Bull. Korean Math. Soc. **36** (1999), No. 3, pp. 419–439

CONDITIONAL ABSTRACT WIENER INTEGRALS OF CYLINDER FUNCTIONS

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ABSTRACT. In this paper, we first develop a general formula for evaluating conditional abstract Wiener integrals of cylinder functions. We next use our formula to evaluate the conditional abstract Wiener integral of various cylinder functions and then specialize our results to conditional Yeh-Wiener integrals to show that we can obtain the corresponding results by Park and Skoug. We finally obtain a Cameron-Martin translation theorem for conditional abstract Wiener integrals.

1. Introduction and Preliminaries

Let H be a real separable infinite dimensional Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$. Let B denote the completion of H with respect to a measurable norm $\|\cdot\|$ on H. As H is identified as a dense subspace of B, we identified the topological dual B^* of B as a dense subspace of $H^* \approx H$ in the sense that, for all y in B^* and x in H, $\langle y,x\rangle = (y,x)$, where $(\cdot,\cdot)$ is the $B^* - B$ pairing. Thus we have a triple $B^* \subset H^* \approx H \subset B$. Gross [5] proved that B carries a mean zero Gaussian measure ν, called as the abstract Wiener measure, which is characterized by the probability measure on the Borel $\sigma-$algebra $\mathcal{B}(B)$ of B such that

$$\int_B e^{i(y,x)} d\nu(x) = \exp\left\{-\frac{1}{2}|y|^2\right\} \quad \text{for} \quad \text{every} \quad y \in B^*.$$

Received June 15, 1998.

1991 Mathematics Subject Classification: Primary 28C20.

Key words and phrases: abstract Wiener space, conditional abstract Wiener integral, Yeh-Wiener space, conditional Yeh-Wiener integral, Cameron-Martin translation theorem.

1. Research supported by the Korea Research Foundation, 1997.

2. Research supported by BSRI 97-1412.

The triple (H, B, ν) is called an *abstract Wiener space.* The integration on B is performed with respect to ν. For more details, see [5,6]. Let $\mathbb{R}^n$ and $\mathbb{C}$ denote an n-dimensional Euclidean space and the complex numbers, respectively.

Let $Q = \{(s,t) : 0 \le s \le p,\ \ 0 \le t \le q\}$ and $(C_2[Q], \mathcal{B}(C_2[Q]), m_y)$ denote Yeh-Wiener space, i.e., $C_2[Q]$ denotes the Banach space $\{x(\cdot,\cdot) :$ x is a real valued continuous function on Q with $x(0,t) = x(s,0) = 0\}$ with the supremum norm and m_y denotes the Yeh-Wiener measure on the Borel $\sigma-$algebra $\mathcal{B}(C_2[Q])$ of $C_2[Q]$ (see [11]). Let $C_2'[Q] = \{x \in C_2[Q] : x(s,t) = \int_0^t \int_0^s f(u,v)dudv, f \in L^2[Q]\}$; then it is a real valued separable infinite dimensional Hilbert space with inner product

$$\langle x_1, x_2 \rangle = \int_0^q \int_0^p D^2 x_1(s,t) \cdot D^2 x_2(s,t) ds dt,$$

where $D^2 x = \frac{\partial^2 x}{\partial t \partial s}$. As is well known, $(C'[Q], C_2[Q], m_y)$ is an example of abstract Wiener spaces.

Let $\{e_j : j \ge 1\}$ be a complete orthonormal set in H such that e_j's are in B^*. For each $h \in H$ and $x \in B$, let

$$(h, \tilde{x)} = \lim_{n \to \infty} \sum_{j=1}^{n} \langle h, e_j \rangle (e_j, x)$$

if the limit exists and equals zero otherwise. It is shown that for each $h (h \ne 0)$ in H, $(h, \tilde{\cdot)}$ is a Gaussian random variable on B with mean zero, variance $|h|^2$, and that $(h, \tilde{\lambda x)} = (\lambda h, \tilde{x)} = \lambda (h, \tilde{x)}$ for all $\lambda \in \mathbb{R}^1$. It is known [2,5,10] that if $\{h_1, h_2, \cdots, h_n\}$ is an orthogonal set in H, then the random variable $(h_i, \tilde{x)}$'s are independent, and that if $B = C_2[Q], H = C_2'[Q]$, then

$$(h, \tilde{x)} = \int_0^t \int_0^s D^2 h(u,v) \tilde{d} x(u,v)$$

where $\int_0^t \int_0^s D^2 h(u,v) \tilde{d} x(u,v)$ is the Paley-Wiener-Zygmund integral of $D^2 h$.

Let A be a self-adjoint, trace class operator with eigenvalues $\{\alpha_k\}$ and the corresponding eigenvectors $\{e_k\}$. Let

$$(x, Ax)\tilde{} = \lim_{n\to\infty} \sum_{j=1}^{n} \alpha_j [(e_j, x)\tilde{}]^2,$$

if the limit exists and equals zero otherwise. For more details, see [5,9].

Let X be an $\mathbb{R}^n$-valued measurable function and Y a $\mathbb{C}$-valued integrable function on $(B, \mathcal{B}(B), \nu)$. Let $\mathcal{F}(X)$ denote the $\sigma-$algebra generated by X. Then by the definition of conditional expectation, the conditional expectation of Y given $\mathcal{F}(X)$, written $E[Y|X]$, is any real valued $\mathcal{F}(X)-$measurable function on B such that

$$\int_E Y d\nu = \int_E E[Y|X] d\nu \quad \text{for} \quad E \in \mathcal{F}(X).$$

It is well known that there exists a Borel measurable and P_X-integrable function ψ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), P_X)$ such that $E[Y|X] = \psi \circ X$, where $\mathcal{B}(\mathbb{R}^n)$ denotes the Borel $\sigma-$algebra of $\mathbb{R}^n$ and P_X is the probability distribution of X defined by $P_X(A) = \nu(X^{-1}(A))$ for $A \in \mathcal{B}(\mathbb{R}^n)$. The function $\psi(\vec{\xi}),\ \ \vec{\xi} \in \mathbb{R}^n$ is unique up to Borel null sets in $\mathbb{R}^n$. Following Yeh [12] the function $\psi(\vec{\xi})$, written $E[Y|X = \vec{\xi}]$, is called the *conditional abstract Wiener integral* of Y given X.

J. Yeh [11] derived several Fourier inversion formulas for conditional Wiener integrals and then used the formulas to evaluate conditional Wiener integrals. Recently, Park and Skoug [7,8] obtained a simple formula for evaluating conditional Wiener and Yeh-Wiener integrals. Chung and Kang [3] considered abstract Wiener space version of conditional Wiener integrals and then obtained evaluation formula for conditional abstract Wiener integral of various functions which include some of results given in [7,8].

The purpose of this paper is to give a general formula, for evaluating conditional abstract Wiener integrals of cylinder functions which includes as special cases results by Park and Skoug [7,8] for Wiener and Yeh-Wiener spaces. We then use this formula to evaluate the conditional abstract Wiener integral of various cylinder functions and specialize our results to conditional Yeh-Wiener integrals to show that we can obtain the corresponding results of Park and Skoug's given in [8]. We finally obtain a Cameron-Martin translation theorem for conditional abstract Wiener integrals.

2. A Formula for Conditional Abstract Wiener Integrals

In this section we give a general formula for evaluating conditional abstract Wiener integrals of cylinder functions that includes as special cases the results of Park and Skoug's given in [7] and [8].

LEMMA 2.1. *Let $\{g_1, g_2, \cdots, g_n\}$ be an orthonormal set in H. Let Y and Z be B-valued random variables defined on B by $Y(x) = x - \sum_{j=1}^n (g_j, \tilde{x}) g_j$ and $Z(x) = \sum_{j=1}^n (g_j, \tilde{x}) g_j$. Then Y and Z are independent.*

Proof. It suffices to show that $(y_1, Y(x))$ and $(y_2, Z(x))$ are independent $\mathbb{R}^1$-valued random variables for all $y_1, y_2 \in B^*$. But it is easily shown that $E[(y_1, Y(x))(y_2, Z(x))] = 0$. So $(y_1, Y(x))$ and $(y_2, Z(x))$ are uncorrelated, so that they are independent. □

THEOREM 2.2. *Let $\{g_1, g_2, \cdots, g_n\}$ be an orthonormal set in H. Let $X : B \to \mathbb{R}^n$ be defined by*

$$X(x) = ((g_1, \tilde{x}), (g_2, \tilde{x}), \cdots, (g_n, \tilde{x})) \tag{2.1}$$

Then for any integrable function F on B,

$$E[F(x)|X(x) = \vec{\xi}] = E\left[F\left(x - \sum_{j=1}^n (g_j, \tilde{x}) g_j + \sum_{j=1}^n \xi_j g_j\right)\right]$$

where $\vec{\xi} = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n$.

Proof. Since B-valued random variables $x - \sum_{j=1}^n (g_j, \tilde{x}) g_j$ and $\sum_{j=1}^n (g_j, \tilde{x}) g_j$ are independent by Lemma 2.1, we have, using Corollary 4.38 of [1, p. 80]

$$
\begin{aligned}
&E[F(x)|(g_1,\tilde{x}) = \xi_1, \cdots, (g_n,\tilde{x}) = \xi_n] \\
&= E\Big[F\Big(x - \sum_{j=1}^{n}(g_j,\tilde{x})g_j + \sum_{j=1}^{n}(g_j,\tilde{x})g_j\Big)\Big|\sum_{j=1}^{n}(g_j,\tilde{x})g_j = \sum_{j=1}^{n}\xi_j g_j\Big] \\
&= E\Big[F\Big(x - \sum_{j=1}^{n}(g_j,\tilde{x})g_j + \sum_{j=1}^{n}\xi_j g_j\Big)\Big].
\end{aligned}
$$

□

3. Conditional Abstract Wiener Integrals of Cylinder Functions

In this section we will give a general theorem by which most of the results given in [3,4,7,8,12] are proved as corollaries.

THEOREM 3.1. *Let $(Y, \mathcal{Y}, \eta)$ be a measure space where η is either $\sigma-$finite measure or a $\mathbb{C}-$valued measure. Let $\phi : Y \to H$ be $\mathcal{Y} - \mathcal{B}(H)$ measurable. Assume that F is a $\mathbb{C}-$valued Borel measurable function on $\mathbb{R}$ such that $F((\phi(y),\tilde{x}))$ is integrable on $(Y \times B, \mathcal{Y} \times \mathcal{B}(B), \eta \times \nu)$. Let X be as in* Theorem 2.2. *If $\psi : B \to \mathbb{C}$ is given by*

$$\psi(x) = \int_Y F((\phi(y),\tilde{x}))d\eta(y).$$

Then we have

(3.1)

$$
\begin{aligned}
&E[\psi(x)|X(x) = \vec{\xi}] \\
&= \int_Y \frac{1}{\sqrt{2\pi|p_y|^2}} \int_{-\infty}^{\infty} F\Big(u + \sum_{j=1}^{n}\langle\phi(y), g_j\rangle\xi_j\Big) \exp\Big\{-\frac{u^2}{2|p_y|^2}\Big\} du\, d\eta(y)
\end{aligned}
$$

where $|p_y|^2 = |\phi(y)|^2 - \sum_{j=1}^{n}\langle\phi(y), g_j\rangle^2$, $\vec{\xi} = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n$.

Proof. We first note that

$$E[\psi(x)|X(x)=\vec{\xi}]=\int_Y E[F((\phi(y),\tilde{x}))|X(x)=\vec{\xi}]d\eta(y).$$

Using Theorem 2.2, we have

$$\begin{aligned}
&E[F((\phi(y),\tilde{x}))|X(x)=\vec{\xi}]\\
&=E\Big[F\Big(\Big\langle\phi(y),x-\sum_{j=1}^{n}(g_j,\tilde{x})g_j\Big\rangle+\sum_{j=1}^{n}g_j\xi_j\Big)\Big]\\
&=E\Big[F\Big((\phi(y)-\sum_{j=1}^{n}\langle\phi(y),g_j\rangle g_j,\tilde{x})+\sum_{j=1}^{n}\langle\phi(y),g_j\rangle\xi_j\Big)\Big]\\
&=\int_{-\infty}^{\infty}F\Big(u+\sum_{j=1}^{n}\langle\phi(y),g_j\rangle\xi_j\Big)\frac{1}{\sqrt{2\pi|p_y|^2}}\exp\Big\{-\frac{u^2}{2|p_y|^2}\Big\}du
\end{aligned}$$

where $p_y=\phi(y)-\sum_{j=1}^{n}\langle\phi(y),g_j\rangle g_j$. Hence we obtain (3.1) as desired.□

REMARK 3.1. (1) The function $F((\phi(y),\tilde{x}))$ in Theorem 3.1 may be replaced by $F(y,(\phi(y),\tilde{x}))$ depending on y, with the same proof.

(2) It is worth to note in the proof of the theorem that $p_y=\phi(y)-\sum_{j=1}^{n}\langle\phi(y),\ g_j\rangle g_j$ is in $[g_1,g_2,\cdots,g_n]^{\perp}$ where $[g_1,g_2,\cdots,g_n]^{\perp}$ is the orthogonal complement of the subspace of H spanned by $\{g_1,g_2,\cdots,g_n\}$. Since $\phi(y)=p_y+\sum_{j=1}^{n}\langle\phi(y),g_j\rangle g_j$, it follows that

$$\begin{aligned}
&E[F((\phi(y),\tilde{x}))|X(x)=\vec{\xi}]\\
&=E\Big[F\Big((p_y+\sum_{j=1}^{n}\langle\phi(y),g_j\rangle g_j,\tilde{x})\Big)\Big|X(x)=\vec{\xi}\Big]\\
&=E\Big[F\Big((p_y,\tilde{x})+\sum_{j=1}^{n}\langle\phi(y),g_j\rangle(g_j,\tilde{x})\Big)\Big|X(x)=\vec{\xi}\Big]\qquad(3.2)\\
&=E\Big[F\Big((p_y,\tilde{x})+\sum_{j=1}^{n}\langle\phi(y),g_j\rangle\xi_j\Big)\Big].
\end{aligned}$$

Thus, in order to prove Theorem 3.1 we may use the above argument in (3.2) instead of using Theorem 2.2 (see [3,4]).

COROLLARY 3.2. *Let X be as in* Theorem 3.1. *Let*

$$\psi(x) = \int_H \exp\{i\widetilde{(h,x)}\}d\sigma(h)$$

where σ is a $\mathbb{C}$-valued measure on $(H, \mathcal{B}(H))$. Then

$$\begin{aligned} &E[\psi(x)|X(x) = \vec{\xi}] \\ &= \int_H \exp\{i\langle X(h), \vec{\xi}\rangle\} \exp\left\{-\frac{1}{2}(|h|^2 - |X(x)|^2)\right\} d\sigma(h) \end{aligned} \tag{3.3}$$

where $X(h) = (\langle g_1, h\rangle, \langle g_2, h\rangle, \cdots, \langle g_n, x\rangle)$.

Proof. We apply Theorem 3.1 after taking $(Y, \mathcal{Y}, \eta) = (H, \mathcal{B}(H), \sigma)$, $F(t) = e^{it}$ and $\psi(y) = y$. Then we have

$$\begin{aligned} &E[\psi(x)|X(x) = \vec{\xi}] \\ &= \int_H \frac{1}{\sqrt{2\pi|p_y|^2}} \int_{-\infty}^{\infty} \exp\left\{i\sum_{j=1}^{n}\langle h,\ g_j\rangle\xi_j\right\} e^{iu} \exp\left\{-\frac{u^2}{2|p_y|^2}\right\} du d\sigma(h) \\ &= \int_H \exp\{i\langle X(h), \xi\rangle\} \exp\left\{-\frac{1}{2}|p_y|^2\right\} d\sigma(h) \end{aligned}$$

where $p_y = h - \sum_{j=1}^{n}\langle h,\ g_j\rangle g_j$. Hence we obtain (3.3) as desired. □

COROLLARY 3.3. *Let X be as in* Theorem 3.1. *Let $h \in H$ and S be a bounded linear operator on H. Assume that $F(\widetilde{(Sh,x)})$ is integrable on B. Then*

$$\begin{aligned} &E[F(\widetilde{(Sh,x)})|X(x) = \vec{\xi}] \\ &= \frac{1}{\sqrt{2\pi|p|^2}} \int_{-\infty}^{\infty} F\left(u + \sum_{j=1}^{n}\langle Sh, g_j\rangle\xi_j\right) \exp\left\{-\frac{u^2}{2|p|^2}\right\} du \end{aligned} \tag{3.4}$$

where $|p|^2 = |Sh|^2 - \sum_{j=1}^{n}\langle Sh, g_j\rangle^2$.

Proof. We apply Theorem 3.1 after taking $(Y,\mathcal{Y},\eta)=(H,\mathcal{B}(H),\eta)$, where η has the unity mass at $Sh \in H$. □

COROLLARY 3.4. *Let X and S be as in* Corollary 3.3. *Then for any positive integer n,*

$$E[((Sh,\tilde{x}))^n|X(x)=\vec{\xi}]=\sum_{j=0}^{n}\binom{n}{k}v^k a^{n-k} \tag{3.5}$$

where $a=\sum_{j=1}^{n}\langle Sh,g_j\rangle\xi_j$, $v^0=1$ and

$$v^k=\begin{cases}1\cdot 3\cdots(k-1)\Big(|Sh|^2-\sum_{j=1}^{n}\langle Sh,g_j\rangle^2\Big)^{k/2}, & k=2,4,6,\cdots\\ 0, & k=1,3,5\cdots.\end{cases}$$

Proof. The proof follows from Corollary 3.3 with F defined by $F(u)=u^n$ and the fact that

$$\frac{1}{\sqrt{2\pi|p|^2}}\int_{-\infty}^{\infty}u^k\exp\Big\{-\frac{u^2}{2|p|^2}\Big\}du$$
$$=\begin{cases}1\cdot 3\cdots(k-1)|p|^k, & k=2,4,6,\cdots\\ 0, & k=1,3,5,\cdots\end{cases}$$

where $|p|^2=|Sh|^2-\sum_{j=1}^{n}\langle Sh,g_j\rangle^2$. □

COROLLARY 3.5. *Let X be as in* Theorem 3.1. *Let A be a self-adjoint, trace class operator on H. Then*

$$\begin{aligned}&E[(x,A\tilde{x})|X(x)=\vec{\xi}]\\ &=TrA+\Big\langle\sum_{j=1}^{n}\xi_j g_j,A\Big(\sum_{j=1}^{n}\xi_j g_j\Big)\Big\rangle-\sum_{j=1}^{n}\langle g_j,Ag_j\rangle\end{aligned} \tag{3.6}$$

where TrA stands for the trace of A.

Proof. Let $\{e_m\}$ be the orthonormal eigenvectors and $\{\alpha_m\}$ be the corresponding eigenvalues of A. Let $\langle g_j, e_m\rangle = a_{mj}$. Since $(x, Ax)\tilde{} = \sum_{m=1}^{\infty} \alpha_m((e_m, x)\tilde{})^2$, a.e. $x \in B$, we have by letting $S = I$, $h = e_m$ and $n = 2$ in Corollary 3.4

$$\begin{aligned}&E[(x, Ax)\tilde{}\,|X = \vec{\xi}]\\&= \sum_{m=1}^{\infty} \alpha_m E[((e_m, x)\tilde{})^2|X = \vec{\xi}]\\&= \sum_{m=1}^{\infty} \alpha_m \left[\left(\sum_{j=1}^{n} a_{mj}\xi_j\right)^2 + \left(1 - \sum_{j=1}^{n} a_{mj}^2\right)\right].\end{aligned}$$

Hence we obtain (3.6) as desired. □

COROLLARY 3.6. *Let X and S be as in* Corollary 3.4. *Then*

$$\begin{aligned}&E[\exp\{\lambda(Sh, x)\tilde{}\}|X(x) = \vec{\xi}]\\&= \exp\left\{\lambda\sum_{j=1}^{n}\langle Sh, g_j\rangle\xi_j + \frac{\lambda^2}{2}\left(|Sh|^2 - \sum_{j=1}^{n}\langle Sh, g_j\rangle^2\right)\right\}\end{aligned} \tag{3.7}$$

where $\lambda \in \mathbb{C}$.

Proof. Applying Corollary 3.3 with F defined by $F(u) = \exp\{\lambda u\}$,

$$\begin{aligned}&E[\exp\{\lambda(Sh, x)\tilde{}\}|X(x) = \vec{\xi}]\\&= \frac{1}{\sqrt{2\pi|p|^2}}\int_{-\infty}^{\infty} \exp\left\{\lambda u + \lambda\sum_{j=1}^{n}\langle Sh, g_j\rangle\xi_j\right\}\exp\left\{-\frac{u^2}{2|p|^2}\right\}du,\end{aligned}$$

where $|p|^2 = |Sh|^2 - \sum_{j=1}^{n}\langle Sh, g_j\rangle^2$. But

$$\frac{1}{\sqrt{2\pi|p|^2}}\int_{-\infty}^{\infty}\exp\{\lambda u\}\exp\left\{-\frac{u^2}{2|p|^2}\right\}du = \exp\left\{\frac{\lambda^2}{2}|p|^2\right\}.$$

Hence we obtain (3.7) as desired. □

The following theorem is a generalization of Theorem 5 in [8] and Theorem 4 in [12] to abstract Wiener space.

THEOREM 3.7. *Let $\{g_1, g_2, \cdots, g_n\}$ be an orthonormal set in H, and let $h_1, h_2, \cdots, h_k \in H$ be such that h_j's are in $[g_1, g_2, \cdots, g_n]^\perp$. Let X be as in* Theorem 2.2. *If $F(\tilde{(h_1, x)}, \cdots, \tilde{(h_k, x)})$ is integrable on B, then*

(3.8)
$$E[F(\tilde{(h_1, x)}, \cdots, \tilde{(h_k, x)})|X(x) = \vec{\xi}] = E[F(\tilde{(h_1, x)}, \cdots, \tilde{(h_k, x)})].$$

Furthermore if $\{h_1, h_2, \cdots, h_k\}$ is an orthogonal set in H, then

(3.9)
$$E[F(\tilde{(h_1, x)}, \cdots, \tilde{(h_k, x)})|X(x) = \vec{\xi}]$$
$$= \prod_{j=1}^{k} (2\pi|h_j|^2)^{-1/2} \int_{\mathbb{R}^k} F(u_1, \cdots, u_k) \exp\left\{-\frac{1}{2}\sum_{j=1}^{k} \frac{u_j^2}{|h_j|^2}\right\} du_1 \cdots du_k.$$

Proof. Since h_j's are in $[g_1, g_2, \cdots, g_n]^\perp$, it follows that $X(x)$ and $(\tilde{(h_1, x)}, \cdots, \tilde{(h_k, x)})$ are independent, so that X and $F(\tilde{(h_1, x)}, \cdots, \tilde{(h_k, x)})$ are independent. Hence (3.8) follows immediately. Equation (3.9) is obvious. □

THEOREM 3.8. *Let $\{g_1, g_2, \cdots, g_n\}$ be as in* Theorem 3.7. *Let $h_1, h_2, \cdots, h_k \in H$ be such that h_j's are in $[g_1, g_2, \cdots, g_n]$. If $F(\tilde{(h_1, x)}, \cdots, \tilde{(h_k, x)})$ is integrable on B, then*

(3.10)
$$E[F(\tilde{(h_1, x)}, \cdots, \tilde{(h_k, x)})|X(x) = \vec{\xi}]$$
$$= F\left(\sum_{j=1}^{n} \langle h_1, g_j\rangle \xi_j, \cdots, \sum_{j=1}^{n} \langle h_k, g_j\rangle \xi_j\right).$$

Proof. Since h_j's are in $[g_1, g_2, \cdots, g_n]$, the function $F(\tilde{(h_1, x)}, \cdots, \tilde{(h_k, x)})$ is clearly $\mathcal{F}(X)$-measurable. Hence we have

$$E[F(\tilde{(h_1, x)}, \cdots, \tilde{(h_k, x)})|X(x) = \vec{\xi}]$$
$$= E\left[F\left(\sum_{j=1}^{n} \langle h_1, x\rangle \tilde{(g_j, x)}, \cdots, \sum_{j=1}^{n} \langle h_k, g_j\rangle \tilde{(g_j, x)}\right)\Bigg|X(x) = \vec{\xi}\right]$$
$$= F\left(\sum_{j=1}^{n} \langle h_1, g_j\rangle \xi_j, \cdots, \sum_{j=1}^{n} \langle h_{k,j}\rangle \xi_j\right) E[1|X(x) = \vec{\xi}]$$

from which (3.10) follows. □

4. Examples

Throughout this section, we consider the cases where $B = C[Q]$, $H = C'[Q]$ and $\nu = m_y$. Let S be the operator on $C'[Q]$ defined by

$$Sf(s,t) = \int_0^t \int_0^s f(u,v)dudv. \tag{4.1}$$

Then S is a bounded linear operator and the adjoint operator S^* of S is given by

(4.2)

$$S^*f(s,t) = stf(p,q) - s\int_0^t f(p,v)dv - t\int_0^s f(u,q)du + \int_0^t \int_0^s f(u,v)dudv.$$

and then the operator $A = S^*S$ is given by

$$Af(s,t) = \int_Q \min\{s,u\}\min\{t,v\}f(u,v)dudv. \tag{4.3}$$

Then we note that A is a self-adjoint, trace class operator on $C'[Q]$ and that $\langle f, Ag\rangle = \langle Sf, Sg\rangle = \int_Q f(u,v)g(u,v)dudv$ for all $f, g \in C'[Q]$ and so A is positive definite, i.e., $\langle f, Af\rangle \geq 0$, for all $f \in C'[Q]$.

Let us a partition $\tau_{m,n} = \{(s_i,t_j), i = 1,\cdots,m; j = 1,\cdots,n\}$ of Q with $0 = s_0 < s_1 \cdots < s_m = p$ and $0 = t_0 < t_1 < \cdots < t_n = q$. For this $\tau_{m,n}$, let $Q_{ij} = (s_{i-1}, s_i] \times (t_{j-1}, t_j], i = 1,\cdots,m; j = 1,\cdots,n$ and $\Delta_i s = s_i - s_{i-1}$, $\Delta_j t = t_j - t_{j-1}$. For a fixed $\vec{\xi} = (\xi_{1,1},\cdots,\xi_{m,n}) \in \mathbb{R}^{mn}$, define a function $[\vec{\xi}]$ on Q by

$$\begin{aligned}[\vec{\xi}](s,t) =& \xi_{i-1,j-1} + [(s-s_{i-1})(t-t_{j-1})/(\Delta_i s\Delta_j t)]\Delta_{ij}\vec{\xi} \\ &+ [(s-s_{i-1})/\Delta_i s](\xi_{i,j-1} - \xi_{i-1,j-1}) \\ &+ [(t-t_{j-1})/\Delta_j t](\xi_{i-1,j} - \xi_{i-1,j-1})\end{aligned} \tag{4.4}$$

on each Q_{ij} where $\Delta_{ij}\vec{\xi} = \xi_{i,j} - \xi_{i-1,j} - \xi_{i,j-1} + \xi_{i-1,j-1}$, $\xi_{0,j} = \xi_{i,0} = 0$ for all i and j, and $[\vec{\xi}](s,t) = 0$ if $st = 0$.

Let $g_{ij} \in C'[Q]$ be defined by

$$g_{ij}(s,t) = \frac{1}{\sqrt{\Delta_i s \Delta_j t}} \int_0^t \int_0^s 1_{Q_{ij}}(u,v)\,dudv. \tag{4.5}$$

Then $\{g_{11}, \cdots, g_{mn}\}$ is an orthonormal set in $C'[Q]$ and

$$(g_{ij}, x)\tilde{} = \frac{1}{\sqrt{\Delta_i s \Delta_j t}} \Delta_{ij} x(s,t) \tag{4.6}$$

where $\Delta_{ij}x(s,t) = x(s_i,t_j) - x(s_{i-1},t_j) - x(s_i,t_{j-1}) + x(s_{i-1},t_{j-1})$.

EXAMPLE 4.1. Let S be the operator as in (4.1). Let $h \in C'[Q]$ be defined by $h(s,t) = st$. Then it was shown in [3] that

$$(S^*h, x)\tilde{} = \int_Q x(s,t)\,dsdt. \tag{4.7}$$

Using Corollary 3.4 together with (4.6) and (4.7), we obtain

(4.8)

$$\begin{aligned}
&E\left[\int_Q x(s,t)dsdt \middle| x(s_1,t_1) = \xi_{1,1}, \cdots, x(s_m,t_n) = \xi_{m,n}\right] \\
&= E\left[(S^*h, x)\tilde{} \middle| (g_{ij}, x)\tilde{} = \frac{\Delta_{ij}\vec{\xi}}{\sqrt{\Delta_i s \Delta_j t}}, i = 1, \cdots, m; j = 1, \cdots, n\right] \\
&= \sum_{i=1}^m \sum_{j=1}^n \langle h, Sg_{ij}\rangle \frac{1}{\sqrt{\Delta_i s \Delta_j t}} \Delta_{ij}\vec{\xi} \\
&= \int_Q \left(\int_0^t \int_0^s \sum_{i=1}^m \sum_{j=1}^n \frac{\Delta_{ij}\vec{\xi}}{\Delta_i s \Delta_j t} 1_{Q_{ij}}(u,v)\,dudv \right) dsdt \\
&= \int_Q [\vec{\xi}](s,t)\,dsdt
\end{aligned}$$

which agree with the results of Park and Skoug's [8].

EXAMPLE 4.2. Let A be the operator as in (4.3). Then it was shown in [3] that

$$(x, A\tilde{x}) = \int_Q x^2(s,t)dsdt \quad \text{and} \quad \mathrm{Tr}A = \int_Q stdsdt. \tag{4.9}$$

Using Corollary 3.5 together with (4.6) and (4.9), we obtain

(4.10)

$$\begin{aligned}
&E\Big[\int_Q x^2(s,t)dsdt | x(s_1,t_1) = \xi_{1,1}, \cdots, x(s_m,t_n) = \xi_{m,n}\Big] \\
&= E\Big[(x, A\tilde{x}) | (g_{ij}, \tilde{x}) = \frac{\Delta_{ij}\vec{\xi}}{\sqrt{\Delta_i s \Delta_j t}}, i = 1, \cdots, m; j = 1, \cdots, n\Big] \\
&= \mathrm{Tr}A + \Big\langle \sum_{i=1}^m \sum_{j=1}^n \frac{\Delta_{ij}\vec{\xi}}{\sqrt{\Delta_i s \Delta_j t}} g_{ij}, A\Big(\sum_{i=1}^m \sum_{j=1}^n \frac{\Delta_{ij}\vec{\xi}}{\sqrt{\Delta_i s \Delta_j t}} g_{ij}\Big)\Big\rangle \\
&\qquad - \sum_{i=1}^m \sum_{j=1}^n \langle g_{ij}, A g_{ij}\rangle \\
&= \frac{(pq)^2}{4} + \int_Q \Big(\int_0^t \int_0^s \sum_{i=1}^m \sum_{j=1}^n \frac{\Delta_{ij}\vec{\xi}}{\Delta_i s \Delta_j t} 1_{Q_{ij}}(u,v)dudv\Big)^2 dsdt \\
&\qquad - \int_Q \sum_{i=1}^m \sum_{j=1}^n \Big(\int_0^t \int_0^s \frac{1}{\sqrt{\Delta_i s \Delta_j t}} 1_{Q_{ij}}(u,v)dudv\Big)^2 dsdt \\
&= \frac{(pq)^2}{4} + \int_Q [\vec{\xi}]^2(s,t)dsdt \\
&\qquad - \sum_{i=1}^m \sum_{j=1}^n \int_{Q_{ij}} \Big(s_{i-1}t_{j-1} + \frac{(s-s_{i-1})^2(t-t_{j-1})^2}{\Delta_i s \Delta_j t} \\
&\qquad\qquad + \frac{t_{j-1}(s-s_{i-1})^2}{\Delta_i s} + \frac{s_{i-1}(t-t_{j-1})^2}{\Delta_j t}\Big) dsdt
\end{aligned}$$

which agrees with the result of Park and Skoug's [8].

EXAMPLE 4.3. Let $\phi : Q \to C'[Q]$ be defined by

$$\phi(s,t)(u,v) = \int_0^v \int_0^u f \cdot 1_{[0,s]\times[0,t]}(y,z)dydz$$

where $f \in L^2[Q]$ with $\|f\| > 0$. Let $b(s,t) \equiv |\phi(s,t)|^2$. Then b is a strictly increasing, continuous function on Q. It is interesting to note that the function $(\phi(s,t), \tilde{x})$ on $Q \times C'[0,t]$ is a two parameter generalized Brownian motion with mean zero and covariance $r((s,t),(s',t')) = b(\min\{s,s'\}, \min\{t,t'\})$. By using Theorem 3.1, we obtain

$$\begin{aligned} & E\left[\int_Q (\phi(s,t), \tilde{x})dsdt | (\phi(s,t), \tilde{x}) = \xi\right] \\ & = \int_Q \left(\langle \phi(s,t), \phi(p,q) \rangle \frac{\xi}{b(p,q)} \right) dsdt \\ & = \frac{\xi}{b(p,q)} \int_Q b(s,t)dsdt \end{aligned}$$

and

$$\begin{aligned} & E\left[\int_Q ((\phi(s,t), \tilde{x}))^2 dsdt | (\phi(p,q), \tilde{x}) = \xi\right] \\ & = \int_Q \left(b(s,t) - \frac{1}{b(p,q)} b^2(s,t) + \frac{\xi^2}{b(p,q)} b^2(s,t) \right) dsdt. \end{aligned}$$

In particular, if we take $f \equiv 1$, then the results above are special cases of Example 4.1 and 4.2. Thus Example 4.1 and 4.2 also can be evaluated by using Theorem 3.1.

EXAMPLE 4.4. Let S and h be as in Example 4.1. Then using Corollary 3.6 we obtain

(4.11)

$$\begin{aligned} & E\left[\exp\left\{\int_Q x(s,t)dsdt\right\} | x(s_1,t_1) = \xi_{1,1}, \cdots, x(s_m,t_n) = \xi_{m,n}\right] \\ & = E\left[\exp\{(S^*h, \tilde{x})\} | (g_{ij}, \tilde{x}) = \frac{\Delta_{ij}\vec{\xi}}{\sqrt{\Delta_i s \Delta_j t}},\ i = 1, \cdots, m;\ j = 1, \cdots, n\right] \end{aligned}$$

$$= \exp\left\{\frac{1}{2}\left(|S^*h|^2 - \sum_{i=1}^{m}\sum_{j=1}^{n}\langle S^*h, g_{ij}\rangle^2\right) + \sum_{i=1}^{m}\sum_{j=1}^{n}\langle h, Sg_{ij}\rangle\frac{\Delta_{ij}\vec{\xi}}{\sqrt{\Delta_i s\Delta_j t}}\right\}$$

$$= \exp\left\{\frac{1}{2}\left|S^*h - \sum_{i=1}^{m}\sum_{j=1}^{n}\langle S^*h, g_{ij}\rangle g_{ij}\right|^2 + \int_Q [\vec{\xi}](s,t)dsdt\right\}.$$

Let P^{mn} be the orthogonal projection onto $[g_{11}, \cdots, g_{mn}]$. Then it is easily shown that for $f \in C'[Q]$, $P^{mn}f(s,t)$ is the quadratic approximation of f based on the partition τ_{mn} which is given by

$$P^{mn}f(s,t) = \sum_{i=1}^{m}\sum_{j=1}^{n}\langle h, g_{ij}\rangle g_{ij}(s,t) = [f](s,t) \tag{4.12}$$

where

$$\begin{aligned}[f](s,t) =& f(s_{i-1},t_{j-1}) + [(s-s_{i-1})(t-t_{j-1})/(\Delta_i s\Delta_j t)]\Delta_{ij}f(s,t)\\ &+ [(s-s_{i-1})/\Delta_i s](f(s_i,t_{j-1}) - f(s_{i-1},t_{j-1}))\\ &+ [(t-t_{j-1})/\Delta_j t](f(s_{i-1},t_j) - f(s_{i-1},t_{j-1}))\end{aligned}$$

on each Q_{ij} and $[f](s,t) = 0$ if $st = 0$. It can be checked that if $\|\tau_{mn}\| \to 0$ as $m \to \infty$ and $n \to \infty$, then P^{mn} converges to the identity operator I on $C'[Q]$. Hence we conclude that

$$|(I - P^{mn})S^*h|^2 = |S^*h - \sum_{i=1}^{m}\sum_{j=1}^{n}\langle S^*h, g_{ij}\rangle g_{ij}|^2 \to 0$$

as $\|\tau_{mn}\| \to 0$. Therefore for each $w \in C[Q]$, we have, from the last equality of (4.11),

$$\begin{aligned}&\lim_{\|\tau_{mn}\|\to 0} E\left[\exp\left\{\int_Q x(s,t)dsdt\right\}|x(s_1,t_1)\right.\\ &\qquad\qquad \left. = w(s_1,t_1), \cdots, x(s_m,t_n) = w(s_m,t_n)\right]\\ &= \lim_{\|\tau_{mn}\|\to 0} E\left[\exp\left\{|(I-P^{mn})S^*h|^2\right\}\right]\exp\left\{\int_Q [\vec{w}](s,t)dsdt\right\}\\ &= \exp\left\{\int_Q w(s,t)dsdt\right\}\end{aligned}$$

which agrees with the result in [8].

EXAMPLE 4.5. Let $f \in L^2[Q]$ and $h \in C'[Q]$ be defined by $h(s,t) = \int_0^t \int_0^s f(u,v)dudv$. Since

$$\left| h - \sum_{i=1}^{m} \sum_{j=1}^{n} \langle h, g_{ij} \rangle g_{ij} \right|^2 = |h|^2 - \sum_{i=1}^{m} \sum_{j=1}^{n} \langle h, g_{ij} \rangle^2 = |h|^2 - \left| \sum_{i=1}^{m} \sum_{j=1}^{n} \langle h, g_{ij} \rangle g_{ij} \right|^2,$$

using (4.5) it follows that

$$\left| h - \sum_{i=1}^{m} \sum_{j=1}^{n} \langle h, g_{ij} \rangle g_{ij} \right|^2 = \int_Q (f - \tilde{f})^2 dsdt = \int_Q f^2 dsdt - \int_Q \tilde{f}^2 dsdt \tag{4.13}$$

and

$$\left\langle h, \sum_{i=1}^{m} \sum_{j=1}^{n} \langle h, g_{ij} \rangle g_{ij} \right\rangle = \int_Q f(s,t) \cdot \tilde{f}(s,t) dsdt = \int_Q \tilde{f}^2 dsdt \tag{4.14}$$

where $\tilde{f}(s,t)$ is defined by

$$\tilde{f}(s,t) = \frac{1}{\Delta_i s \Delta_j t} \int_{Q_{ij}} f(s,t) dsdt$$

on each Q_{ij} and $\tilde{f}(s,t) = 0$ if $st = 0$. Next we note that

$$\left\langle h, \sum_{i=1}^{m} \sum_{j=1}^{n} (g_{ij}, \tilde{x}) g_{ij} \right\rangle = \int_Q f \tilde{d}[x] \tag{4.15}$$

$$\left(\sum_{i=1}^{m} \sum_{j=1}^{n} \langle h, g_{ij} \rangle g_{ij}, x \right)^{\sim} = \int_Q \tilde{f} \tilde{d}x \tag{4.16}$$

$$\left\langle \sum_{i=1}^{m} \sum_{j=1}^{n} \langle h, g_{ij} \rangle g_{ij}, \sum_{i=1}^{m} \sum_{j=1}^{n} (g_{ij}, \tilde{x}) g_{ij} \right\rangle = \int_Q \tilde{f} \tilde{d}[x] \tag{4.17}$$

where for $x \in C[Q]$, $[x]$ is defined as in (4.12). Using (4.13)-(4.17), we obtain Theorem 4 of [8].

EXAMPLE 4.6. Let f and h be as in Example 4.4. Assume that $F((h, \tilde{x}))$ is integrable. Then using Theorem 3.1 with $S = I$, we obtain

(4.18)

$$E\left[F\left(\int_Q f\tilde{d}x\right)|x(s_1,t_1) = \xi_{1,1}, \cdots, x(s_m,t_n) = \xi_{m,n}\right]$$

$$= E[f((h, \tilde{x}))|(g_{ij}, \tilde{x}) = \frac{\Delta_{ij}\vec{\xi}}{\sqrt{\Delta_i\Delta_j t}}, i = 1, \cdots, m; j = 1, \cdots, n]$$

$$= \frac{1}{\sqrt{2\pi|p|^2}} \int_{-\infty}^{\infty} F\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\langle h, g_{ij}\rangle \frac{\Delta_{ij}\vec{\xi}}{\sqrt{\Delta_i s\Delta_j t}} + u\right) \exp\left\{-\frac{u^2}{2|p|^2}\right\} du$$

where $p = h - \sum_{i=1}^{m}\sum_{j=1}^{n}\langle h, g_{ij}\rangle g_{ij}$. Since

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\langle h, g_{ij}\rangle \frac{\Delta_{ij}\vec{\xi}}{\sqrt{\Delta_i s\Delta_j t}} = \int_Q \tilde{f}\tilde{d}[\vec{\xi}]$$

using (4.13) it follows that (4.18) agrees with the result of Park and Skoug's given in [8].

5. Translation of Conditional Abstract Wiener Integrals

The abstract Wiener space version of Cameron-Martin translation theorem states that if $h \in H$ and if $T : B \to B$ is a transformation defined by $T(x) = x + h$, then for any integrable function F on B and Γ in $\mathcal{B}(B)$

$$\int_\Gamma F(y)d\nu(y) = \int_{T^{-1}(\Gamma)} F(x+h)J(h,x)d\nu(x)$$

where

$$J(h,x) = \exp\left\{-\frac{1}{2}|h|^2 - (h, \tilde{x})\right\}. \tag{5.1}$$

In particular, if $\Gamma = B$, then

$$E[F(y)] = E[F(x+h)J(h,x)].$$

The following is an abstract Wiener space version of translation theorem for conditional Wiener integrals.

THEOREM 5.1. *Let X be as in* Theorem 2.2 *and Let $h \in H$. Suppose that $T : B \to B$ is a transformation defined by $T(x) = x + h$. For any $F \in L^1(B, \mathcal{B}(B), \nu)$, we have*

(5.2)
$$E[F(y)|X(y) = \vec{\xi}]$$
$$= E[F(x+h)J(h,x)|X(x+h) = \vec{\xi}]\exp\left\{-\frac{1}{2}|X(h)|^2 + \langle X(h), \vec{\xi}\rangle\right\}$$

where $X(h) = (\langle g_1, h\rangle, \cdots, \langle g_n, h\rangle)$ and $\vec{\xi} = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$.

Proof. Using Theorem 2.2 and Cameron-Martin translation theorem, we have

$$\begin{aligned}
&E[F(y)|X(y) = \vec{\xi}] \\
&= E\left[F\left(y - \sum_{j=1}^{n}(g_j, \tilde{y})g_j + \sum_{j=1}^{n}\xi_j g_j\right)\right] \\
&= E\left[F\left(x + h - \sum_{j=1}^{n}(g_j, x \tilde{+} h)g_j + \sum_{j=1}^{n}\xi_j g_j\right)J(h,x)\right].
\end{aligned}$$

Note that

$$\begin{aligned}
&J(h,x) \\
&= \exp\left\{-\frac{1}{2}|h|^2 - (h, \tilde{x})\right\} \\
&= \exp\left\{-\frac{1}{2}|h|^2\right\}\exp\left\{-\left(h, x + h - \sum_{j=1}^{n}(g_j, x \tilde{+} h)g_j + \sum_{j=1}^{n}\xi_j g_j\right)^{\sim}\right\} \\
&\qquad \cdot \exp\left\{-\left\langle h, -h + \sum_{j=1}^{n}(g_j, x \tilde{+} h)g_j - \sum_{j=1}^{n}\xi_j g_j\right\rangle\right\}.
\end{aligned}$$

And thus we have

$$\begin{aligned}
&E[F(y)|X(y)=\vec{\xi}]\\
&=E\Big[F\Big(x+h-\sum_{j=1}^{n}(g_j,x+\tilde{h)}g_j+\sum_{j=1}^{n}\xi_j g_j\Big)\exp\Big\{-\frac{1}{2}|h|^2\Big\}\\
&\quad\cdot\exp\Big\{-\Big(h,x+h-\sum_{j=1}^{n}(g_j,x+\tilde{h)}g_j+\sum_{j=1}^{n}\xi_j g_j\Big)^{\sim}\Big\}\\
&\quad\cdot\exp\Big\{-\Big\langle h,\sum_{j=1}^{n}(g_j,\tilde{x)}g_j\Big\rangle\Big\}\Big]\\
&\quad\cdot\exp\Big\{-\Big\langle h,-h+\sum_{j=1}^{n}\langle g_j,h\rangle g_j-\sum_{j=1}^{n}\xi_j g_j\Big\rangle\Big\}.
\end{aligned}$$

Since $x-\sum_{j=1}^{n}(g_j,\tilde{x)}g_j$ and $\sum_{j=1}^{n}(g_j,\tilde{x)}g_j$ are independent,

$$\exp\Big\{-\Big\langle h,\sum_{j=1}^{n}(g_j,\tilde{x)}g_j\Big\rangle\Big\}$$

and

$$\begin{aligned}
&F\Big(x+h-\sum_{j=1}^{n}(g_j,x+\tilde{h)}g_j+\sum_{j=1}^{n}\xi_j g_j\Big)\\
&\qquad\cdot\exp\Big\{-\Big(h,x+h-\sum_{j=1}^{n}(g_j,x+\tilde{h)}g_j+\sum_{j=1}^{n}\xi_j g_j\Big)^{\sim}\Big\}
\end{aligned}$$

are also independent. Therefore we have

$$\begin{aligned}
&E[F(y)|X(y)=\vec{\xi}]\\
&=E\Big[F(x+h)\exp\Big\{-\frac{1}{2}|h|^2\Big\}\exp\{-(h,x+\tilde{h)}\}\Big|X(x+h)=\vec{\xi}\Big]\\
&\quad\cdot E\Big[\exp\Big\{-\Big\langle h,\sum_{j=1}^{n}(g_j,\tilde{x)}g_j\Big\rangle\Big\}\Big]\\
&\quad\cdot\exp\Big\{-\Big\langle h,-h+\sum_{j=1}^{n}\langle g_j,h\rangle g_j-\sum_{j=1}^{n}\xi_j g_j\Big\rangle\Big\}
\end{aligned}$$

$$= E\left[F(x+h)J(h,x)|X(x+h)=\vec{\xi}\right]E\left[\exp\left\{-\left\langle h, \sum_{j=1}^{n}(g_j, \tilde{x})g_j\right\rangle\right\}\right]$$
$$\cdot \exp\left\{-\left\langle h, \sum_{j=1}^{n}\langle g_j, h\rangle g_j\right\rangle\right\}\exp\left\{\left\langle h, \sum_{j=1}^{n}\xi_j g_j\right\rangle\right\}.$$

And hence we have

$$\begin{aligned}
E\left[\exp\left\{-\left\langle h, \sum_{j=1}^{n}(g_j, \tilde{x})g_j\right\rangle\right\}\right] &= \int_B \exp\left\{-\left(\sum_{j=1}^{n}\langle h, g_j\rangle g_j, x\right)^{\sim}\right\}d\nu(x)\\
&= \exp\left\{\frac{1}{2}\left|\sum_{j=1}^{n}\langle h, g_j\rangle g_j\right|^2\right\}\\
&= \exp\left\{\frac{1}{2}|X(h)|^2\right\}.
\end{aligned}$$

Note that

$$\left\langle h, \sum_{j=1}^{n}(g_j, \tilde{h})g_j\right\rangle = |X(h)|^2 \qquad \text{and} \qquad \left\langle h, \sum_{j=1}^{n}\xi_j g_j\right\rangle = \langle X(h), \vec{\xi}\rangle.$$

Hence we obtain (5.2) as desired. □

EXAMPLE 5.1. Let X be as in Theorem 3.2 and let $h \in H$. Then

$$\begin{aligned}
&E[\exp\{(h, \tilde{x})\}|X(x)=\vec{\eta}]\\
&= \exp\left\{\frac{1}{2}\left\langle h, h - \sum_{j=1}^{n}\langle g_j, h\rangle g_j + 2\sum_{j=1}^{n}\eta_j g_j\right\rangle\right\}
\end{aligned}$$

where $\vec{\eta} = (\eta_1, \cdots, \eta_n) \in \mathbb{R}^n$.

Proof. By setting $F \equiv 1$ and $\vec{\xi} = \vec{\eta} + X(h)$ in Theorem 5.1,

$$1 = E[J(h,x)|X(x)=\vec{\eta}]\exp\left\{\frac{1}{2}|X(h)|^2 + \langle X(h), \vec{\eta}\rangle\right\}.$$

Hence we have, by using (5.1)

$$\begin{aligned}
&E[\exp\{-\widetilde{(h,x)}\}|X(x)=\vec{\eta}]\\
&=\exp\left\{\frac{1}{2}|h|^2-\frac{1}{2}|X(h)|^2+\langle X(h),\vec{\eta}\rangle\right\}\\
&=\exp\left\{\frac{1}{2}|h|^2-\frac{1}{2}\left\langle h,\sum_{j=1}^{n}\langle g_j,h\rangle g_j\right\rangle-\left\langle h,2\sum_{j=1}^{n}\eta_j g_j\right\rangle\right\}\\
&=\exp\left\{\frac{1}{2}\left\langle h,h-\sum_{j=1}^{n}\langle g_j,h\rangle g_j-2\sum_{j=1}^{n}\eta_j g_j\right\rangle\right\}.
\end{aligned}$$

The result now follows by replacing h by $-h$. □

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