COGRADIENTS IN FUZZY $BCK$-ALGEBRAS

Hee Sik Kim

Abstract. In this paper we apply the notion of $\triangleright_\mu$ and $\triangleleft_\mu$ to fuzzy $BCK$-algebra, and show that $\triangleleft_\mu$ is cogradient to a partial order of the $BCK$-algebra.

1. Introduction

J. Neggers ([7]) has defined a pogroupoid and he obtained a functorial connection between posets and pogroupoids and associated structure mappings. J. Neggers and H. S. Kim ([8]) demonstrated that a pogroupoid $X(\cdot)$ is modular* if and only if its associated poset $X(\leq)$ is $(C_2 + 1)$-free, a condition which corresponds naturally to the notion of sublattice (in the sense of Kelly-Rival [3, 5]) isomorphic to $N_5$, and that this is equivalent to the associativity of the pogroupoid. J. Neggers and H. S. Kim ([10]) introduced the notion of the relation $\triangleright_\mu$ on fuzzy pogroupoid, and proved that for given a pogroupoid $X(\cdot)$, the associated poset $X(\leq)$ is $(C_2 + 1)$-free iff the relation $\triangleright_\mu$ is transitive for any fuzzy subset $\mu$ of $X$. In this paper we apply the notion of $\triangleright_\mu$ and $\triangleleft_\mu$ to fuzzy $BCK$-algebra, and show that $\triangleleft_\mu$ is cogradient to a partial order of the $BCK$-algebra.

2. A relation $\triangleright_\mu$

The notion of $BCK$-algebras was formulated first in 1966 by K. Iséki. This notion was originated from two different ways. One is based on set theory, and the other is propositional calculi. A $BCK$-algebra is a

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non-empty set $X$ together with a binary operation $*$ and a constant 0 satisfying the following axioms: for all $x, y, z \in X$,

(I) $((x * y) * (x * z)) * (z * y) = 0$,

(II) $(x * (x * y)) * y = 0$,

(III) $x * x = 0$,

(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$,

(V) $0 * x = 0$.

The concept of a fuzzy set was introduced by L. A. Zadeh ([16]). A fuzzy subset of a set $X$ is a function $\mu : X \rightarrow [0, 1]$. The applications of fuzzy concepts to posets and groupoids have been investigated by several authors (including [2, 10, 13, 15, 17]). A map $\mu : X \rightarrow [0, 1]$ is called a fuzzy subalgebra of a BCK-algebra $X$ if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$, for any $x, y \in X$. Note that if $\mu$ is a fuzzy subalgebra of a BCK-algebra $X$ then $\mu(0) \geq \mu(x)$ for all $x \in X$.

Suppose $(X, *, 0)$ and $(Y, \ast', 0')$ are two BCK-algebras. A mapping $f : X \rightarrow Y$ is called a BCK-homomorphism if for any $x, y \in X$, $f(x * y) = f(x) * f(y)$. Moreover, if $f$ is one-one and onto, then we can say $f$ a BCK-isomorphism and denote it by $X \cong Y$. With this concept we have the following properties: (i) $f(0) = 0'$, and (ii) if $x * y = 0$ in $X$, then $f(x) \ast' f(y) = 0'$ in $Y$.

On the while, the concept of isomorphism in the poset theory is a little bit different from the concept of BCK-algebras. Even though there is a one-one and onto order-preserving mapping between two posets, the two posets need not be isomorphic ([1]). We say two posets $X$ and $Y$ are (poset)-isomorphic if there is a one-one and onto order preserving mapping $f$ and its inverse mapping $f^{-1}$ is also order preserving. There are two ways to define a partially ordered set: (i) weak inclusion; reflexive, anti-symmetric, transitive (ii) strong inclusion; irreflexive, transitive, and they are equivalent ([12, pp. 1-3]).

In a BCK-algebra $X$ we define a binary operation $\leq$ by $x \leq y$ if and only if $x * y = 0$. We can see that a BCK-algebra contains a poset structure in it. The poset $(X, \leq)$ is said to be the associated poset with the BCK-algebra $(X; *, 0)$. The association is not bi-unique, i.e., non-isomorphic BCK-algebras may have order-isomorphic posets associated with them.
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EXAMPLE 2.1. Consider the following two $BCK$-algebras having the same poset structure:

\[
\begin{array}{c|cccc}
*_1 & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 & 0 \\
3 & 3 & 3 & 3 & 0 & 0 \\
4 & 4 & 4 & 4 & 3 & 0 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
*_2 & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 & 0 \\
3 & 3 & 3 & 3 & 0 & 0 \\
4 & 4 & 3 & 4 & 1 & 0 \\
\end{array}
\]

Define a map $f : X := \{0, 1, 2, 3, 4\} \to X$ by $f(i) = i \ (i = 0, 1, 2, 3, 4)$. Then $f$ is a poset isomorphism, but not a $BCK$-isomorphism, since $f(4 *_{1} 1) = 4 \neq 3 = f(4) *_{2} f(1)$.

Let $\mu : X \to [0, 1]$ be a fuzzy subset of a $BCK$-algebra $X$. Define a relation $\triangleright_{\mu}$ on $X$ by

\[x \triangleright_{\mu} y \iff \mu(x * y) < \mu(y * x).\]

Since $x * x = 0$, $\mu(x * x) < \mu(x * x)$ does not hold, and hence the relation $\triangleright_{\mu}$ is irreflexive. Similarly, we define a relation $\triangleleft_{\mu}$ on $X$ by $x \triangleleft_{\mu} y \iff \mu(y * x) < \mu(x * y)$.

EXAMPLE 2.2. Consider the following $BCK$-algebra $X$ ([6, pp. 273]).
Define a map $\mu : X \to [0, 1]$ by $0 \leq \mu(0) < \mu(3) < \mu(4) < \mu(1) < \mu(2) \leq 1$. Then the transitivity of $\triangleright_\mu$ does not hold, since $1 \triangleright_\mu 3$ and $3 \triangleright_\mu 4$, but not $1 \triangleright_\mu 4$. If we define a map $\nu : X \to [0, 1]$ by $1 \geq \nu(0) > \nu(4) > \nu(3) > \nu(2) > \nu(1) \geq 0$, then $X(\triangleright_\nu)$ is a poset as following left Hasse diagram:

Moreover, if we define a fuzzy subset $\xi : X \to [0, 1]$ on the BCK-algebra $(X, \ast_1)$ described in Example 2.1 by $0 \leq \xi(0) = \xi(3) < \xi(1) = \xi(2) < \xi(4) \leq 1$, then $X(\triangleright_\xi)$ is a poset as the above right Hasse diagram.

**Theorem 2.3.** Let $(X; \ast, 0)$ be a BCK-algebra. Define a fuzzy subset $\mu : X \to [0, 1]$ by

$$\mu(x) := \begin{cases} a & \text{if } x = 0, \\ b & \text{otherwise}. \end{cases}$$

where $0 \leq a < b \leq 1$. Then $X(\triangleright_\mu)$ is a poset.

**Proof.** Let $x \triangleright_\mu y$ and $y \triangleright_\mu z$. Then $\mu(x \ast y) < \mu(y \ast x)$, $\mu(y \ast z) < \mu(z \ast y)$. This means $x \ast y = 0$ and $y \ast z = 0$, since $\mu$ is two-valued. It
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follows from $X(\leq)$ is a poset that $x \leq z$. By (IV) we obtain $x \ast z = 0$
and $z \ast x \neq 0$. Hence $\mu(x \ast z) = a < b = \mu(z \ast x)$, i.e., $x \triangleright_{\mu} z$. Thus
$X(\triangleright_{\mu})$ is a poset. \hfill \Box

In Theorem 2.3 we introduced two-valued fuzzy subset $\mu$ of a $BCK$-
algebra for $X(\triangleright_{\mu})$ to be a poset. We pose the following open problem:

**Problem.** Under what other condition(s) for $X(\triangleright_{\mu})$ to be a poset?

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Suppose $R_1$ and $R_2$ are relations on a set $X$. We shall consider relations $R_1$ and $R_2$ to be cogradient provided that $(x,y) \in R_i$ (or $xR_iy$) implies $(y,x) \notin R_j$, $i,j = 1,2$, $i \neq j$, where $x \neq y$. We then obtain the following result.

**Theorem 3.1.** If $(X;*,0)$ is a $BCK$-algebra, and if $\mu : X \to [0,1]$ is a fuzzy subalgebra of this $BCK$-algebra, then the relations $x \leq y$ iff $x \ast y = 0$ and $x \triangleleft_{\mu} y$ iff $\mu(y \ast x) < \mu(x \ast y)$ are cogradient.

**Proof.** Let $x,y \in X$ with $x \triangleleft_{\mu} y$. If $y < x$, then $x \ast y \neq 0$, but $y \ast x = 0$. Hence $\mu(0) = \mu(y \ast x) < \mu(x \ast y) \leq \mu(0)$, a contradiction. This means that $y < x$ does not hold. On other hand, let $x \leq y$ in $X(\leq)$. We may assume $x < y$ in $X(\leq)$, since $x \triangleleft_{\mu} x$ does not hold. Assume $y \triangleleft_{\mu} x$. Then $\mu(x \ast y) < \mu(y \ast x)$. Since $x < y$, $x \ast y = 0$, but $y \ast x \neq 0$. Hence $\mu(0) = \mu(x \ast y) < \mu(y \ast x) \leq \mu(0)$, a contradiction. It follows that $y \triangleleft_{\mu} x$ does not hold. This proves the theorem. \hfill \Box

Of course, in the general situation $X(\leq)$ and $X(\triangleright_{\mu})$ (or $X(\triangleleft_{\mu})$) may fail to be cogradient. A question arises to what extent the cogradience of $X(\leq)$ and $X(\triangleright_{\mu})$ (or $X(\triangleleft_{\mu})$) influences the "approximate" fuzzy subalgebra structure of the fuzzy subset $\mu$ of $X$.

Suppose that $(X;*,0)$ is a $BCK$-algebra and suppose that the fuzzy
subset $\mu$ is defined as follows:

$$
\mu(x) := \begin{cases}
  a & \text{if } x = 0, \\
  b & \text{otherwise.}
\end{cases}
$$
where $0 \leq a < b \leq 1$. Now suppose $x < y$. Then $x \ast y = 0$ and $y \ast x \neq 0$. Hence $\mu(x \ast y) = a < b = \mu(y \ast x)$, i.e., $x \triangleright^\mu y$. This means that $\triangleright^\mu$ is an extension of $\prec$. Conversely, if $x \triangleright^\mu y$, then $\mu(x \ast y) < \mu(y \ast x)$, whence $x \ast y = 0$ and $x < y$, since $y \ast x \neq 0$. Thus $\triangleright^\mu = \prec$, i.e., $X(\prec) = X(\triangleright^\mu)$ precisely. Thus we summarize:

**Theorem 3.2.** If $(X; \ast, 0)$ is a BCK-algebra and if $\mu$ is a fuzzy subset of $X$ where if $x \neq 0$, $\mu(0) = a < b = \mu(x)$, then $X(\prec) = X(\triangleright^\mu)$.

Thus we may "code" $X(\prec)$ precisely by taking $a = 0$ and $b = 1$, and within the class $X(\triangleright^\mu)$, $X(\prec)$ will be uniquely determined in this fashion.

Actually, if $(X; \ast, 0)$ is a $d$-algebra ([11]), i.e., if it satisfies conditions (III), (IV) and (V) for the BCK-algebra, then we may use the same scheme, i.e., we set

$$x \lessdot^\mu y \text{ provided } \mu(y \ast x) < \mu(x \ast y).$$

Thus, if $\mu : X \rightarrow [0, 1]$ is a fuzzy subalgebra of the $d$-algebra, then $\mu(x \ast y) \geq \min\{\mu(x), \mu(y)\}$ and $\mu(0) \geq \mu(x)$ for all $x \in X$.

Suppose now that we define $x < y$ iff $x \ast y = 0$ in a $d$-algebra $(X; \ast, 0)$. Then $X(\prec)$ is not necessarily a poset. However, if $\mu$ is a fuzzy subalgebra of $X$ and if $x < y$ then $x \ast y = 0$ and $y \ast x \neq 0$, and hence $\mu(y \ast x) \leq \mu(x \ast y) = \mu(0)$. It means that either $x \lessdot^\mu y$ or $\mu(y \ast x) = \mu(x \ast y)$, i.e., $y \lessdot^\mu x$ does not hold. Conversely, if $x \lessdot^\mu y$, then $y < x$ is impossible. It follows that:

**Corollary 3.3.** Theorem 3.1 holds if $(X; \ast, 0)$ is a $d$-algebra.

Similarly, we obtain:

**Corollary 3.4.** Theorem 3.2 holds if $(X; \ast, 0)$ is a $d$-algebra.

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**References**


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Hee Sik Kim, Department of Mathematics, Hanyang University, Seoul 133-791, Korea
E-mail: heekim@email.hanyang.ac.kr

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