\nu\text{-PATHS OF ARCS IN REGULAR MULTIPARTITE TOURNAMENTS}

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ABSTRACT. A \nu\text{-path of an arc } xy \text{ in a multipartite tournament } T \text{ is an oriented path in } T - y \text{ which starts at } x \text{ such that } y \text{ does not dominate the end vertex of the path. We show that if } T \text{ is a regular } n\text{-partite } (n \geq 7) \text{ tournament, then every arc of } T \text{ has a } \nu\text{-path of length } m \text{ for all } m \text{ satisfying } 2 \leq m \leq n - 2. \text{ Our result extends the corresponding result for regular tournaments, due to Alspach, Reid and Roselle [2] in 1974, to regular multipartite tournaments.}

1. Introduction

The vertex set of a digraph \(D\) is denoted by \(V(D)\). If \(xy\) is an arc of a digraph \(D\), then we say that \(x\) dominates \(y\). More generally, if \(A\) and \(B\) are two disjoint subdigraphs of \(D\) such that every vertex of \(A\) dominates every vertex of \(B\), then we say that \(A\) dominates \(B\), denoted by \(A \rightarrow B\). The outset \(N^{+}(x)\) of a vertex \(x\) is the set of vertices dominated by \(x\), and the inset \(N^{-}(x)\) is the set of vertices dominating \(x\). A digraph \(D\) is said to be regular if there is an integer \(r\) such that \(|N^{+}(x)| = |N^{-}(x)| = r\) holds for every \(x \in V(D)\).

A digraph obtained by replacing each edge of a complete \(n\)-partite graph with an arc or a pair of mutually opposite arcs is called a semicomplete \(n\)-partite digraph or a semicomplete multipartite digraph. A multipartite tournament is a semicomplete multipartite digraph without

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a cycle of length 2, and a tournament is an \( n \)-partite tournament having exactly \( n \) vertices.

Paths and cycles in a digraph are always assumed to be directed. A \textit{bypath} of an arc \( xy \) is a path from \( x \) to \( y \). Alspach, Reid and Roselle [2] proved that every arc of a regular tournament with \( n \geq 7 \) vertices has bypaths of all lengths \( \ell, 3 \leq \ell \leq n - 1 \). Further results about bypaths in tournaments (respectively, in local tournaments) can be found in [5] and [6] (respectively, in [4]).

It is not difficult to construct a regular \( n \)-partite \( (n \geq 3) \) tournament \( T \) such that \( T \) contains an arc having no bypath of length \( \ell \) for some \( \ell \) with \( 3 \leq \ell \leq n - 1 \). So, the result in [2] cannot be extended to multipartite tournaments in this way.

Note that the concept of bypaths defined as above has another representation, i.e., an arc \( xy \) of a tournament \( T \) has a bypath of length \( \ell \) if and only if \( T - y \) contains a path of length \( \ell - 1 \) which starts at \( x \), and \( y \) does not dominate the end vertex of the path.

In general, we define a \( \nu \)-\textit{path} of an arc \( xy \) in a digraph \( D \) as a path in \( D - y \) which starts at \( x \) such that \( y \) dominates the end vertex of the path only if the end vertex also dominates \( y \). Thus, the concept of \( \nu \)-paths in digraphs is a generalization of that of bypaths in tournaments.

In this paper, we prove the following theorem, and it is clear that our result generalizes the result of [2] for regular tournaments.

\textbf{Theorem.} \textit{Let} \( T \) \textit{be a regular} \( n \)-\textit{partite tournament with} \( n \geq 7 \). \textit{Then every arc of} \( T \) \textit{has a} \( \nu \)-\textit{path of length} \( m \) \textit{for all} \( m \) \textit{satisfying} \( 2 \leq m \leq n - 2 \).

\section{2. Proof of the theorem}

Let \( V_0, V_1, \ldots, V_{n-1} \) be the partite sets of \( T \). From the regularity of \( T \), it is not difficult to check that all partite sets of \( T \) have the same cardinality, say \( k \). So, it is clear that

\[ |N^+(x)| = |N^-(x)| = \frac{(n-1)k}{2} \quad \text{for each} \quad x \in V(T). \]

Let \( a_1a_0 \) be an arbitrary arc of \( T \) and assume without loss of generality that \( a_i \in V_i \) for \( i = 0, 1 \). We first show that \( a_1a_0 \) has a \( \nu \)-path of length 2. Since \( n \geq 7 \), there are at least two vertices \( b, c \) in \( N^+(a_1) - V_0 \) such
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that \( T[\{a_0, b, c\}] \) is a tournament. Without loss of generality, we assume \( b \to c \). If \( c \to x_0 \) for some \( x_0 \in V_0 \), then \( a_1bc \) (when \( x_0 = a_0 \)) or \( a_1cx_0 \) (when \( x_0 \neq a_0 \)) is a desired \( \nu \)-path of \( a_1a_0 \). So, we may assume that \( V_0 \to c \). Now, we see from the regularity of \( T \) that there exists a vertex \( x \) with \( c \to x \to a_0 \), and hence, \( a_1cx \) is a \( \nu \)-path of \( a_1a_0 \).

Suppose that \( a_1a_0 \) has a \( \nu \)-path \( P \) of length \( m-1 \) (say \( P = a_1a_2 \cdots a_m \)), but it has no \( \nu \)-path of length \( m \) for some \( m \) satisfying \( 3 \leq m < n - 1 \). Let

\[
\begin{align*}
A &= \{ x \mid x \in V_i, V_i \cap V(P) = \emptyset, x \to a_0, 2 \leq i \leq n - 1 \}, \\
B &= \{ y \mid y \in V_j, V_j \cap V(P) = \emptyset, a_0 \to y, 2 \leq j \leq n - 1 \}.
\end{align*}
\]

Let \( P_0 = \{ a_0, a_1, a_2, \ldots, a_m \} \). It is clear that \( A \cup B \neq \emptyset \) and every vertex in \( A \cup B \) is adjacent with all vertices in \( P_0 \). If \( T \) is a tournament, then the theorem holds by [2]. So, we prove the theorem for \( k \geq 2 \) and consider the following two cases:

**Case 1.** \( A \neq \emptyset \).

From the initial hypothesis that \( a_1a_0 \) has no \( \nu \)-path of length \( m \), we see that \( A \to a_m \), and consequently, \( A \to P \) holds.

Let \( a \) be an arbitrary vertex of \( A \). Since \( T \) is regular, it is easy to check that there is a vertex \( a' \) such that \( a_{m-2} \to a' \), but \( a \not\to a' \). Clearly, \( a' \notin P_0 \). If \( a \) and \( a' \) are adjacent, then we have \( a' \to a \), and hence, \( a_1a_0 \) has a \( \nu \)-path \( a_1 \cdots a_{m-2}a'aa_m \) of length \( m \), a contradiction. Assume now that \( a \) and \( a' \) belong to the same partite set of \( T \). Since \( |N^+(a')| = |N^+(a)| \) and \( a \to a_{m-2} \to a' \), there is a vertex \( u \) with \( a' \to u \to a \). Obviously, \( u \notin P_0 \). Thus, \( a_1a_0 \) has a \( \nu \)-path \( a_1 \cdots a_{m-2}a'ua \) of length \( m \), a contradiction.

**Case 2.** \( A = \emptyset \).

In this case we have \( B \neq \emptyset \). Assume without loss of generality that \( V_{n-1} \subseteq B \). From the regularity of \( T \) and the definition of \( B \), it is not difficult to check that every arc from \( a_0 \) to \( B \) is in a cycle of length 3.

(a): \( B \to a_1 \).

If there is a vertex \( b \in B \) with \( a_1 \to b \), then \( a_i \to b \) for all \( i \geq 2 \). Since \( a_0b \) is in a cycle of length 3, there is a vertex \( x \) such that \( b \to x \to a_0 \). It is easy to see that \( x \notin V(P) \). But now, \( a_1a_2 \cdots a_{m-1}bx \) is a \( \nu \)-path of \( a_1a_0 \) which is of length \( m \), a contradiction.

(b): \( B \to a_2 \).

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Assume, on the contrary, that there is a vertex \( b \in B \) such that \( a_2 \to b \), then \( a_i \to b \) for all \( i \geq 3 \). Thus, we have \( V_0 \to b \). It follows that \( a_0 \) is adjacent with each vertex of \( N^+(b) \), and furthermore, \( a_0 \to (N^+(b) - a_1) \cup V_j \), where \( V_j \) is the partite set of \( T \) which contains \( b \). From the assumption that \( |V_j| = k \geq 2 \), we have \( |N^+(a_0)| > |N^+(b)| \), a contradiction to the regularity of \( T \).

(c): \( m \geq 4 \).

If \( m = 3 \), then, by (b) and the assumption that \( n \geq 7 \), we have \( |N^-(a_2)| > |N^+(a_2)| \), a contradiction.

(d): \( a_m \to B \).

If there is a vertex \( b \in B \) (assume without loss of generality that \( b \in V_{n-1} \)) such that \( b \to a_m \), then we have \( b \to \{a_3, \cdots, a_{m-1}\} \). Since \( V_{n-1} \to a_2 \), \( b \) is adjacent to each vertex of \( N^+(a_2) \). Since \( |N^+(b) \cap P_0| = m \) and \( |N^+(a_2) \cap P_0| \leq m - 1 \), there is a vertex \( x \notin V(P) \) such that \( a_2 \to x \to b \). Hence, \( a_1a_2xba_4 \cdots a_m \) is of length \( m \), a contradiction.

(e): \( a_{m-1} \to B \).

Suppose, on the contrary, that \( b \to a_{m-1} \) for some \( b \in B \) (assume without loss of generality that \( b \in V_{n-1} \)). If there is a vertex \( u \in N^+(a_1) \setminus P_0 \) with \( u \to b \), then \( a_1uba_3 \cdots a_m \) is of length \( m \), a contradiction. Hence, we have that \( b \to N^+(a_1) \setminus P_0 \). From \( V_{n-1} \to a_1 \) and the regularity of \( T \) we conclude that

\[ |N^+(a_1) \cap P_0| \geq |N^+(b) \cap (P_0 \cup B)| \geq m - 1. \]

Suppose that \( m \geq 5 \). Then it is easy to see that \( b \to N^+(a_2) \setminus P_0 \). Since \( |N^+(a_2)| = |N^+(b)| \) and \( B \to a_2 \), \( |N^+(a_2) \cap P_0| \geq |N^+(b) \cap (P_0 \cup B)| \geq m - 1 \) holds. This implies that

\[ a_2 \to \{a_0, a_3, a_4, \cdots, a_m\} \] and \( N^+(b) \cap B = \emptyset \).

It is a simple matter to verify by (2) that \( a_0 \to \{a_3, a_4, \cdots, a_{m-1}\} \), and furthermore, \( a_1 \not\to a_3 \) (otherwise, \( a_1a_3 \cdots a_mba_2 \) yields a contradiction).

So, by (1), the following holds:

\[ a_1 \to \{a_4, a_5, \cdots, a_m\} \]

Assume that \( B \) contains at least two partite sets of \( T \). By (2), there is a vertex \( b' \in B \) with \( b' \to b \). So, we see from (3) and (2) that \( a_1a_4a_5 \cdots a_mb'ba_2 \) is a \( \nu \)-path of \( a_1a_0 \), a contradiction. Therefore, \( B = V_{n-1} \). Clearly, \( m = n - 2 \) and we may assume without loss of generality that \( a_i \in V_i \) for \( i = 2, 3, \cdots, n - 2 \).
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Let \( H = N^-(a_0) \setminus (P_0 \cup V_{m-1}) \). If \( a_{m-1} \to x \) for some \( x \in H \cup (V_0 - a_0) \), then \( a_1 a_m b a_3 a_4 \cdots a_{m-1} x \) is of length \( m \), a contradiction. Hence, we have that \( H \cup V_0 \to a_{m-1} \). But now, the following two inequalities

\[
\begin{align*}
|N^-(a_{m-1})| & \geq |V_0| + |H| + |\{a_1, a_2, a_{m-2}, b\}| = k + |H| + 4, \\
|N^-(a_0)| & \leq |V_{m-1} \setminus \{a_{m-1}\}| + |H| + |\{a_1, a_2, a_m\}| \\
& = k + |H| + 2
\end{align*}
\]

imply a contradiction to the regularity of \( T \).

Suppose now that \( m = 4 \). Since \( n \geq 7 \), \( B \) contains at least two partite sets of \( T \) and there is a vertex \( b' \in B \) which is adjacent with the vertex \( b \).

If \( b \to b' \), then \( a_1 \to \{a_2, a_3, a_4\} \) holds by (1). It follows that \( a_0 \to \{a_2, a_3\} \). Let \( F = N^-(a_0) \setminus P_0 \). Clearly, \( |F| \geq |N^-(a_0)| - 2 \). If there is a vertex \( x \in F \) with \( b' \to x \), then \( a_1 a_2 b' x \) is a \( \nu \)-path of \( a_1 a_0 \), a contradiction. Hence, \( F \to b' \). Now we see that \( |N^-(b')| \geq |F| + |\{a_0, a_m, b\}| \geq |N^-(a_0)| + 1 \) contradicts to the regularity of \( T \).

Assume now that \( b' \to b \). From (1) and (d), it is easy to check that \( a_0 \to a_2 \). Since \( |N^-(a_0)| = |N^-(b)| \), we see by the same arguments as above and (1) that \( |N^-(a_0) \cap P_0| \geq 3 \), i.e. \( \{a_3, a_4\} \to a_0 \). From \( V_{n-1} \to a_2 \) and the regularity of \( T \), we conclude that there is a vertex \( y \) with \( a_2 \to y \to b \). Obviously, \( y \notin \{a_0, a_1, a_3\} \) and \( a_1 a_2 y b a_3 \) is a \( \nu \)-path of \( a_1 a_0 \), a contradiction.

(f): \( a_3 \to B \) if \( m \geq 5 \).

Note by (e) that \( N^+(B) \cap (V_0 \setminus P_0) = \emptyset \). Suppose that \( b \to a_3 \) for some \( b \in B \). It is obvious that \( (N^+(a_1) \setminus P_0) \cap N^-(b) = \emptyset \). Hence, if \( N^+(a_1) \setminus P_0 \neq \emptyset \), we have \( b \to N^+(a_1) \setminus P_0 \), and furthermore, \( a_0 \to N^+(a_1) \setminus P_0 \).

If there is a vertex \( a_j \) (\( 3 \leq j \leq m \)) such that \( a_1 \to a_j \), but \( a_0 \neq a_{j-1} \), then \( a_1 a_j \cdots a_m b a_3 a_4 \cdots a_{j-1} \) is a \( \nu \)-path of \( a_1 a_0 \), a contradiction. This means that \( |N^+(a_0) \cap P_0| \geq |N^+(a_1) \cap P_0| - 2 \). It follows by the regularity of \( T \) that \( |B| \leq 2 \). So, by the assumption that \( k \geq 2 \), we have \( |B| = |V_{n-1}| = 2 \). Note that \( m = n - 2 \) and \( T[P_0] \) is a tournament. Let \( a'_0 \) be the vertex in \( V_0 - a_0 \). Then, it is easy to see that \( a' \to V(P) \cup B \), a contradiction to the regularity of \( T \). This completes the proof of (e).

According to (a)-(f), we have that \( \{a_3, a_4, \cdots, a_m\} \to B \to \{a_1, a_2\} \). Since \( k \geq 2 \) and \( m \leq n - 2 \), we have \( N^-(a_0) \setminus P_0 \neq \emptyset \). By (c) and (e),

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$N^-(a_0) \setminus P_0 \to B$ holds. Since $|N^-(a_0)| = |N^-(b)|$, we have

$$|N^-(a_0) \cap P_0| \geq |N^-(b) \cap (P_0 \cup B)| \geq m - 1$$

for any vertex $b \in B$.

If $T[B]$ contains an arc, say $b' \to b$, then, by (4), we have $|N^-(a_0) \cap P_0| = m$, this means that $\{a_1, a_2, \cdots, a_m\} \to a_0$. It is easy to show that $N^+(a_1) \cap V(P) = \{a_2\}$. So, $N^+(a_1) \setminus P_0 \neq \emptyset$. Clearly, $B \to N^+(a_1) \setminus P_0$. But now, $|N^+(b')| > |N^+(a_1)|$ yields a contradiction.

Suppose now that $B = V_{n-1}$ and let $b$ be a vertex of $B$. Note that $m = n - 2$ and $|V_i \cap V(P)| = 1$ for $i = 0, 1, 2, \cdots, n - 2$. By (e), it is easy to see that $a_0 \to N^+(b) \setminus P_0$. Since $|N^+(a_0)| = |N^+(b)|$ and $b$ has exactly two out-neighbors in $P_0$, $|B| = 2$, i.e., $k = 2$. Let $a'_0$ be the other vertex in $V_0$. Then we see that $a'_0 \to V(P) \cup B$, a contradiction to the regularity of $T$.

The proof of the theorem is complete.

References


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