z) - g((A_\xi^2 + A_C^2)Y, Z) \]

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for any $Y, Z$ in $D$. Then the tensor $A^2_{\xi} + A^2_{\phi}$ of the pseudo-Einstein submanifold $M(t)$ can be constructed in such a way that

$$
\begin{aligned}
(A^2_{\xi} + A^2_{\phi})U &= \lambda U, \\
(A^2_{\xi} + A^2_{\phi})\phi U &= \lambda \phi U, \\
(A^2_{\xi} + A^2_{\phi})X &= \mu X, \quad X \in D \perp U, \phi U,
\end{aligned}
$$

(3.6)

where $\lambda$ and $\mu$ are smooth functions on $M(t)$. So its Ricci tensor $S^t$ of $M(t)$ is given by

$$
S^t = \left(\frac{n}{2}c - \mu\right)I + (\mu - \lambda)\{U \otimes U^* + \phi U \otimes (\phi U)^*\}.
$$

Then from the formula (3.5) it follows

**Lemma 3.1.** (See [12]) Let $M$ be a pseudo-Einstein and not Einstein ruled real hypersurfaces in $M_n(c)$, $c \neq 0$, $n \geq 3$. Then we have

$$
\begin{aligned}
AU &= \beta \xi + \gamma U + \delta \phi U, \\
A\phi U &= \delta U - \gamma \phi U, \quad \lambda = 2(\gamma^2 + \delta^2)
\end{aligned}
$$

(3.7)

In particular, if it is totally geodesic, we have $\gamma = \delta = 0$.

**Proof.** Naturally let us put

$$
\begin{aligned}
A\xi &= \alpha \xi + \beta U, \\
AU &= \beta \xi + \gamma U + \delta \phi U + \epsilon X, \\
A\phi U &= -\gamma \phi U + \delta U - \epsilon \phi X,
\end{aligned}
$$

(3.8)

for some vector field $X$ orthogonal to $\xi, U$ and $\phi U$ where in the third equation we have used the condition (3.1), because the distribution $D$ is integrable. Since $M$ is supposed to be proper pseudo-Einstein, we may put $\lambda \neq \mu$. In order to prove $\epsilon = 0$, firstly let us prove the following

$$
A^2 U = (\alpha + \gamma)\beta \xi + \left(\beta^2 + \frac{\lambda}{2}\right)U.
$$

(3.9)
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Indeed, (3.2), (3.3) and the first formula of (3.6) imply

\[ \lambda U = -A_\xi \phi AU + A_C(AU - \beta \xi) \]
\[ = \phi A \phi AU + A(AU - \beta \xi) - \beta g(AU - \beta \xi, U) \xi \]
\[ = 2\{A^2U - \beta A \xi - \beta g(AU, U) \xi\}, \]

where in the third equality we have used the condition (3.4). Secondly, we calculate the following

(3.10) \[ A^2 \phi U = \beta \delta \xi + \frac{\lambda}{2} \phi U. \]

In fact, (3.2), (3.3) and the second formula of (3.6) give

\[ \lambda \phi U = (A^2_\xi + A^2_C) \phi U \]
\[ = \phi A^2 U + A^2 \phi U - \beta^2 \phi U - \beta g(A \phi U, U) \xi. \]

So by (3.8) we get the above (3.10). Finally we give the following for any \( X \) orthogonal to \( \xi, U \) and \( \phi U \).

(3.11) \[ A^2 X = \beta \epsilon \xi + \frac{\mu}{2} X, \]

because the third formula of (3.6) and the condition (3.1) imply that

\[ \mu X = -A_\xi \phi AX + A_C\{AX - \beta g(X, U) \xi\} \]
\[ = 2(A^2 X - \beta g(AX, U) \xi). \]

Now let us apply the shape operator \( A \) to the second formula of (3.8) and use also (3.8) and (3.9). Then

\[ \epsilon AX = \left( \frac{\lambda}{2} - \gamma^2 - \delta^2 \right) U - \gamma \epsilon X + \delta \epsilon \phi X \]
\[ = \epsilon^2 U - \gamma \epsilon X + \delta \epsilon \phi X, \]
where we have used

\[(3.12) \quad \|A\phi U\|^2 = \gamma^2 + \delta^2 + \epsilon^2 = \frac{\lambda}{2},\]

which can be obtained from (3.8) and (3.10). So let us assume \(\epsilon \neq 0\), then \(AX = \epsilon U - \gamma X + \delta \phi X\). This implies

\[
A^2 X = \epsilon AU - \gamma AX + \delta A\phi X \\
= (\beta \xi + \gamma U + \delta \phi U + \epsilon X) - \gamma(\epsilon U - \gamma X + \delta \phi X) \\
- (\epsilon \phi U - \gamma \phi X - \delta X) \\
= \epsilon \beta \xi + (\epsilon^2 + \gamma^2 + \delta^2) X.
\]

From this together with (3.11) it follows

\[\mu = 2(\gamma^2 + \delta^2 + \epsilon^2).\]

Then by (3.12) we have \(\lambda = \mu\), which makes a contradiction. So we should have \(\epsilon = 0\). It completes the proof of Lemma 3.1. \(\square\)

**Remark 3.1.** When both the functions \(\lambda\) and \(\mu\) in (3.6) vanish identically, (3.10) and (3.11) imply respectively

\[A\phi U = 0 \text{ and } AX = 0\]

for any \(X\) orthogonal to \(\xi\), \(U\) and \(\phi U\). Then from this together with (3.12) it follows the function \(\lambda = 0\), that is \(\gamma = \delta = 0\). Then naturally, \(M\) is congruent to totally geodesic ruled real hypersurfaces in \(M_n(c)\), \(c \neq 0\).

**Remark 3.2.** When the function \(\mu\) in (3.6) vanishes identically, (3.11) gives \(\|AX\| = 0\). This implies \(\epsilon = 0\). So it naturally satisfies

\[
\begin{align*}
AU &= \beta \xi + \gamma U + \delta \phi U, \\
A\phi U &= \delta U - \gamma \phi U, \\
AX &= 0, \quad X \perp \xi, U, \phi U.
\end{align*}
\]
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When \( \mu = 0 \) and at least one of the function \( \gamma \) and \( \delta \) vanishes identically, \( M \) is said to be non-proper pseudo-Einstein ruled real hypersurfaces. So for convenience sake let us say the function \( \delta \) vanishes identically. Then by Remark 3.2 we can put

\[
\begin{align*}
A\xi &= \alpha\xi + \beta U, \\
AU &= \beta\xi + \gamma U, \\
A\phi U &= -\gamma\phi U, \\
AX &= 0
\end{align*}
\]

(3.13)

for any \( X \in T_1 \), where \( T_1 \) denotes a distribution defined by a subspace \( T_1(x) = \{ u \in T_0(x) : g(u, U(x)) = g(u, \phi U(x)) = 0 \} \).

Next the covariant derivative \((\nabla_X A)Y\) with respect to \( X \) and \( Y \) in \( T_0 \) is explicitly expressed. The equation (2.3) of Codazzi gives us

\[
(\nabla_X A)\xi - (\nabla_{\xi} A)X = -\frac{c}{4}\phi X.
\]

By the direct calculation of the left hand side of the above relation and using the second equation of (3.2) we get

\[
\begin{align*}
d\alpha(X)\xi + \alpha\phi AX + d\beta(X)U + \beta\nabla_X U - A\phi AX \\
- \nabla_{\xi}(AX) + A\nabla_{\xi}X + \frac{c}{4}\phi X = 0, \quad X \in T_0.
\end{align*}
\]

(3.14)

Now from (3.1), (3.2) and the above equation we can derive the following

**Lemma 3.2.** Let \( M \) be a non-proper pseudo-Einstein ruled real hypersurfaces in \( M_n(c), c \neq 0, n \geq 2 \). Then it follows

\[
\beta\nabla_X U = \begin{cases}
\{ \beta^2 - \gamma^2 - \alpha\gamma - \frac{\xi}{4} + 2g(\nabla_\xi U, \phi U)\gamma \}\phi U, & X = U \\
-(\xi\gamma)\phi U - \gamma\phi\nabla_\xi U, & X = \phi U \\
-\frac{\xi}{4}\phi X - g(X, \phi \nabla_\xi U)\gamma\phi U, & X \in T_1,
\end{cases}
\]

(3.15)

and

\[
\begin{align*}
d\beta(X) &= \begin{cases}
0, & X = U \\
\gamma^2 - \alpha\gamma + \beta^2 + \frac{\xi}{4} - 2\gamma g(\nabla_\xi U, U), & X = \phi U \\
-\gamma g(\nabla_\xi U, U), & X \in T_1.
\end{cases}
\]

(3.16)

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**Proof.** Putting $X = U$ in (3.14) and taking an inner product with $U$ imply

$$
d_{\beta}(U) = 0.
$$

Moreover, by taking an orthogonal part from $\xi$ and $U$, we know that

$$
\beta \nabla_{U} U + \alpha \phi A U - A \phi A U + \frac{c}{4} \phi U - \beta^{2} \phi U + A \nabla_{\xi} U - \gamma \nabla_{\xi} U = 0.
$$

From this, together with (3.13), it follows that

$$
\beta \nabla_{U} U = \left\{ \beta^{2} - \gamma^{2} - \alpha \gamma - \frac{c}{4} + 2 g(\nabla_{\xi} U, \phi U) \gamma \right\} \phi U.
$$

So we have the first formula of (3.15). Also putting $X = \phi U$ in (3.14) and using (3.13), we have

$$
d\alpha(\phi U)\xi + \alpha \gamma U + d\beta(\phi U)U + \beta \nabla_{\phi U} U - A \phi A \phi U
$$

(3.17)

$$
+ \nabla_{\xi}(\gamma \phi U) + A \nabla_{\xi} \phi U = -\frac{c}{4} \phi^{2} U.
$$

On the other hand, we can put

$$
\nabla_{\xi} U = g(\nabla_{\xi} U, \phi U) \phi U + g(\nabla_{\xi} U, Z) Z
$$

for some unit $Z$ in $T_{1} = [\xi, U, \phi U]^{\perp}$ which is the orthogonal complement of the distribution $[\xi, U, \phi U]$ spanned by $\xi, U$ and $\phi U$. So by (3.13), it follows that

$$
A \phi \nabla_{\xi} U = -g(\nabla_{\xi} U, \phi U) A U.
$$

Then the last two terms of the left side of (3.17) becomes

$$
\nabla_{\xi}(\gamma \phi U) + A \nabla_{\xi} (\phi U) = d\gamma(\xi) \phi U + \gamma\{-\beta \xi + \phi \nabla_{\xi} U\}
$$

(3.18)

$$
- g(\nabla_{\xi} U, \phi U) A U - \beta\{\alpha \xi + \beta U\}.
$$

Substituting (3.18) into (3.17) and comparing an orthogonal part from $\xi$ and $U$ imply

$$
\beta \nabla_{\phi U} U = -d\gamma(\xi) \phi U - \gamma \phi \nabla_{\xi} U.
$$

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So, this gives the second formula of (3.15). Moreover, if we take an inner product (3.17) with $U$, we also get the second formula of (3.16).

On the other hand, (3.13) and (3.14) imply

\[(3.19) \quad d\alpha(X)\xi + d\beta(X)U + \beta\nabla_X U + A\nabla_\xi X + \frac{c}{4}\phi X = 0\]

for any $X \in T_1$. From this, if we take an inner product with $U$, it follows

$$d\beta(X) = -\gamma g(\nabla_\xi X, U),$$

which gives the third formula of (3.16).

Now for any $X \in T_1$ we can express the vector $\nabla_\xi X$ in such a way that

$$\nabla_\xi X = g(\nabla_\xi X, U)U + g(\nabla_\xi X, \phi U)\phi U + g(\nabla_\xi X, Z)\dot{Z}$$

for some unit $Z$ in $T_1$. Then by applying the shape operator $A$ to this formula and substituting this into (3.19), we have

$$d\alpha(X)\xi + d\beta(X)U + \beta\nabla_X U + g(\nabla_\xi X, U)AU$$
$$+ g(X, \phi \nabla_\xi U)\gamma \phi U + \frac{c}{4}\phi X = 0.$$ 

From this, if we compare the part orthogonal to $\xi$ and $U$, we have

$$\beta \nabla_X U = -\frac{c}{4}\phi X - g(X, \phi \nabla_\xi U)\gamma \phi U, \quad X \in T_1.$$ 

Accordingly, we have completed the proof of Lemma 3.2. \qed

Now let us calculate

$$g((\nabla_X A)Y, \xi)$$
$$= g((\nabla_X A)\xi, Y) = g(\nabla_X (A\xi) - A\nabla_X \xi, Y)$$
$$= g((X\alpha)\xi + a\nabla_X \xi + (X\beta)U + \beta \nabla_X U - A\phi AX, Y)$$
$$= a g(\phi AX, Y) + d\beta(X) g(Y, U) + \beta g(\nabla_X U, Y) - g(A\phi AX, Y).$$

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Now any \( X \in T_0 \) can be written in such a way that

(3.20) \[ X = g(X, U)U + g(X, \phi U)\phi U + g(X, Z)Z \]

for some unit \( Z \in T_1 \). Then for any \( X \) in \( T_1 \) this expression and Lemma 3.2 imply the followings:

\[ \beta \nabla_X U = \beta g(X, U)\nabla_U U + \beta g(X, \phi U)\nabla_{\phi U} U + g(X, Z)\beta \nabla_Z U \]

\[ = g(X, U)\{\beta^2 - \gamma^2 - \alpha \gamma - \frac{c}{4} + 2\gamma g(\nabla_\xi U, \phi U)\}\phi U \]

\[ + g(X, \phi U)\{-d\gamma(\xi)\phi U - \gamma \phi \nabla_\xi U\} \]

\[ + g(X, Z)\{-\frac{c}{4}\phi Z - \gamma g(Z, \phi \nabla_\xi U)\phi U\}, \]

and

\[ d\beta(X) = g(X, \phi U)\{\gamma^2 - \alpha \gamma + \beta^2 + \frac{c}{4} - 2\gamma g(\nabla_\xi \phi U, U)\} \]

\[ - \gamma g(X, Z)g(\nabla_\xi Z, U). \]

So it follows that

(3.21) \[ g((\nabla_X A)Y, \xi) \]

\[ = d\beta(X)g(U, Y) + \beta g(\nabla_X U, Y) + \alpha g(\phi AX, Y) \]

\[ - g(A\phi AX, Y) \]

\[ = [g(X, U)\{\beta^2 - \gamma^2 - \alpha \gamma - \frac{c}{4} + 2\gamma g(\nabla_\xi U, \phi U)\} \]

\[ - g(X, \phi U)d\gamma(\xi) - \gamma g(X, Z)g(Z, \phi \nabla_\xi U)]g(\phi U, Y) \]

\[ + \{-\gamma g(X, \phi U)g(\phi \nabla_\xi U, Y) - \frac{c}{4}g(X, Z)g(\phi Z, Y)\} \]

\[ + \{\gamma^2 - \alpha \gamma + \beta^2 + \frac{c}{4} - 2\gamma g(\nabla_\xi \phi U, U)\}g(X, \phi U)g(U, Y) \]

\[ - \gamma g(X, Z)g(\nabla_\xi Z, U)g(U, Y) + \alpha g(\phi AX, Y) - g(A\phi AX, Y). \]

On the other hand, the expression (3.20) and (3.13) implies

\[ \phi AX = \gamma g(X, U)\phi U + \gamma g(X, \phi U)U, \]

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and
\[
\begin{align*}
A\phi AX &= \gamma g(X, U)A\phi U + \gamma g(X, \phi U)AU \\
&= -\gamma^2 g(X, U)\phi U + \gamma g(X, \phi U)(\beta \xi + \gamma U) \\
&= -\gamma^2 g(X, U)\phi U + \gamma\beta g(X, \phi U)\xi + \gamma^2 g(X, \phi U)U.
\end{align*}
\]

Substituting these formulas into (3.21) and taking account of the equation of Codazzi (2.3), we have for any \(X, Y \in T_0\) and \(Z \in T_1\)
\[
g((\nabla_\xi A)X, Y) \\
= g((\nabla X A)Y, \xi) \\
= \beta^2\{g(X, U)g(\phi U, Y) + g(X, \phi U)g(U, Y)\} \\
+ 2\gamma g(\nabla_\xi U, \phi U)\{g(X, U)g(\phi U, Y) + g(X, \phi U)g(U, Y)\} \\
- g(X, \phi U)g(Y, \phi U)\gamma(\xi) - \gamma g(X, Z)g(Z, \phi \nabla_\xi U)g(\phi U, Y) \\
- \gamma g(X, \phi U)g(\phi \nabla_\xi U, Y) - \gamma g(X, Z)g(\nabla_\xi Z, U)g(U, Y),
\]
where we have used the fact that
\[
-\frac{c}{4}\{g(X, U)g(\phi U, Y) - g(X, \phi U)g(U, Y) + g(X, Z)g(\phi Z, Y)\}
= -\frac{c}{4}g(\phi X, Y)
\]
because of (3.20). From this we can assert
\[
(\nabla_\xi A)U = \{\beta^2 + 2\gamma g(\nabla_\xi U, \phi U)\}\phi U + \theta(U)\xi,
\]
and
\[
(\nabla_\xi A)\phi U = \beta^2 U + 2\gamma g(\nabla_\xi U, \phi U)U \\
- d\gamma(\xi)\phi U - \gamma \phi \nabla_\xi U + \theta(\phi U)\xi.
\]

Now summing up the formulas in above, non-proper pseudo Einstein ruled real hypersurfaces in \(M_n(c)\) satisfy the followings:
(3.22)
\[
\begin{align*}
(\nabla_\xi A)U &\equiv \lambda \phi U \pmod{\xi}, \\
(\nabla_\xi A)\phi U &\equiv \lambda U - d\gamma(\xi)\phi U - \gamma \phi \nabla_\xi U \pmod{\xi}, \\
(\nabla_\xi A)X &\equiv -\gamma g(X, \phi \nabla_\xi U)\phi U + \gamma g(X, \nabla_\xi U)U \pmod{\xi}.
\end{align*}
\]
When we assume that the vector field $U$ is parallel along the direction $\xi$, then $\nabla_\xi U = 0$ and $\gamma = g(AU, U)$ is constant along the direction $\xi$. Then (3.22) together with this assumption imply that a non-proper pseudo Einstein ruled real hypersurface satisfies

$$ (\nabla_\xi A)X \equiv f\phi AX, $$

where $f\gamma = \beta^2$.

4. Proof of the Theorem

In this section we want to prove the main Theorem. It will be turned out that non-proper pseudo Einstein ruled real hypersurfaces in complex space form $M_n(c), c \neq 0$ are only totally geodesic ruled real hypersurfaces in $M_n(c)$ foliated in such a way that its structure vector $U$ is parallel along the direction of $\xi$. Namely it will be congruent to one of ruled real hypersurfaces in th sense of Kimura [8] for $c > 0$ and Ahn, Lee and Suh [1] for $c < 0$.

Let $M$ be a real hypersurface in complex space forms $M_n(c), c \neq 0$. Now let us denote by $T_0$ be a distribution defined by the subspace

$$ T_0(x) = \{X \in T_x M | g(X, \xi_x) = 0\} $$

in the tangent space $T_x M$ at any point $x$ in $M$, which is called a holomorphic distribution.

On the other hand, we have seen in section 3 that non-proper pseudo Einstein ruled real hypersurfaces in $M_n(c)$ with the vector field $U$ is parallel along the direction of $\xi$ satisfies

$$ (\nabla_\xi A)X \equiv f\phi AX \mod \xi $$

for any vector field $X$ in $T_0$ and a smooth function $f$ without zero points. Moreover, we have known that its structure vector $\xi$ is not principal.

First of all, we assert the following
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**Lemma 4.1.** Let $M$ be a real hypersurface satisfying

\begin{equation}
(\nabla_{\xi} A)X \equiv f\phi AX \pmod{\xi},
\end{equation}

for any vector field $X$ in $T_0$ and a smooth function $f$ without zero points, then the distribution $T_0$ is integrable.

**Proof.** By the assumption (4.1) and

$$g((\nabla_{\xi} A)X, Y) = g((\nabla_{\xi} A)Y, X)$$

it turns out to be

$$fg((A\phi + \phi A)X, Y) = 0$$

for any vector fields $X$ and $Y$ in $T_0$. Since the function $f$ has no zero points,

\begin{equation}
(4.2)
g((A\phi + \phi A)X, Y) = 0
\end{equation}

for any vector fields $X$ and $Y$ in $T_0$. It completes the proof. \qed

Let $M$ be a real hypersurface in $M_n(c)$, $c \neq 0$, $n \geq 2$ satisfying (4.1). Then the distribution $T_0$ is integrable by Lemma 4.1.

Now we can put $A_{\xi} = \alpha_{\xi} + \beta U$, where $U$ is a unit vector field in the holomorphic distribution $T_0$, and $\alpha$ and $\beta$ are smooth functions on $M$. So we may consider that the function $\beta$ does not vanish identically on $M$. Let $M_0$ be the non-empty open subset of $M$ consisting of points $x$ at which $\beta(x) \neq 0$. Moreover, the set $M_0$ is a dense subset of $M$.

In fact, we suppose that the interior of the subset $M - M_0$ is not empty. On the interior we see $\beta = 0$. Namely, $\xi$ is a principal vector. with principal curvature $\alpha$. Then by (4.2) we have

$$A\phi + \phi A = 0.$$ 

Since Lemma 2.1 is a local property, it implies $c = 0$ on the interior and hence on the whole $M$, a contradiction. So the interior of the subspace $M - M_0$ is empty, namely $M_0$ is dense in $M$. 

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By (2.3) and (4.1) there is a 1-form $\theta$ such that

$$(\nabla_Y A)\xi = f\phi AX - \frac{1}{4} c\phi Y + \theta(Y)\xi$$

for any vector field $Y$ in $T_0$.

Differentiating $A\xi = \alpha\xi + \beta U$ covariantly with respect to any vector field $X$ in $T_0$, we have

$$(4.3) \quad \beta \nabla_X U = A\phi AX + (f - \alpha)\phi AX - \frac{1}{4} c\phi X$$

$$- d\alpha(X)\xi - d\beta(X)U + \theta(X)\xi,$$

where we have used (4.1) and (4.2). By (4.2) the above equation can be reformed as

$$\beta \nabla_X U = - \phi A^2 X + (f - \alpha)\phi AX - \frac{1}{4} c\phi X$$

$$+ \{-d\alpha(X) - \beta g(AX, \phi U) + \theta(X)\} \xi - d\beta(X)U.$$

From this, if we take an inner product of the above equation with $\xi$, we get

$$-d\alpha(X) - \beta g(AX, \phi U) + \theta(X) = \beta g(AX, \phi U).$$

Thus it follows

$$(4.4) \quad \beta \nabla_X U = - \phi A^2 X + (f - \alpha)\phi AX - \frac{1}{4} c\phi X$$

$$+ \beta g(AX, \phi U)\xi - d\beta(X)U$$

for any vector field $X$ in $T_0$.

On the open subset $M_0$ we can put $AU = \beta \xi + \gamma U + \delta V$, where $\xi, U$ and $V$ are orthonormal vector fields, and $\gamma$ and $\delta$ are smooth functions on $M_0$. Now let us denote by $L(V, W, \cdots, X, Y)$ a subspace in the tangent subspace $T_x M$ spanned by any vectors $V, W, \cdots, X, Y$ at any point $x$. Then by Lemma 4.1 we also have the following.
Lemma 4.2. Let $M$ be a real hypersurface in $M_n(c)$, $n \geq 3$. If it satisfies

\begin{equation}
(\nabla_\xi A)X \equiv f\phi AX \pmod{\xi}
\end{equation}

for any vector field $X$ in $T_0$ and a smooth function $f$ without zero points, then $L(\xi, A\xi)$ is not $A$-invariant.

Proof. By Lemma 2.1 and the above remark we know that on the subset $M_0$ the vector $A\xi$ can be expressed as $A\xi = \alpha \xi + \beta U$, where $\beta$ is a smooth function defined on $M$ and $U$ is a unit vector orthogonal to $\xi$.

Now let us suppose that the linear subspace $L(\xi, A\xi)$ in the tangent space of $M$ is $A$-invariant. Then the vector $AU$ can be written in such a way that $AU = \beta \xi + \gamma U$. From this, together with the integrability of the distribution $T_0$ in Lemma 4.1, we have

\begin{equation}
A\phi U = -\gamma \phi U,
\end{equation}

because $(A\phi + \phi A)U = 0$. Differentiating $A\xi = \alpha \xi + \beta U$ along $X$ in $T_0$, by the assumption (4.1) we also have the formula (4.3). Then taking an inner product (4.3) with $\phi U$ and using (4.5) imply

\begin{equation}
\beta g(\nabla_X U, \phi U) = (f - \alpha - \gamma)g(AX, U) - \frac{1}{4}cg(X, U).
\end{equation}

Taking an inner product of (4.4) with $\phi U$, we obtain

$$
\beta g(\nabla_X U, \phi U) = -g(A^2 X, U) + (f - \alpha)g(AX, U) - \frac{1}{4}cg(X, U)
$$

for any vector field $X$ in $T_0$. From (4.5) and the above equation it follows that

$$
g(A^2 X, U) = \gamma g(AX, U).
$$

So it implies

$$
\beta g(AX, \xi) = 0
$$

for any vector field $X$ in $T_0$, a contradiction. Thus we have the conclusion. 

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Now we are going to prove our main Theorem in introduction. Let $M$ be a non-proper pseudo-Einstein ruled real hypersurfaces in $M_n(c)$. Then it satisfies

\[
\begin{aligned}
(\nabla_{\xi}A)U & \equiv \lambda \phi U \pmod{\xi} \\
(\nabla_{\xi}A)\phi U & \equiv \lambda U - d\tau(\xi)\phi U - \gamma \phi \nabla_{\xi} U \pmod{\xi} \\
(\nabla_{\xi}A)X & \equiv -g(X, \phi \nabla_{\xi} U)\gamma \phi U + \gamma g(X, \nabla_{\xi} U)U \pmod{\xi}.
\end{aligned}
\]

When we assume that $U$ is parallel along the direction of $\xi$, then $\nabla_{\xi}U = 0$ and the smooth function $\gamma$ is constant along the direction of $\xi$.

Now let us suppose that the function $\gamma$ has no zero points. Then by the assumption $\nabla_{\xi}U = 0$ and the formula (*), we know that non-proper pseudo-Einstein ruled real hypersurfaces in $M_n(c)$ satisfy (4.1) for a smooth function $f$ which has no zero points in such a way that

\[
(\nabla_{\xi}A)X \equiv f \phi AX \pmod{\xi}, \quad f = \frac{\beta^2}{\gamma}.
\]

Then Lemma 4.2 implies that $L(\xi, A\xi)$ is not $A$-invariant. But in section 3 we know that $AU = \beta \xi + \gamma U$ for a non-proper pseudo Einstein ruled real hypersurfaces. This makes a contradiction. So we should have that the function $\gamma$ has some zero points.

Now let us denote by $M'$ a subset in $M$ consisting of points at which $\gamma$ has the value 0. That is, the set $M' = \{x \in M | \gamma(x) = 0\}$ should be non-empty. Now we suppose that $M - M' \neq \phi$. Then on $M - M'$ we know that the function $\gamma$ has no zero points. Accordingly, by using the same arguments as in above, we can also makes a contradiction. So we should have $M - M' = \phi$. That is, the set $M'$ is dense in $M$. Then by the continuity, the function $\gamma$ vanishes identically on $M$. This means that $M$ is totally geodesic pseudo-Einstein ruled real hypersurface in the sense of Kimura [8] for $c > 0$ and Ahn, Lee and the present author [1] for $c < 0$. Consequently, we complete the proof of our Theorem.

References

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