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SOME PERMANENTAL FORMULAS

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ABSTRACT. In this paper, we represent the permanent of a matrix A as inner products of some vectors or functions by noting that the permanent of A equals certain coefficients of some polynomials associated with A.

1. Introduction

For an $m \times n$ matrix $A = [a_{ij}]$, $m \leq n$, the *permanent* of A, per$A$, is defined as

$$\text{per}A = \sum_{\sigma} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{m\sigma(m)},$$

where the summation runs over all m-permutations of $1, 2, \cdots, n$ [4]. The notion of the permanent plays very important roles in various combinatorial problems such as system of distinct representatives (SDR), matching theory, permutations with forbidden positions, etc. In spite of such a usefulness, the permanent has the adversity that its evaluation is not easy. In 1961 Marcus and Newman represented the permanent function as an inner product on the symmetry class of completely symmetric tensors [2], and used it to prove some permanental inequalities later in [3]. In this paper, we represent the permanent of a matrix A as inner products of some vectors or functions by observing that the permanent of A equals certain coefficients of some polynomials associated with A.

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2. Permanent as Inner Product of Two Vectors

We begin this section with a lemma.

LEMMA 1. *Let $k_0, k_1, \cdots, k_{n-1}$ be nonnegative integers such that $k_0 + k_1 + \cdots + k_{n-1} = n$. Then $k_0 2^0 + k_1 2^1 + \cdots + k_{n-1} 2^{n-1} = 2^n - 1$ if and only if $(k_0, k_1, \cdots, k_{n-1}) = (1, 1, \cdots, 1)$.*

Proof. For an integer s, let $\omega(s)$ denote the number of 1's in the expression of s in the binary number system. Let $l_1, l_2, \cdots, l_m$ be nonnegative integers which may or may not be all distinct. We first note that $\omega(2^{l_1} + 2^{l_2} + \cdots + 2^{l_m})$ cannot exceed m. For, if $l_1, l_2, \cdots, l_m$ are all distinct, then certainly $\omega(2^{l_1} + 2^{l_2} + \cdots + 2^{l_m}) = m$. If there are two same numbers among $l_1, l_2, \cdots, l_m$, say, $l_1 = l_2$ for example, then $2^{l_1} + 2^{l_2} = 2^{l_2+1}$ so that $2^{l_1} + 2^{l_2} + \cdots + 2^{l_m} = 2^{l_2+1} + 2^{l_3} + \cdots + 2^{l_m}$, and the assertion follows by induction. Assume that

$$k_0 2^0 + k_1 2^1 + \cdots + k_{n-1} 2^{n-1} = 2^n - 1. \tag{1}$$

Since $2^n - 1$ is an odd number, we see that k_0 is an odd number. Suppose that $k_0 \neq 1$. Then $k_0 2^0 + k_1 2^1 + \cdots + k_{n-1} 2^{n-1} = 2^0 + ((k_0 - 1)/2 + k_1) 2^1 + \cdots + k_{n-1} 2^{n-1}$. Thus $n = \omega(2^n - 1) = \omega(2^0 + ((k_0 - 1)/2 + k_1) 2^1 + \cdots + k_{n-1} 2^{n-1}) \leq 1 + ((k_0 - 1)/2 + k_1) + k_2 + \cdots + k_{n-1} = n - (k_0 - 1)/2 < n$, a contradiction. Hence it must be that $k_0 = 1$, and (1) reduces to $k_1 2^0 + k_2 2^1 + \cdots + k_{n-1} 2^{n-2} = 2^{n-1} - 1$. Thus it follows that $(k_1, k_2, \cdots, k_{n-1}) = (1, 1, \cdots, 1)$ by induction, and the proof of 'only if ' part is complete. The 'if' part is trivial. □

Let $X_n = \operatorname{diag}(1, x, x^{2^{2-1}-1}, \cdots, x^{2^{n-1}-1})$ where x is a real variable. For an $n \times n$ matrix A, let

$$f_A(x) = \prod_{i=1}^{n} r_i(AX_n),$$

where and in the sequel, for a matrix B, $r_i(B)$ denotes the ith row sum of B. The degree of the polynomial $f_A(x)$ is less than or equal to $n(2^{n-1} - 1)$. Let n be fixed and let $d = n(2^{n-1} - 1) + 1$ from now on in the sequel. Let $\mathbf{a} = (\alpha_1, \alpha_2, \cdots, \alpha_d)^T$ be any fixed real d-vector with $\alpha_1, \alpha_2, \cdots, \alpha_d$ being mutually distinct real numbers, and let

$$L_i(x) = \frac{(x-\alpha_1)\cdots(x-\alpha_{i-1})(x-\alpha_{i+1})\cdots(x-\alpha_d)}{(\alpha_i-\alpha_1)\cdots(\alpha_i-\alpha_{i-1})(\alpha_i-\alpha_{i+1})\cdots(\alpha_i-\alpha_d)},$$
$$\text{for } i = 1, 2, \cdots, d.$$

Let c_i be the coefficient of x^{2^n-n-1} in $L_i(x)$ for $i = 1, 2, \cdots, d$, and let $\mathbf{c_a} = [c_1, c_2, \cdots, c_d]^T$. For vectors $\mathbf{u}$ and $\mathbf{v}$, let $\mathbf{u} \cdot \mathbf{v}$ denote the Euclidean inner product of $\mathbf{u}$ and $\mathbf{v}$. Now we are ready to give one of our main theorems.

THEOREM 2. *Let* $\mathbf{a} = (\alpha_1, \alpha_2, \cdots, \alpha_d)^T$ *and* $\mathbf{c_a} = [c_1, c_2, \cdots, c_d]^T$ *be vectors defined as above. Then for any* $n \times n$ *matrix* A, *we have*

$$\text{per}A = [f_A(\alpha_1), f_A(\alpha_2), \cdots, f_A(\alpha_d)]^T \cdot \mathbf{c_a}. \tag{2}$$

Proof. Let $A = [a_{ij}]$. We first show that the coefficient of x^{2^n-1} in

$$x^n f_A(x) = \prod_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x^{2^j-1} \tag{3}$$

equals per$A$. Note that $\prod_{j=1}^{n} x^{2^{j-1}} = x^{2^n-1}$. The degree of each of the n^n terms in the product (3) is a number of the form $k_0 2^0 + k_1 2^1 + \cdots + k_{n-1}2^{n-1}$ with $k_0, k_1, \cdots, k_{n-1}$ being some nonnegative integers such that $k_0 + k_1 + \cdots + k_{n-1} = n$. In order for a term, thus appeared, to be of degree $2^n - 1$ it is necessary and sufficient that $k_0 2^0 + k_1 2^1 + \cdots + k_{n-1}2^{n-1} = 2^n - 1$, or equivalently that $(k_0, k_1, \cdots, k_{n-1}) = (1, 1, \cdots, 1)$ by Lemma 1. Thus we see that per$A$ equals the coefficient of x^{2^n-1} in $x^n f_A(x)$ which equals that of x^{2^n-n-1} in $f_A(x)$. Since

$$f_A(x) = \sum_{j=1}^{d} f_A(\alpha_i) L_i(x),$$

by Lagrange interpolation formula, we have the theorem. □

In 1963, H. Ryser gave the following formula for evaluating the permanent, with a proof based on the inclusion-exclusion principle [5].

[Ryser's formula] For an $m \times n$ matrix $A = [a_{ij}]$, $m \leq n$,

$$\text{per}A = \sum_{j=0}^{m-1} (-1)^j \binom{n-m+j}{j} \sum_{B \in \Lambda_{m-j}} \prod_{i=1}^{m} r_i(B), \tag{4}$$

where Λ_k denote the set of all $m \times k$ submatrices of A, and $r_i(B)$ denotes the ith row sum of B.

In case that $m = n$, the formula (4) reduces to

$$\text{per}A = \sum_{j=0}^{n-1} (-1)^j \sum_{B \in \Lambda_{n-j}} \prod_{i=1}^{n} r_i(B). \tag{5}$$

The number of multiplications needed in (5) is $(n-1)(2^n - 1)$ which is roughly exponential in n. One might have expected some other formula in which the number of necessary multiplications is polynomial in n. However it has turned out that such an expectation is impossible. In fact, in 1979, L. G. Valiant [6] proved that the evaluation of the permanent of a square $(0,1)$-matrix is a $\#P$-complete problem, a problem which is as hard as an NP-complete problem. He also showed that the permanent of a square matrix X of order n can be evaluated by calculating the determinant of a matrix Y of order m associated with X where m is a constant times $n^2 2^n$. So far Ryser's formula is the best known computational procedure for the permanent [1]. In view of the above paragraph, we believe that the formula given in Theorem 2 has also some importance as far as the computational complexity is concerned. Note that the numbers α_i in the formula (2) can be taken freely. So we may take, for example, $\alpha_i = i$, for $i = 1, 2, \cdots, d$, in which case the formula (2) becomes

$$\text{per}A = [f_A(1), f_A(2), \cdots, f_A(d)]^T \cdot [c_1, c_2, \cdots, c_d]^T,$$

where

$$c_k = \left(\prod_{j=1}^{k-1} \frac{1}{k-j}\right)\left(\prod_{j=k+1}^{d} \frac{1}{k-j}\right)$$
$$\cdot (-1)^{\beta}\sigma_{\beta}([1,2,\cdots,k-1,k+1,\cdots,d])$$
$$= \frac{(-1)^{n-k+1}}{(k-1)!(d-k)!}\sigma_{\beta}([1,2,\cdots,k-1,k+1,\cdots,d])$$

with $\beta = d - 2^n + n + 1$ and σ_β being the β-th elementary symmetric function. If we calculate $c_1, c_2, \cdots, c_d$ in advance and store them in a computer, then to evaluate the permanent of a matrix A, all we need is to calculate the values $f_A(1), f_A(2), \cdots, f_A(d)$ and the inner product. The number of multiplications needed here is $(n-1)^2 d + d = (n^2 - 2n + 2)(n2^{n-1} - n + 1)$.

Let B be a square matrix of order $m \le n$, and let

$$C = \begin{bmatrix} B & O \\ O & I_{n-m} \end{bmatrix}.$$

Then $\mathrm{per}B = \mathrm{per}C$. Let D be a rectangular matrix of size $m \times k$ with $m < k \le n$, and let

$$E = \begin{bmatrix} D & O \\ J_{n-m,k} & J_{n-m,n-k} \end{bmatrix},$$

where $J_{p,q}$ denotes the $p \times q$ matrix of 1's. Then $\mathrm{per}D = (1/(n-m)!)\mathrm{per}E$. Thus we see that the stored vector $[c_1, c_2, \cdots, c_d]^T$ can be used to calculate the permanent of every matrix of size $m \times k$ with $m \le k \le n$.

3. Permanent as Inner Product of Gradients and Vectors

In this section we derive the permanent of a square matrix in connection with the differentiation and inner product.

Again let n be a fixed positive integer and let $x_1, x_2, \cdots, x_n$ be independent variables. Let $X = \mathrm{diag}(x_1, x_2, \cdots, x_n)$. For an $n \times n$ matrix $A = [a_{ij}]$, let

$$p_A(x_1, x_2, \cdots, x_n) = \prod_{i=1}^{n} r_i(AX). \tag{6}$$

Observe that

$$\mathrm{per}A = \frac{\partial^n}{\partial x_n \partial x_{n-1} \cdots \partial x_1} p_A(x_1, x_2, \cdots, x_n), \tag{7}$$

because the coefficient of the term $x_1 x_2 \cdots x_n$ in the expansion of (6) equals $\mathrm{per}A$.

THEOREM 3. *Let $f(y_1, y_2, \cdots, y_n) = y_1 y_2 \cdots y_n$. Then for any $n \times n$ matrix $A = [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n]$, we have*

$$\mathrm{per}A = \nabla(\cdots \nabla(\nabla f \cdot \mathbf{a}_1) \cdot \mathbf{a}_2) \cdots) \cdot \mathbf{a}_n.$$

Proof. Let $A[x_1, x_2, \cdots, x_n]^T = [y_1, y_2, \cdots, y_n]^T$. Then $f(y_1, y_2, \cdots, y_n) = p_A(x_1, x_2, \cdots, x_n)$ and

$$\frac{\partial y_i}{\partial x_j} = a_{ij}, (i, j = 1, 2, \cdots, n)$$

so that

$$\left[\frac{\partial y_1}{\partial x_j}, \frac{\partial y_2}{\partial x_j}, \cdots, \frac{\partial y_n}{\partial x_j}\right] = \mathbf{a}_j, (j = 1, 2, \cdots, n). \tag{8}$$

From (7) and (8), we have

$$\begin{aligned}\mathrm{per}A &= \frac{\partial^n f}{\partial x_n \partial x_{n-1} \cdots \partial x_1} \\ &= \sum_{i_n=1}^{n} \frac{\partial}{\partial y_{i_n}} \cdots \sum_{i_2=1}^{n} \frac{\partial}{\partial y_{i_2}} \sum_{i_1=1}^{n} \frac{\partial f}{\partial y_{i_1}} \frac{\partial y_{i_1}}{\partial x_1} \frac{\partial y_{i_2}}{\partial x_2} \cdots \frac{\partial y_{i_n}}{\partial x_n}\end{aligned}$$

$$= \sum_{i_n=1}^{n} \frac{\partial}{\partial y_{i_n}} \cdots \sum_{i_3=1}^{n} \frac{\partial}{\partial y_{i_3}} \sum_{i_2=1}^{n} \frac{\partial}{\partial y_{i_2}} (\nabla f \cdot \mathbf{a}_1) \frac{\partial y_{i_2}}{\partial x_2} \frac{\partial y_{i_3}}{\partial x_3} \cdots \frac{\partial y_{i_n}}{\partial x_n}$$

$$= \sum_{i_n=1}^{n} \frac{\partial}{\partial y_{i_n}} \cdots \sum_{i_3=1}^{n} \frac{\partial}{\partial y_{i_3}} \nabla(\nabla f \cdot \mathbf{a}_1) \cdot \mathbf{a}_2 \frac{\partial y_{i_3}}{\partial x_3} \cdots \frac{\partial y_{i_n}}{\partial x_n}$$

$$\vdots$$

$$= \nabla(\cdots \nabla(\nabla f \cdot \mathbf{a}_1) \cdot \mathbf{a}_2) \cdot \cdots) \cdot \mathbf{a}_n,$$

and the proof is complete. □

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