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EVALUATION OF CONDITIONAL WIENER INTEGRALS USING PARK AND SKOUG'S FORMULA

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ABSTRACT. In this paper we first evaluate the conditional Wiener integral of certain functionals using a Park and Skoug's formula. And we also evaluate the conditional Wiener integral $E(F \mid X_\alpha)$ of functional F on $C[0,T]$ given by

$$F(x) = \exp\left\{\int_0^T s^k\, x(s)\, ds\right\}$$

for a general conditioning function X_α on $C[0,T]$.

1. Introduction

Let $(C[0,T], \mathcal{F}, m_w)$ denote Wiener measure space where $C[0,T]$ is the space of all continuous functions x on $[0,T]$ vanishing at the origin. For each partition $\tau = \tau_n = \{t_1, \cdots, t_n\}$ of $[0,T]$ with $0 = t_0 < t_1 < \cdots < t_n = T$, let $X_\tau : C[0,T] \longrightarrow R^n$ be defined by $X_\tau(x) = (x(t_1), \cdots, x(t_n))$. Let $\mathcal{B}^n$ be the σ–algebra of Borel sets in R^n. Then, by the definition of conditional Wiener integral (see [9, 10]), for each Wiener integrable function $F(x)$,

$$\int_{X_\tau^{-1}(B)} F(x)\, m_w\, (dx) = \int_B E\, (F|X_\tau)(\vec{\xi})\, P_{X_\tau}\, (d\vec{\xi})$$

where $B \in \mathcal{B}^n$, $P_{X_\tau}(B) = m_w\, (X_\tau^{-1}(B))$, and $E\, (F|X_\tau)(\vec{\xi})$ is a Borel measurable function of $\vec{\xi}$ which is unique up to Borel null sets in R^n. Here $E\, (F|X_\tau)$ is called a conditional Wiener integral of F given condition X_τ.

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The purpose of this paper is to evaluate the conditional Wiener integral of certain functionals using a Park and Skoug's formula. And we also evaluate the conditional Wiener integral $E(F \mid X_\alpha)$ of functional F on $C[0,T]$ given by

$$F(x) = \exp\left\{\int_0^T s^k\, x(s)\, ds\right\}$$

and the conditioning function X_α on $C[0,T]$ given by

$$X_\alpha(x) = \left(\int_0^T \alpha_1(t)\, dx(t), \cdots, \int_0^T \alpha_n(t)\, dx(t)\right)$$

for $\alpha_j(t) = I_{[t_{j-1},t_j]}(t)$, $0 = t_0 < t_1 < \cdots < t_n = T$, the indicator function of $[t_{j-1}, t_j]$, $j = 1, 2, \cdots, n$.

2. Preliminaries

For a given partition $\tau = \tau_n$ of [0,T] and $x \in C[0,T]$, define the polygonal function $[x]$ on [0,T] by

$$[x](t) = x(t_{j-1}) + \frac{t - t_{j-1}}{t_j - t_{j-1}}\,(x(t_j) - x(t_{j-1})) \tag{2.1}$$

where $t \in [t_{j-1}, t_j]$ and $j = 1, \cdots, n$. Similarly, for each $\vec{\xi} = (\xi_1, \cdots, \xi_n) \in R^n$, define the polygonal function $[\vec{\xi}]$ of $\vec{\xi}$ on [0,T] by

$$[\vec{\xi}](t) = \xi_{j-1} + \frac{t - t_{j-1}}{t_j - t_{j-1}}\,(\xi_j - \xi_{j-1}) \tag{2.2}$$

where $t \in [t_{j-1}, t_j]$, $j = 1, \cdots, n$ and $\xi_0 = 0$. Then both $[x]$ and $[\vec{\xi}]$ are continuous on [0,T], their graphs are line segments on each subinterval $[t_{j-1}, t_j]$, and $[x](t_j) = x(t_j)$ and $[\vec{\xi}](t_j) = \xi_j$ at each $t_j \in \tau$.

For a polygonal function $[x]$ we have the following lemma.

LEMMA 2.1. *For x in $C[0,T]$, we have*

$$E\left(\left\{\int_0^T [x](t)\,dt\right\}^2\right) \tag{2.3}$$
$$= E\left(\int_0^T\int_0^T x(u)\,[x](v)\,du\,dv\right)$$
$$= \frac{1}{4}\sum_{j=1}^{n}\{2\,T\,(t_j+t_{j-1})-(t_j^2+t_{j-1}^2)\}\,(t_j-t_{j-1}).$$

Proof. By the Fubini theorem, we have

$$E\left(\left\{\int_0^T [x](t)\,dt\right\}^2\right) \tag{2.4}$$
$$= \int_0^T\int_0^T E([x](u)\,[x](v))\,du\,dv$$
$$= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\left\{\sum_{i=1}^{j-1}\int_{t_{i-1}}^{t_i} E([x](u)\,[x](v))\,du\right.$$
$$+\int_{t_{j-1}}^{v} E([x](u)\,[x](v))\,du + \int_{v}^{t_j} E([x](u)\,[x](v))\,du$$
$$\left.+\sum_{i=j+1}^{n}\int_{t_{i-1}}^{t_i} E([x](u)\,[x](v))\,du\right\}\,dv.$$

Since $E(x(s)\,x(t)) = \min\{s,\ t\}$ and

$$[x](u)\,[x](v) = \left\{x(t_{i-1}) + \frac{u-t_{i-1}}{t_i-t_{i-1}}\,(x(t_i)-x(t_{i-1}))\right\}\left\{x(t_{j-1}) + \frac{v-t_{j-1}}{t_j-t_{j-1}}\,(x(t_j)-x(t_{j-1}))\right\} \tag{2.5}$$

for (u,v) in $[t_{i-1},t_i]\times[t_{j-1},t_j]$, $i,\ j=1,2,\cdots,n$, the right hand side of the last equality in (2.4) becomes

$$\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\left\{\int_0^{t_{j-1}} u\,du + \int_{t_{j-1}}^{v}\left(t_{j-1}+\frac{v-t_{j-1}}{t_j-t_{j-1}}\,(u-t_{j-1})\right)du\right. \tag{2.6}$$

$$+\int_{v}^{t_j}\left(t_{j-1}+\frac{(v-t_{j-1})^2}{t_j-t_{j-1}}+\frac{v-t_{j-1}}{t_j-t_{j-1}}\,(u-v)\right)\,du$$

$$+\sum_{i=j+1}^{n}\int_{t_{i-1}}^{t_i} v\,du\Bigg\}\,dv$$

$$=\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\left\{\frac{1}{2}\,t_{j-1}^2+\frac{1}{2}\,(v+t_{j-1})\,(t_j-t_{j-1})+v\,(T-t_j)\right\}\,dv.$$

Now, using the Fubini theorem, we have

$$(2.7)\ E\left(\int_0^T\int_0^T x(u)\,[x](v)\,du\,dv\right)$$

$$=\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\left\{\int_0^T E(x(u)\,[x](v))du\right\}\,dv$$

$$=\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\Bigg\{\int_0^{t_{j-1}} u\,du$$

$$+\int_{t_{j-1}}^{t_j}\left(t_{j-1}+\frac{v-t_{j-1}}{t_j-t_{j-1}}\,(u-t_{j-1})\right)\,du+\int_{t_j}^{T} v\,du\Bigg\}\,dv.$$

The right hand side of the last equality in (2.7) becomes the right hand side of (2.6) and so we have (2.3) from (2.6) and (2.7).

In [5], Park and Skoug obtained that for a Wiener integrable and Borel measurable function F on C[0,T], they have

$$(2.8)\qquad E\left(F(x)|X_\tau(x)=\vec{\xi}\right)=E\left(F\left(x-[x]+[\vec{\xi}]\right)\right)$$

where the equality in (2.8) means that both sides are Borel measurable function of $\vec{\xi}$ and they are equal except for Borel null sets. We call the formula (2.8) as a Park and Skoug's formula for conditional Wiener integral.

Let $\{\alpha_1(t),\cdots,\alpha_n(t)\}$ be an orthogonal set of functions in $L_2[0,T]$ with $\|\alpha_j\| = \left[\int_0^T (\alpha_j(t))^2\,dt\right]^{\frac{1}{2}} \neq 0$ for $j=1,2,\cdots,n$. Then the corresponding stochastic integrals

$$\gamma_j(x) = \int_0^T \alpha_j(t)\, dx(t), \qquad j = 1, 2, \cdots, n \tag{2.9}$$

form a set of Gaussian random variables on $C[0,T]$ with

$$\begin{aligned} E\,[\,\gamma_j(x)\,\gamma_k(x)] &= \int_0^T \alpha_j(t)\,\alpha_k(t)\,dt, \\ E\,[\,x(t)\,\gamma_j(x)] &= \int_0^t \alpha_j(s)\,ds. \end{aligned} \tag{2.10}$$

Let $X_\alpha : C[0,T] \longrightarrow R^n$ be the conditioning function defined by

$$X_\alpha(x) = (\gamma_1(x), \cdots, \gamma_n(x)) \tag{2.11}$$

and let

$$\beta_j(t) = \int_0^t \alpha_j(s)\,ds, \; 0 \le t \le T \text{ and } j = 1, 2, \cdots, n. \tag{2.12}$$

For $x \in C[0,T]$ and $\vec{\xi} = (\xi_1, \cdots, \xi_n) \in R^n$, let

$$\begin{aligned} x_n(t) &= \sum_{j=1}^{n} \|\alpha_j\|^{-2}\,\beta_j(t)\,\gamma_j(x) \\ \vec{\xi}_n(t) &= \sum_{j=1}^{n} \|\alpha_j\|^{-2}\,\beta_j(t)\,(\xi_j - \xi_{j-1}). \end{aligned} \tag{2.13}$$

The following lemma comes from [6] which will be used in last theorem.

Lemma 2.2. *Let $g \in L_2[0,T]$. Then*

$$\begin{aligned} &E\left(\exp\left\{\int_0^T g(s)\,x(s)\,ds\right\} | X_\alpha(x) = \vec{\xi}\right) \\ &= \exp\left\{\sum_{j=1}^{n} \frac{(\xi_j - \xi_{j-1})\,(g, \beta_j)}{(\alpha_j, \alpha_j)} \right. \\ &\left. + \frac{1}{2}\int_0^T \left[\int_s^T g(t)\,dt\right]^2 ds - \frac{1}{2}\sum_{j=1}^{n} \frac{(g, \beta_j)^2}{(\alpha_j, \alpha_j)}\right\}. \end{aligned} \tag{2.14}$$

3. Conditional Wiener Integrals for Vector-valued Conditioning Function

In this section we evaluate the conditional Wiener integral of certain functionals using a Park and Skoug's formula (2.8).

THEOREM 3.1. *For a positive integer m, let* $F_m(x) = \int_0^T ([x](t))^m\, dt$ *for* x *in* $C[0,T]$. *Then we have*

$$E(F_m|X_\tau)(\vec{\xi}) = \frac{1}{m+1} \sum_{j=1}^{n} \left\{ \sum_{i=0}^{m} \xi_j^{m-i} \xi_{j-1}^{i} \right\} (t_j - t_{j-1}) \tag{3.1}$$

for $\vec{\xi} = (\xi_1, \cdots, \xi_n)$ *in* R^n.

Proof. Using a formula (2.8) we have

$$\begin{aligned} E(F_m|X_\tau)(\vec{\xi}) &= E\left(\int_0^T ([x - [x] + [\vec{\xi}]]\ (t))^m\, dt\right) \\ &= \int_0^T ([\vec{\xi}]\ (t))^m\, dt \end{aligned} \tag{3.2}$$

where the second equality in (3.2) comes from the fact that the polygonal function satisfies the linearity and $[[x]]\ (t) = [x]\ (t)$ for t in [0,T]. Now we have, by a simple change of variable,

$$\begin{aligned} &\int_0^T ([\vec{\xi}]\ (t))^m\, dt \\ &= \sum_{j=1}^{n} \frac{t_j - t_{j-1}}{\xi_j - \xi_{j-1}} \int_{\xi_{j-1}}^{\xi_j} u^m\, du \\ &= \frac{1}{m+1} \sum_{j=1}^{n} (\xi_j^m + \xi_j^{m-1}\, \xi_{j-1} + \cdots + \xi_{j-1}^m)\ (t_j - t_{j-1}). \end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3), we have (3.1) as we desire. □

THEOREM 3.2. *Let* $F(x) = \left\{\int_0^T x(t)\, dt\right\}^2$ *for* x *in* $C[0,T]$. *Then we have*

$$\text{(3.4)} \qquad E\,(F\,|X_\tau)(\vec{\xi}) = \frac{T^3}{3} - \frac{1}{4}\sum_{i=1}^{n}\{2\,T\,(t_i + t_{i-1}) - (t_i^2 + t_{i-1}^2)\}\,(t_i - t_{i-1}) + \frac{1}{4}\sum_{j=1}^{n}\sum_{i=1}^{n}(\xi_i + \xi_{i-1})\,(\xi_j + \xi_{j-1})(t_i - t_{i-1})\,(t_j - t_{j-1})$$

for $\vec{\xi} = (\xi_1, \ldots, \xi_n)$ *in* R^n.

Proof. Using a formula (2.8), we have

$$\text{(3.5)} \qquad E\,(F\,|X_\tau)(\vec{\xi}) = E\left(\left\{\int_0^T (x(t) - [x](t) + [\vec{\xi}](t))\,dt\right\}^2\right) = \int_0^T\int_0^T E\,(x(u)\,x(v) - x(u)\,[x](v) - [x](u)\,x(v) + [x](u)\,[x](v) + [\vec{\xi}](u)\,[\vec{\xi}](v))\,du\,dv$$

where the second equality in (3.5) comes from the Fubini theorem and the linearity of the Wiener integral. Now we have

$$\text{(3.6)} \qquad \int_0^T\int_0^T E\,(x(u)\,x(v))\,du\,dv = \int_0^T\left\{\int_0^v u\,du + \int_v^T v\,du\right\}\,dv = \frac{T^3}{3}$$

and

$$\text{(3.7)} \qquad \int_0^T\int_0^T E\,([\vec{\xi}](u)\,[\vec{\xi}](v))\,du\,dv = \frac{1}{4}\sum_{j-1}^{n}\sum_{j-1}^{n}(\xi_i + \xi_{i-1})\,(\xi_j + \xi_{j-1})\,(t_i - t_{i-1})\,(t_j - t_{j-1}).$$

Combining (3.5), (3.6), (3.7) and Lemma 2.1, we obtain (3.4) as we desire. □

Finally we treat the conditional Wiener integral for general conditioning function.

THEOREM 3.3. *Let* X_α *be the general conditioning function on* $C[0,T]$ *defined by* $X_\alpha(x) = \left(\int_0^T \alpha_1(x)\,dx(t), \cdots, \int_0^T \alpha_n(x)\,dx(t)\right)$ *where* $\alpha_j(t) = I_{[t_{j-1},t_j]}(t)$. *Then we have*

$$E\left(\exp\left\{\int_0^T s^k\, x(s)\, ds\right\} | X_\alpha(x) = \vec{\xi}\right) \tag{3.8}$$
$$= \exp\left\{\frac{T^{2k+3}}{(k+2)\,(2k+3)} - \frac{1}{2\,(k+1)^2}\sum_{j=1}^{n}\frac{1}{t_j - t_{j-1}}\right.$$
$$\left\{T^{k+1}\,(t_j - t_{j-1}) - \frac{1}{k+2}\,(t_j^{k+2} - t_{j-1}^{k+2})\right\}^2 + \frac{1}{k+1}$$
$$\left.\sum_{j=1}^{n}\frac{\xi_j - \xi_{j-1}}{t_j - t_{j-1}}\left\{T^{k+1}\,(t_j - t_{j-1}) - \frac{1}{k+2}\,(t_j^{k+2} - t_{j-1}^{k+2})\right\}\right\}$$

for $\vec{\xi} = (\xi_1, \cdots, \xi_n) \in R^n$.

Proof. Since $s^k \in L_2[0,T]$ and $\{\alpha_j(t)\}_{j-1}^n$ is an orthogonal set, we have

$$E\left(\exp\left\{\int_0^T s^k\, x(s)\, ds\right\} | X_\alpha(x) = \vec{\xi}\right) \tag{3.9}$$
$$= \exp\left\{\sum_{j=1}^{n}\frac{(\xi_j - \xi_{j-1})\,(s^k, \beta_j)}{(\alpha_j, \alpha_j)}\right.$$
$$\left. + \frac{1}{2}\int_0^T\left[\int_s^T t^k\, dt\right]^2 ds - \frac{1}{2}\sum_{j=1}^{n}\frac{(s^k, \beta_j)^2}{(\alpha_j, \alpha_j)}\right\}$$

using Lemma 2.2. From (2.13), we get

$$\sum_{j=1}^{n}\frac{(\xi_j - \xi_{j-1})\,(s^k, \beta_j)}{(\alpha_j, \alpha_j)} \tag{3.10}$$

$$= \sum_{j=1}^{n} \frac{\xi_j - \xi_{j-1}}{t_j - t_{j-1}} \int_0^T s^k \left[\int_0^s \alpha_j(t)\, dt \right] ds$$

$$= \sum_{j=1}^{n} \frac{\xi_j - \xi_{j-1}}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} \int_t^T s^k \, ds\, dt$$

$$= \frac{1}{k+1} \sum_{j=1}^{n} \frac{\xi_j - \xi_{j-1}}{t_j - t_{j-1}} \left\{ T^{k+1}(t_j - t_{j-1}) - \frac{1}{k+2} (t_j^{k+2} - t_{j-1}^{k+2}) \right\}$$

where the first equality in (3.10) comes from (2.12) and the second equality follows from the change of the order of integration. Now, we easily obtain

$$\int_0^T \left[\int_s^T t^k \, dt \right]^2 ds = \frac{2T^{2k+3}}{(k+2)\,(2k+3)}. \tag{3.11}$$

Using the fact

$$(s^k, \beta_j) = \frac{1}{k+1} \{ T^{k+1}\, (t_j - t_{j-1}) - \frac{1}{k+2}\, (t_j^{k+2} - t_{j-1}^{k+2}) \} \tag{3.12}$$

and $\|\alpha_j\|^2 = t_j - t_{j-1}$, we have

(3.13)

$$-\frac{1}{2} \sum_{j=1}^{n} \frac{(s^k, \beta_j)^2}{(\alpha_j, \alpha_j)}$$

$$= -\frac{1}{2\,(k+1)^2} \sum_{j=1}^{n} \frac{1}{t_j - t_{j-1}} \left\{ T^{k+1}\, (t_j - t_{j-1}) - \frac{1}{k+2}\, (t_j^{k+2} - t_{j-1}^{k+2}) \right\}^2 .$$

Combining (3.9), (3.10), (3.11), and (3.13), we have the desired result (3.8). □

REMARK. Corollaries 8 and 9 in [6] are special cases of Theorem 3.3 for k = 0 and 1, respectively.

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