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NOTES ON THREE-DIMENSIONAL WEAKLY SYMMETRIC SPACES

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ABSTRACT. In the present paper, we describe the action of isometry groups of 3-dimensional weakly symmetric spaces and classify 3-dimensional connected weakly symmetric spaces. Further, we determine 3-dimensional weakly symmetric spaces in terms of the eigenvalues of the Ricci transformation.

1. Introduction

The notion of a weakly symmetric space has been introduced by A. Selberg in 1956 ([4]). His motivation was to generalize the so-called Poisson summation formula. From his definition, we can easily see that any weakly symmetric space is Riemannian homogeneous, and furthermore, that any Riemannian symmetric space is weakly symmetric. Much work has been done on the harmonic analysis on weakly symmetric spaces. However, his definition seems to be rather abstract and only a few examples of non-symmetric weakly symmetric space were known. Recently, J. Berndt and L. Vanhecke gave the following geometric characterization of weakly symmetric spaces.

THEOREM 1 ([1]). *A Riemannian manifold $M = (M, g)$ is weakly symmetric if and only if for any two points p, $q \in M$ there exists an isometry φ of M satisfying $\varphi(p) = q$ and $\varphi(q) = p$.*

They also classified 3-dimensional, connected, simply connected weakly symmetric spaces.

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THEOREM 2 ([1]). *A 3-dimensional, connected, simply connected Riemannian manifold is a weakly symmetric space if and only if it is isometric to one of the following spaces:*

(i) *a 3-dimensional Riemannian symmetric space,*
(ii) *the group $SU(2) = S^3$ with a left-invariant metric which is known as the Berger sphere,*
(iii) *the universal covering space of $SL(2;\mathbb{R})$ with a left-invariant metric,*
(iv) *the Heisenberg group with any left-invariant.*

(*More details for* (ii)$\sim$(iv) *will be stated in section* 3.)

However, non-simply connected ones have not been classified. One of the purpose of the present paper is to classify them.

In [1], the proof of Theorem 2 owes a great deal to the classification of D'Atri spaces, i.e., Riemannian manifolds whose geodesic symmetries around each point are volume-preserving up to sign (for the definition of a D'Atri space and further information, see for example [10]). In the present paper, we give an explicit alternative proof of Theorem 2. We describe the action of isometry groups and isotropy representations to construct a reflection in the tangent space (see Proposition 4 in the next section). This explicit description enables us to determine 3-dimensional connected weakly symmetric spaces. We will discuss this in the last section. In section 4, we determine 3-dimensional weakly symmetric spaces in terms of the eigenvalues of the Ricci transformation.

2. Some Properties of Weakly Symmetric Spaces

In this section, we introduce some characterizations and properties of weakly symmetric spaces. A trivial consequence of Theorem 1 is

PROPOSITION 3 ([1]). *A Riemannian manifold $M = (M, g)$ is a weakly symmetric space if and only if for every geodesic γ in M there exists an isometry φ on M such that $\varphi(\gamma(t)) = \gamma(-t)$.*

From this characterization, it is obvious that if $M = (M, g)$ is a weakly symmetric space, then for each $v \in T_pM$ there exists an f in H_p, the full isotropy subgroup at p, such that $df_p(v) = -v$. Since a weakly symmetric space is homogeneous and the isotropy subgroup at any two points are conjugate to each other, we obtain the following.

PROPOSITION 4 ([11]). *Let $M = G/H$ be a homogeneous space. For a point $p \in M$, we denote by $\chi : H_p \to GL(T_pM)$ the isotropy representation of the isotropy subgroup H_p at p. If for each $v \in T_pM$ there exists an element $h \in H_p$ satisfying $\chi(h)(v) = -v$, then M is weakly symmetric with respect to any G-invariant Riemannian metric on M.*

For the classification problem, the following proposition is very useful.

PROPOSITION 5 ([1]). *The Riemannian universal covering space of a weakly symmetric space is also a weakly symmetric space.*

Further, the converse is also true, namely

PROPOSITION 6. *A Riemannian homogeneous manifold whose Riemannian universal covering space is weakly symmetric is weakly symmetric.*

3. Three-dimensional Weakly Symmetric Spaces

In this section, we give an explicit alternative proof of Theorem 2. First of all, we review the result of [3] and [4].

Let $M = (M, g)$ be a homogeneous Riemannian manifold. We denote by ∇ and R the Riemannian connection with respect to g and the associated curvature tensor of M. A homogeneous Riemannian manifold satisfies the following condition $P(n)$ for any integer $n \geq 0$:

$P(n)$: for each p, $q \in M$, there exists a linear isometry $\Phi : T_pM \to T_qM$ such that $\Phi^*((\nabla^k R)_q) = (\nabla^k R)_p$ for $k = 0, 1, \ldots, n$.

I. M. Singer dealt with the converse problem and proved that a connected, simply connected and complete Riemannian manifold satisfying the condition $P(n)$ for certain n is Riemannian homogeneous ([5]). The minimum of such integer n depends on the manifold M, but it is smaller than $m(m-1)/2 + 1$, $m = \dim M$. In dimension 3, the third author proved the following:

THEOREM 7 ([3]). *Let $M = (M, g)$ be a 3-dimensional, connected, simply connected, complete Riemannian manifold satisfying the condition $P(1)$. Then M is Riemannian homogeneous, and furthermore, it is* *isometric to one of* (I) S^3, (II) $\mathbb{R}^3$, (III) $\mathbb{H}^3$, (IV) $M^2 \times \mathbb{R}$, (V) *a group manifold with certain left-invariant metrics which are not symmetric ones, where* $M^2 = S^2$ *or* $\mathbb{H}^2$ *in* (IV).

Further, the same author gave a list of Lie algebras of the full isometry groups which acts effectively and transitively on 3-dimensional, connected, simply connected Riemannian manifold ([3]):

	$\mathfrak{i}(M, g)$	$\mathfrak{h}$	M is diffeomorphic to
(a)	$\mathfrak{so}(4)$	$\mathfrak{so}(3)$	S^3
	$\mathfrak{so}(3) + \mathbb{R}^3$ (semi-direct sum)	$\mathfrak{so}(3)$	$\mathbb{R}^3$
	$\mathfrak{so}(3,1)$	$\mathfrak{so}(3)$	$\mathbb{R}^3$
(b)	$\mathfrak{so}(3) \oplus \mathbb{R}$	$\mathfrak{so}(2)$	$S^2 \times \mathbb{R}$
	$\mathfrak{so}(2,1) \oplus \mathbb{R}$	$\mathfrak{so}(2)$	$\mathbb{R}^3$
(c)	$\mathfrak{so}(3) \oplus \mathbb{R}$	$\mathfrak{so}(2)$	S^3
	$\mathfrak{so}(2,1) \oplus \mathbb{R}$	$\mathfrak{so}(2)$	$\mathbb{R}^3$
	$\mathfrak{so}(2) + \mathfrak{b}$ (semi-direct sum)	$\mathfrak{so}(2)$	$\mathbb{R}^3$
(d)	certain Lie algebra	$\{0\}$	$\mathbb{R}^3$ or S^3

Here, $\mathfrak{i}(M, g)$ is the Lie algebra of the full isometry group $I(M, g)$ of (M, g), $\mathfrak{h}$ is the Lie algebra of the isotropy subgroup and $\mathfrak{b}$ is a certain Lie algebra (see [4] for more details). From this table, we see that (a) and (b) correspond to (I)$\sim$(IV) in Theorem 7. In case (d), since we consider a connected and simply connected one, we see that the isotropy group H must be trivial. Then, the isotropy representation of H is also trivial, and hence, from Proposition 4, M is not weakly symmetric. Therefore, it suffices to consider only the case (c) for our purpose. Here we note that the Ricci transformation of a manifold of type (c) has two non-zero distinct eigen-values (see [4]).

Now, we describe the action of isometry groups and isotropy representation of (ii)$\sim$(iv) in Theorem 2 to prove that they are weakly symmetric spaces.

Case (ii)

Let G_1 be the Cartesian product of the special unitary group $SU(2)$ and the additive group $\mathbb{R}$. The group G_1 acts transitively on the 3-dimensional sphere $S^3 = SU(2)$ as follows. Let α and β be arbitrary fixed non-zero real numbers satisfying $\alpha^2 + \beta^2 = 1$, and for $(A, \theta) \in G_1$ and $P \in SU(2)$ we define the action by

$$(A, \theta)P = APE\left(i\frac{\alpha}{\beta}\theta\right)$$

where $E(i\eta) = \begin{pmatrix} e^{i\eta} & 0 \\ 0 & e^{-i\eta} \end{pmatrix}$, $\eta \in \mathbb{R}$, and $i^2 = -1$. The isotropy subgroup H_1 at the origin $I \in SU(2)$ is $H_1 = \{ (E(i\alpha\theta), -\beta\theta) \mid \theta \in \mathbb{R} \}$. Thus, we have $S^3 = G_1/H_1$ and denote by π the natural projection from G_1 onto G_1/H_1. The G_1-invariant vector fields

$$e_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_4 = \frac{\partial}{\partial\theta}$$

span the Lie algebra $\mathfrak{g}_1$ of G_1 and $\alpha e_3 - \beta e_4$ generates the Lie algebra $\mathfrak{h}_1$ of H_1. The inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}_1$ defined by $\langle e_i, e_j \rangle = \delta_{ij}$ $(i, j = 1, \ldots, 4)$ gives rise to a left-invariant metric on G_1. By making use of the orthogonal decomposition $\mathfrak{g}_1 = \mathfrak{m} + \mathfrak{h}_1$, where $\mathfrak{m} = \mathrm{span}\{e_1, e_2, e_0 = \beta e_3 + \alpha e_4\}$, we may obtain a G_1-invariant metric g_1 on $SU(2) = G_1/H_1$ such that $(SU(2), g_1)$ is a normal homogeneous Riemannian space. The Riemannian manifold $S^3 = (SU(2), g_1)$ is known as the Berger sphere. By the formulas due to O'Neill ([2]), we may easily see that the Ricci transformation of $S^3 = (SU(2), g_1)$ has two non-zero distinct eigenvalues $2\beta^2$ and $2(\alpha^2 + 1)$ with multiplicity 2. Now, we define $\mu_1 : G_1/H_1 \to G_1/H_1$ induced by the automorphism $\bar{\mu}_1$ on G_1 defined by $\bar{\mu}_1(A, \theta) = (\bar{A}, -\theta)$. Then, g_1 is μ_1-invariant and the differential map at the origin $\pi(I) \in G_1/H_1$ of μ_1 changes the sign of e_1, e_0 and leaves e_2 invariant. Further, it is easy to see that the isotropy representation of H_1 is $SO(2)$, the rotation around e_0. Therefore, from Proposition 4, we conclude that $S^3 = (SU(2), g_1)$ is a weakly symmetric space.

Case (iii)

As a Riemannian universal covering space of the special linear group $SL(2; \mathbb{R})$, without loss of generality, it suffices to consider

$$\mathbb{H}^2 \times \mathbb{R} = \{ (z, \phi) \in \mathbb{C} \times \mathbb{R} \mid \mathrm{Im}\, z > 0 \}$$

endowed with a Riemannian metric

$$g_2 = \frac{dx^2 + dy^2}{y^2} + \left(d\phi - \frac{dx}{2y}\right)^2$$

where $z = x + iy$, $x, y \in \mathbb{R}$. Let G_2 be the Cartesian product of $SL(2;\mathbb{R})$ and the additive group $\mathbb{R}$. The group G_2 acts transitively on $\mathbb{H}^2 \times \mathbb{R}$ by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, t\right)(z, \phi) = \left(\frac{az+b}{cz+d}, \phi + \arg(cz+d) + t\right)$$

for $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, t\right) \in G_2$ and $(z, \phi) \in \mathbb{H}^2 \times \mathbb{R}$. We see that g_2 is G_2-invariant. The isotropy subgroup at the origin $o = (i, 0) \in \mathbb{H}^2 \times \mathbb{R}$ is

$$H_2 = \left\{ \left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \theta\right) \middle| \theta \in \mathbb{R} \right\}$$

and hence, we have $\mathbb{H}^2 \times \mathbb{R} = G_2/H_2$. It is straightforward to see that the Ricci transformation has two distinct eigenvalues $1/8$ and $-9/8$ with multiplicity 2. The left-invariant vector fields

$$e_1 = y\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial \phi}, \quad e_2 = y\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial \phi}$$

form a global orthonormal frame field on $(\mathbb{H}^2 \times \mathbb{R}, g_2)$. Now, we define a map $\mu_2 : \mathbb{H}^2 \times \mathbb{R} \to \mathbb{H}^2 \times \mathbb{R}$ by $\mu_2(z, \phi) = (-\bar{z}, -\phi) = (-x, y, -\phi)$. Then, g_2 is μ_2-invariant and the differential map of μ_2 at the origin o changes the sign of $(e_1)_o$ and $(e_3)_o$ and leaves $(e_2)_o$ invariant. Further, it is easy to see that the isotropy representation of H_2 is $SO(2)$, the rotation around $(e_3)_o$. Therefore, from Proposition 4, we conclude that the Riemannian universal covering space of $SL(2;\mathbb{R})$ is a weakly-symmetric space.

Case (iv)

Without loss of generality, it suffices to consider the Heisenberg group

$$H = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\} \cong \mathbb{R}^3$$

endowed with a left-invariant metric

$$g_3 = dx^2 + dz^2 + (dy - xdz)^2.$$

We define a homomorphism $\rho : \mathbb{R} \longrightarrow \mathrm{Aut}(H)$ by

$$\rho_t \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x\cos t + z\sin t & y + \sigma_t(x,z) \\ 0 & 1 & -x\sin t + z\cos t \\ 0 & 0 & 1 \end{pmatrix}$$

where $\sigma_t(x,z) = -\frac{1}{2}(x^2 - z^2)\sin t\cos t - xz\sin^2 t$. Let G_3 be the semi-direct product $H \rtimes_\rho \mathbb{R}$, namely, $H \times \mathbb{R}$ with the group structure defined by $(A,t)\cdot(A',t') = (A\rho_t(A'), t+t')$ for (A,t), $(A',t') \in H \times \mathbb{R}$. We see that the Riemannian metric g_3 is G_3-invariant. The group $H \rtimes_\rho \mathbb{R}$ acts transitively on H by

$$(A,t)P = A\rho_t(P)$$

for $(A,t) \in H \rtimes_\rho \mathbb{R}$ and $P \in H$. The isotropy subgroup of the identity $I \in H$ is $H_3 = \left\{ (I,t) \,\middle|\, t \in \mathbb{R} \right\}$ and hence, we have $H = G_3/H_3$. We may easily see that the Ricci transformation of (H, g_3) has two distinct eigenvalues $1/2$ and $-1/2$ with multiplicity 2. We see that the G_3-invariant vector fields

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial y}$$

form a global orthonormal frame field on H. Now, We define a map on $\mu_3 : H \longrightarrow H$ by

$$\mu_3 \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -x & -y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, g_3 is μ_3-invariant and the differential map of μ_3 at I changes the sign of $(e_1)_I$, $(e_3)_I$ and leaves $(e_2)_I$ invariant. Further, it is easy to see that the isotropy representation of H_3 is $SO(2)$, the rotation around $(e_3)_I$. Therefore, from Proposition 4, we conclude that (H, g_3) is a weakly-symmetric space.

4. Eigenvalues of the Ricci Transformation of Weakly Symmetric Spaces

From the arguments in [4] and the preceding section, we obtain the following.

THEOREM 8. *Let $M = (M, g)$ be a 3-dimensional, connected, simply connected homogeneous Riemannian manifold. We denote by λ_i ($i = 1, 2, 3$) the eigenvalues of the Ricci transformation of M. Then we have:*

(1) *Case $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$:*
M is isometric to S^3 ($\lambda > 0$), $\mathbb{R}^3$ ($\lambda = 0$) or $\mathbb{H}^3$ ($\lambda < 0$).

(2) *Case $\lambda_1 = \lambda_2 = \lambda$ and $\lambda_3 = \mu \neq \lambda$:*
If $\mu = 0$, then M is isometric to $S^2 \times \mathbb{R}$ ($\lambda > 0$) or $\mathbb{H}^2 \times \mathbb{R}$ ($\lambda < 0$). If $\lambda = 0$, then M is isometric to a non-symmetric group manifold with invariant metric on which the identity component of the full isometry group acts simply transitively. Otherwise ($\lambda \neq 0$ and $\mu \neq 0$), M is isometric to a 3-dimensional weakly symmetric space.

(3) *Case $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$:*
M is isometric to a group manifold with certain invariant metric which is not symmetric.

For (2), we also remark that if $\lambda + \mu > 0$, then M is isometric to a Berger sphere (case (ii)); if $\lambda + \mu < 0$, then M is isometric to the universal covering space of $SL(2, \mathbb{R})$ (case (iii)) and if $\lambda + \mu = 0$, then M is isometric to the Heisenberg group (case (iv)).

Finally, we give an immediate consequence of the above arguments.

COROLLARY 9. *A 3-dimensional, connected, simply connected Riemannian manifold $M = (M, g)$ is a non-symmetric weakly symmetric space if and only if M is Riemannian homogeneous and its Ricci transformation has two non-zero distinct eigenvalues.*

5. Three-dimensional Weakly Symmetric Spaces

In this section, we give a list of all 3-dimensional, connected weakly symmetric spaces. For our purpose, the following theorem plays an important role.

THEOREM 10 ([10]). *Let $M = (M, g)$ be a Riemannian manifold and Γ a group of isometries of M acting freely and properly discontinuously. Then, M/Γ is homogeneous if and only if the centralizer $C(\Gamma)$ of Γ in $I(M, g)$ acts transitively on M. And, if M/Γ is homogeneous, then every element of Γ is a Clifford translation of M.*

Thus, our problem reduces to determine discrete subgroups Γ satisfying Theorem 10. If M is a simply connected symmetric space, namely if M is isometric to S^3, $\mathbb{R}^3$, E^3, $S^2 \times \mathbb{R}$ or $H^2 \times \mathbb{R}$, then the subgroups Γ have been classified (see [7], [8], [10]). So, it suffices to consider the case where M is a simply connected, non-symmetric weakly symmetric space (case (ii), (iii) and (iv) in Theorem 2).

THEOREM 11. *In case* (ii), Γ *is one of*

$$\{(I,0)\} \quad or \quad \left\{\left(\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, 0\right)\right\} \simeq \mathbb{Z}_2.$$

In case (iii), Γ *is one of*

$$\{(I,0)\} \quad or \quad \left\{\left(\begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}, 0\right)\right\} \simeq \mathbb{Z}.$$

In case (iv), Γ *is one of*

$$\{(I,0)\} \quad or \quad \left\{\left(\begin{pmatrix} 1 & 0 & cn \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, 0\right) \middle| \; n \in \mathbb{Z} \right\} \simeq \mathbb{Z},$$

where c is a non-zero real constant.

Proof. The proof is straightforward. We only prove case (iv). Assume that Γ is not trivial. Since Γ acts freely on H, we see that Γ is a subgroup of $H \rtimes_\rho \{0\} = H \times \{0\}$. The center of $H \times \{0\}$ in $G_3 = H \rtimes_\rho \mathbb{R}$ is

$$\left\{\left(\begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, 0\right) \middle| \; y \in \mathbb{R} \right\}. \text{ Thus, } \left\{\left(\begin{pmatrix} 1 & 0 & cn \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, 0\right) \middle| \; n \in \mathbb{Z} \right\},$$

$c \in \mathbb{R} - \{0\}$, is one of the desired Γ. Taking into account that $H \times \{0\}$ acts simply transitively on H, we find that this Γ is the only non-trivial one. The remaining cases may be proved in a similar way. □

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