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REAL VERSION OF PALEY-WIENER-SCHWARTZ THEOREM FOR ULTRADISTRIBUTIONS WITH ULTRADIFFERENTIABLE SINGULAR SUPPORT

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ABSTRACT. We extend the Paley-Wiener-Schwartz theorem to the space of ultradistributions with respect to ultradifferentiable singular support and obtain its real version. That is, we obtain the growth condition in some tubular neighborhood of $\mathbb{R}^n$ of the Fourier transform of ultradistributions of Roumieu (or Beurling) type with ultradifferentiable singular support contained in a ball centered at the origin, and its real version.

1. Introduction

The Fourier transform $\hat{u}$ of any distribution $u \in \mathcal{E}'(\mathbb{R}^n)$ with compact support can be extended to an entire analytic function in $\mathbb{C}^n$, called *Fourier-Laplace transform* of u, i.e., $\hat{u}(\zeta) = u_x(e^{-i\langle x,\zeta\rangle})$, $\zeta \in \mathbb{C}^n$. Then Paley-Wiener-Schwartz theorem determines the space of generalized function by only the growth condition of its Fourier transform.

Paley-Wiener-Schwartz theorem has been extended by L. Hörmander [5] to the space of distributions $u \in \mathcal{E}'(\mathbb{R}^n)$ with respect to the singular support as follows:

THEOREM 1.1. *Let $u \in \mathcal{E}'(\mathbb{R}^n)$ be a distribution with compact support and K be a convex compact subset of $\mathbb{R}^n$ with supporting function $H_K(\xi) = \sup_{x\in K}\langle x, \xi\rangle$. If $\operatorname{sing\,supp} u \subset K$, then its Fourier-Laplace transform $\hat{u}$ extends to an entire analytic function and there exist a*

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constant N and a sequence of constants C_m, $m = 1, 2, \dots$, such that for every $\epsilon > 0$

$$(1.1) \qquad |\hat{u}(\zeta)| \le C_m(1 + |\zeta|)^N \exp(H_K(\operatorname{Im}\zeta) + \epsilon|\operatorname{Im}\zeta|), \quad \zeta \in \mathbb{C}^n,$$

whenever $|\operatorname{Im}\zeta| \le m \ln(1 + |\zeta|)$.

Conversely, if $\hat{u}$ extends to an analytic function satisfying the condition (1.1) then $\operatorname{sing\,supp} u \subset K$.

The above result has been extended by G. Björck [1] to the space of ultradistributions of Beurling-Björck type with ω-singular support. In this paper we prove the Paley-Wiener-Schwartz type theorem for the space of ultradistributions of Roumieu (or Beurling) type with respect to ultradifferentiable singular support.

On the other direction it has been recently proved by N. Mandache [7] that (1.1) can be replaced by estimates only on the real variables, called the *real version* of Paley-Wiener-Schwartz theorem, as follows:

THEOREM 1.2. *Let $u \in \mathcal{E}'(\mathbb{R}^n)$, K be convex, compact and symmetric with respect to the hyperplanes of coordinates in $\mathbb{R}^n$ and set $C_{K,\epsilon,\alpha} = \sup_{x \in K_\epsilon} |x^\alpha|$, where K_ϵ is an ϵ-neighborhood of K.*

If $\operatorname{sing\,supp} u \subset K$, *then there exist a constant N and a sequence of constants C_m, $m = 1, 2, \dots$, such that for every $\epsilon > 0$*

$$(1.2) \qquad |\partial^\alpha \hat{u}(\xi)| \le C_m C_{K,\epsilon,\alpha}(1 + |\xi|)^N, \quad \xi \in \mathbb{R}^n,$$

whenever $|\alpha| \le m \ln(1 + |\xi|)$.

Conversely, (1.2) implies $\operatorname{sing\,supp} u \in K$.

Here, a compact set K is said to be *symmetric* with respect to the hyperplanes of coordinates in $\mathbb{R}^n$ if for every $x = (x_1, \dots, x_n) \in K$, $(\pm x_1, \dots, \pm x_n) \in K$ for all combinations of signs.

In this paper we extend the above result of N. Mandache to the space of ultradistributions with respect to ultradifferentiable singular support, which is the real version of the Paley-Wiener-Schwartz theorem for ultradistributions with respect to ultradifferentiable singular support.

2. Paley-Wiener-Schwartz Theorem

Let M_p, $p = 0, 1, 2, \ldots$, be a sequence of positive numbers. We consider the following conditions on M_p.

(M.1) $M_p^2 \leq M_{p-1}M_{p+1}$, $p = 1, 2, \ldots$.
(M.2)′ For some C, $L > 0$, $M_{p+1} \leq CL^p M_p$, $p = 0, 1, 2, \ldots$.
(M.2) For some A, $H > 0$, $M_{p+q} \leq AH^{p+q}M_pM_q$, $p, q = 0, 1, 2, \ldots$.
(M.3)′ $\sum_{p=1}^{\infty} M_{p-1}/M_p < \infty$.

For each sequence M_p its *associated function* $M(t)$ on $(0, \infty)$ is defined by

$$M(t) = \sup_p \log \frac{t^p M_0}{M_p}, \quad t > 0.$$

Let Ω be an open subset of $\mathbb{R}^n$. An infinitely differentiable functions φ is called an *ultradifferentiable function of Roumieu type* (resp. *of Beurling type*), denoted by $\mathcal{E}_{\{M_p\}}(\Omega)$ (resp. $\mathcal{E}_{(M_p)}(\Omega)$), if it satisfies the following condition: For every compact subset K there exist positive constants h and C (resp. for every $h > 0$ there exists a constant $C > 0$), depending on φ and K, such that

$$\sup_{x \in K} |\partial^\alpha \varphi(x)| \leq Ch^{|\alpha|}M_{|\alpha|},$$

where $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$, $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\ldots\partial_n^{\alpha_n}$ with $\partial_j = \partial/\partial x_j$ for every multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with nonnegative integers α_j, $j = 1, 2, \ldots, n$. Then we denote by $\mathcal{E}'_{\{M_p\}}(\Omega)$ (resp. $\mathcal{E}'_{(M_p)}(\Omega)$) the strong dual of the space $\mathcal{E}_{\{M_p\}}(\Omega)$ (resp. $\mathcal{E}_{(M_p)}(\Omega)$) and call its elements *ultradistributions of Roumieu type* (resp. *of Beurling type*) with compact support in Ω, even though Ω is not open.

We first introduce the well-known results of Paley-Wiener-Schwartz theorem in the theory of ultradistributions as in Komatsu [6].

THEOREM 2.1 [6]. *Let K be a convex compact subset in $\mathbb{R}^n$ with supporting function $H_K(\xi) = \sup_{x \in K}\langle x, \xi\rangle$. Suppose that M_p satisfies* (M.1), (M.2)′ *and* (M.3)′. *Then an entire function $\hat{\varphi}(\zeta)$ on $\mathbb{C}^n$ is the Fourier transform of an ultradifferentiable function $\varphi(x)$ of Roumieu type* (resp. *of Beurling type*) *with support contained in K if and only* *if there are positive constants* h *and* C (resp. *for any* $h > 0$ *there is a constant* $C > 0$) *such that*

$$|\hat{\varphi}(\zeta)| \le C \exp\{-M(|\zeta|/h) + H_K(\mathrm{Im}\,\zeta)\}, \quad \zeta \in \mathbb{C}^n,$$

where $M(t)$ *is the associated function of* M_p.

THEOREM 2.2 [6]. *Let* K *be a convex compact subset in* $\mathbb{R}^n$ *with supporting function* H_K. *Suppose that* M_p *satisfies* (M.1) *and* (M.2) (resp. (M.2)$'$). *Then the following conditions are equivalent for an entire function* $\hat{u}(\zeta)$ *on* $\mathbb{C}^n$.

(i) $\hat{u}(\zeta)$ *is the Fourier-Laplace transform of an ultradistribution* $u \in \mathcal{E}'_{\{M_p\}}(K)$ (resp. $\mathcal{E}'_{(M_p)}(K)$).

(ii) *For every* $\epsilon > 0$ *and for every* $L > 0$ *there exists a constant* $C > 0$ (resp. *there exist positive constants* L *and* C) *such that*

$$|\hat{u}(\zeta)| \le C \exp\{M(L|\zeta|) + H_K(\mathrm{Im}\,\zeta) + \epsilon|\mathrm{Im}\,\zeta|\}, \quad \zeta \in \mathbb{C}^n.$$

DEFINITION 2.3. For an ultradistribution u the *ultradifferentiable singular support* of u, denoted by $\mathrm{sing}_M\mathrm{supp}\,u$, is the set of points having no open neighborhood to which the restriction of u is an ultradifferentiable function.

We now formulate and prove Paley-Wiener-Schwartz type theorem for the space of ultradistributions with respect to ultradifferentiable singular support, which is the improvement of Theorem 1.1, in fact, its proof follows from the proof in [5].

THEOREM 2.4. *Suppose that* M_p *satisfies* (M.1), (M.2) (resp. (M.2)$'$) *and* (M.3)$'$. *Let* $u \in \mathcal{E}'_{\{M_p\}}$ (resp. $\mathcal{E}'_{(M_p)}$). *Let* K *be a convex compact subset in* $\mathbb{R}^n$ *with supporting function* H_K.

In order that $\mathrm{sing}_M\mathrm{supp}\,u \subset K$, *it is necessary and sufficient that for every* $L > 0$ *there exists a constant* $C_m > 0$ (resp. *there exist positive constants* L *and* C_m), $m = 1, 2, \ldots$, *such that*

$$|\hat{u}(\zeta)| \le C_m \exp\{M(L|\zeta|) + H_K(\mathrm{Im}\,\zeta)\}, \quad \zeta \in \mathbb{C}^n, \tag{2.1}$$

where $|\mathrm{Im}\,\zeta| \le m\,M(|\zeta|)$ *and* $M(t)$ *is the associated function of* M_p.

Proof. Let $u \in \mathcal{E}'_{\{M_p\}}$ with $\operatorname{sing}_M \operatorname{supp} u \subset K$. Choose an ultradifferentiable function $\varphi \in \mathcal{E}_{\{M_p\}}(K_{1/m})$ such that $\operatorname{supp}\varphi \subset K_{1/m} = K + B(0, 1/m)$ and $\varphi = 1$ on a neighborhood of K. Then by Theorem 2.2 we have for every $0 < L_1 < L_2$

$$\begin{aligned}|(\widehat{\varphi u})(\zeta)| &\leq C \exp\{M(L_1|\zeta|) + H_K(\operatorname{Im}\zeta) + (\frac{1}{m} + \epsilon)|\operatorname{Im}\zeta|\} \\ &\leq C \exp\{M(L_2|\zeta|) + H_K(\operatorname{Im}\zeta)\}, \quad \zeta \in \mathbb{C}^n\end{aligned} \tag{2.2}$$

for $|\operatorname{Im}\zeta| \leq m\, M(|\zeta|)$. Since $(1-\varphi)u$ is an ultradifferentiable function with compact support we have for some $h > 0$

$$|(\widehat{(1-\varphi)u})(\zeta)| \leq C \exp\{-M(|\zeta|/h)\} \quad \text{if } |\operatorname{Im}\zeta| \leq m\, M(|\zeta|). \tag{2.3}$$

Combining (2.2) and (2.3) we obtain the estimate (2.1).

To prove the converse we choose an ultradifferentiable function $\varphi \in \mathcal{E}_{\{M_p\}}(B(0,1))$ such that $\operatorname{supp}\varphi \subset B(0,1)$ and $\int \varphi(x)dx = 1$ and put $\varphi_\delta(x) = \delta^{-n}\varphi(x/\delta)$. In order to fit the set where (2.1) is applicable, we define Γ_η to be the cycle

$$\mathbb{R}^n \ni \xi \longmapsto \zeta(\xi) = \xi + i\eta M(\sqrt{1+|\xi|^2}).$$

Note that $M(t) = \int_0^t m(\lambda)/\lambda\, d\lambda$, where $m(\lambda)$ is the number of $m_p = M_p/M_{p-1} \leq \lambda$. Thus (M.3)$'$ implies that $d\zeta_1 \wedge \ldots \wedge d\zeta_n = F(\xi)d\xi_1 \wedge \ldots \wedge d\xi_n$ where $F(\xi) \to 1$ as $\xi \to \infty$, and we have $|\operatorname{Im}\zeta| < |\eta| M(|\zeta|)$ on Γ_η. Then from Cauchy's integral formula we obtain

$$u * \varphi_\delta(x) = \frac{1}{(2\pi)^n} \int_{\Gamma_\eta} e^{i\langle x,\zeta\rangle} \hat{u}(\zeta)\hat{\varphi}_\delta(\zeta)\, d\zeta_1 \wedge \ldots \wedge d\zeta_n. \tag{2.4}$$

Since M_p satisfies (M.3)$'$ we have for $\zeta = \xi + i\eta M(\sqrt{1+|\xi|^2}) \in \Gamma_\eta$ and each $L > 0$

$$|e^{i\langle x,\zeta\rangle}\, \hat{u}(\zeta)| \leq C_\eta \exp\{M(L|\zeta|) + M(\sqrt{1+|\xi|^2})(H_K(\eta) - \langle x,\eta\rangle)\}.$$

If $x_0 \notin K$ we can choose η so that $H_K(\eta) - \langle x,\eta\rangle < -1$ for all x in a neighborhood X_0 of x_0. If we replace η by $t\eta$ the integral (2.4) absolutely converges for each $x \in X_0$ even if the decreasing factor $\hat{\varphi}_\delta$ is omitted. Since $\hat{\varphi}_\delta(\zeta) = \hat{\varphi}(\delta\zeta) \to 1$ as $\delta \to 0$ the restriction of u to X_0 is the ultradifferentiable function

$$u(x) = (2\pi)^{-n} \int_{\Gamma_\eta} e^{i\langle x,\zeta\rangle} \hat{u}(\zeta)\, d\zeta_1 \wedge \ldots \wedge d\zeta_n, \quad x \in X_0,$$

which completes the proof. □

3. Real version of Paley-Wiener-Schwartz theorem

Let K be convex, compact and symmetric with respect to the hyperplanes of coordinates in $\mathbb{R}^n$ with supporting function $H_K(\xi) = \sup_{x\in K}\langle x, \xi\rangle$, where K is said to be *symmetric* with respect to the hyperplanes of coordinates in $\mathbb{R}^n$ if for every $x = (x_1, \ldots, x_n) \in K$, $(\pm x_1, \ldots, \pm x_n) \in K$ for all combinations of signs.

We first introduce the real version of the Paley-Wiener-Schwartz theorem for ultradistributions, that is, the estimate (2.1) can be replaced by estimates only on the real variables, which is the extension of the results of N. Mandache [7] to the space of ultradistributions of Roumieu (or Beurling) class.

THEOREM 3.1 [2]. *Suppose that M_p satisfies* (M.1) *and* (M.2) (resp. (M.2)′). *Then the following conditions are equivalent for an infinitely differentiable function $\hat{u}(\xi)$ on $\mathbb{R}^n$.*

(i) *$\hat{u}(\xi)$ is the Fourier-Laplace transform of an ultradistribution* $u \in \mathcal{E}'_{\{M_p\}}(K)$ (resp. $\mathcal{E}'_{(M_p)}(K)$).

(ii) *For every $\epsilon > 0$ and for every $L > 0$ there exists a constants* $C > 0$ (resp. *there are positive constants L and C*) *such that for each $\alpha \in \mathbb{N}_0^n$*

$$|\partial^\alpha \hat{u}(\xi)| \le C B_{K,\epsilon,\alpha} \exp M(L|\xi|), \quad \xi \in \mathbb{R}^n$$

where $B_{K,\epsilon,\alpha} = \inf_{\rho\in\mathbb{R}^n_+} \alpha! \exp(H_K(\rho) + \epsilon|\rho|)/\rho^\alpha$.

Note that from the definition of the associated function $M(t)$ of each sequence M_p of positive numbers we have $M(et) = \sup_p \sum_{k=1}^{p} (1 +$

$\log t/m_k) \geq m(t)$, where $m(t)$ denotes the number of $m_p = M_p/M_{p-1}$ less than or equal to t and the condition (M.2) is equivalent to $m_{k+1} \leq HM_k^{1/k}$. We refer to W. Matsumoto [8] for more details.

THEOREM 3.2 [2]. *Suppose that M_p satisfies* (M.1), (M.2)′ *and* (M.3)′. *Then the following conditions are equivalent for an infinitely differentiable function $\hat{f}(\xi)$ on $\mathbb{R}^n$.*

(i) *$\hat{f}(\xi)$ is the Fourier-Laplace transform of an ultradifferentiable function of Roumieu type* (resp. *of Beurling type*) *with support contained in K.*

(ii) *For every $\epsilon > 0$ there exist positive constants $L > 0$ and $C > 0$* (resp. *for every $L > 0$ there exists a constant $C > 0$*) *such that*

$$|\partial^\alpha \hat{f}(\xi)| \leq C B_{K,\epsilon,\alpha} \exp(-M(L|\xi|)),$$

where $B_{K,\epsilon,\alpha} = \inf_{\rho \in \mathbb{R}^n_+} \alpha! \exp(H_K(\rho) + \epsilon|\rho|)/\rho^\alpha$.

Note as in [7] that for $C_{K,\epsilon,\alpha} = \sup_{x \in K_\epsilon} |x^\alpha|$ we have

$$C_{K,\epsilon+\epsilon',\alpha+\beta} \geq \left(\frac{\epsilon'}{\sqrt{n}}\right)^{|\beta|} C_{K,\epsilon,\alpha}. \tag{3.1}$$

In particular, for a ball $K = B(0,R)$ centered at the origin with radius $R > 0$ we obtain for each $\epsilon > 0$ and multi-index α

$$B_{K,\epsilon,\alpha} = \sqrt{\frac{|\alpha|^{|\alpha|}}{\alpha^\alpha}} \frac{\alpha!\, e^{|\alpha|}}{|\alpha|^{|\alpha|}} (R+\epsilon)^{|\alpha|} \simeq C_{K,\epsilon,\alpha} \tag{3.2}$$

by the Lagrange multiplier and Stirling's formula.

We introduce the dual space $(\mathcal{S}_{M_p}^{M_p})'$ of the Gelfand-Shilov space of general type S. The space $\mathcal{S}_{M_p}^{M_p}$ consists of all infinitely differentiable functions φ on $\mathbb{R}^n$ satisfying the following: There exist positive constants a, B and C such that

$$\sup_{x \in \mathbb{R}^n} \left| \partial^\alpha \varphi(x) \right| \exp M(a|x|) \leq C\, B^{|\alpha|} N_{|\alpha|}, \quad \alpha \in \mathbb{N}_0^n$$

and we denote by $(\mathcal{S}_{M_p}^{M_p})'$ the strong dual of the space $\mathcal{S}_{M_p}^{M_p}$. Then the Fourier transform is well-defined on this space and, moreover, is an isomorphism of the space $(\mathcal{S}_{M_p}^{M_p})'$. We refer to I. M. Gelfand and G. E. Shilov [4] for more details.

We are now in a position to state and prove the real version of the Paley-Wiener-Schwartz theorem for the space of ultradistributions with respect to ultradifferentiable singular support, which is the extension of Theorem 1.2.

THEOREM 3.3. *Suppose that M_p satisfies* (M.1), (M.2) (resp. (M.2)′) *and* (M.3)′. *Let $u \in \mathcal{E}'_{\{M_p\}}(\mathbb{R}^n)$* (resp. $\mathcal{E}'_{(M_p)}(\mathbb{R}^n)$). *Let K be a ball $B(0,R)$ centered at the origin with radius $R > 0$ in $\mathbb{R}^n$.*

In order that $\operatorname{sing}_M \operatorname{supp} u \subset K$, *it is necessary and sufficient that for every $\epsilon > 0$ and for every $L > 0$ there exists a constant $C_m > 0$* (resp. *there exist positive constants L and C_m*) *for each $m = 1, 2, \ldots$ such that*

$$|\partial^\alpha \hat{u}(\xi)| \le C_m \sqrt{\frac{\alpha^\alpha}{|\alpha|^{|\alpha|}}}\,(R+\epsilon)^{|\alpha|} \exp M(L|\xi|) \tag{3.3}$$

where $|\alpha| \le m\, M(|\xi|)$.

Proof. Let $u \in \mathcal{E}'_{\{M_p\}}$ with $\operatorname{sing}_M \operatorname{supp} u \subset K$. Choose an ultradifferentiable function $\varphi \in \mathcal{E}_{\{M_p\}}(K_{1/m})$ such that $\operatorname{supp}\varphi \subset K_{1/m} = K + B(0, 1/m)$ and $\varphi = 1$ on a neighborhood of K. Then we have for every ϵ, $a > 0$ and $m = 1, 2, \ldots$

$$|(\widehat{\varphi u})(\zeta)| \le C \sqrt{\frac{\alpha^\alpha}{|\alpha|^{|\alpha|}}}\,(R+\epsilon)^{|\alpha|} \exp M(a|\xi|) \tag{3.4}$$

Since $(1-\varphi)u$ is an ultradifferentiable function with compact support we obtain from (3.2) for $\operatorname{supp} u \in B(0, A)$ and for every $a' > 0$

$$|(\widehat{(1-\varphi)u})(\zeta)| \le C \sqrt{\frac{\alpha^\alpha}{|\alpha|^{|\alpha|}}} \exp\{m \log(A+\epsilon) M(|\xi|) - M(a'|\xi|)\},$$

where $|\alpha| \le m\,M(|\xi|)$. Therefore, combining (3.4) and this above formula with sufficiently large $a > 0$ we obtain the wanted estimate (3.3).

To prove the converse we choose an ultradifferentiable function $\varphi \in \mathcal{E}_{(M_p)}(B(0,e^2))$ of Beurling type such that $\operatorname{supp}\varphi \subset B(0,e^2)$ and $\varphi = 1$ on a neighborhood of the ball $B(0,e)$ and let $\varphi_k(x) = \varphi(x/m_k)$, $\varphi_0(x) = \varphi(x)$ and $\psi_k(x) = \varphi_{k+1}(x) - \varphi_k(x)$, where $m_k = M_k/M_{k-1}$ for $k = 1, 2, \dots$.

Let $x_0 \in \mathbb{R}^n \backslash K$. Then there exists a multi-index $\alpha \in \mathbb{N}_0^n$ so that $|x_0^\alpha| > C_{K,0,\alpha}$. In fact, it is equivalent to

$$|x_0^\alpha| > \sqrt{\frac{\alpha^\alpha}{|\alpha|^{|\alpha|}}}\, R^{|\alpha|}\,; \quad \frac{\alpha^\alpha}{|\alpha|^{|\alpha|}} < b_1^{\alpha_1} \cdot \ldots \cdot b_n^{\alpha_n}$$

for some $b_1 + \ldots + b_n > 1$, $b_i \ge 0$, $i = 1, \dots, n$, which follows if we take rational numbers $r_1, \dots, r_n$ such that

$$r_1 + \ldots + r_n = 1, \quad 0 \le r_i < b_i, \quad i = 1, \dots, n.$$

Hence we can choose $\alpha \in \mathbb{N}_0^n$ and $\epsilon > 0$ so that $|x^\alpha| > (1+\epsilon)\sqrt{\alpha^\alpha/|\alpha|^{|\alpha|}}\,(R+\epsilon)^{|\alpha|}$ for any $x \in B(x_0, \epsilon)$. Also, since each $\psi_k(\xi)\hat{u}(\xi)$ has compact support, $f_n(\xi) = \varphi(\xi)\hat{u}(\xi) + \sum_{k=0}^{n} \psi_k(\xi)\hat{u}(\xi)$ is contained in $\big(\mathcal{S}_{M_p}^{M_p}\big)'$. Then $f_n(\xi)$ converges to $\hat{u}(\xi)$ in $\big(\mathcal{S}_{M_p}^{M_p}\big)'$ and $\mathcal{F}^{-1}\big(\psi_k(\xi)\hat{u}(\xi)\big)$ is entire analytic where $\mathcal{F}^{-1}$ denotes the Fourier inversion.

For $x_0 \in \mathbb{R}^n \backslash K$, it suffices to show that there exists a constant $0 < \rho < 1$ such that for every k, β and for some $C > 0$

$$\text{(3.5)} \qquad \big|\partial^\beta \mathcal{F}^{-1}\big(\psi_k(\xi)\hat{u}(\xi)\big)(x)\big| \le C C^{|\beta|} M_{|\beta|} \rho^k \quad \text{for } x \in B(x_0, \epsilon).$$

In fact, since $\operatorname{supp}\psi_k \subset \{\xi \in \mathbb{R}^n \mid e\,m_k \le |\xi| \le e^2 m_{k+1}\}$ we obtain from (3.1), (3.2) and (3.3) that for each $\delta > 0$

$$\begin{aligned}
\big|\partial^\beta \mathcal{F}^{-1}\big(\psi_k(\xi)\hat{u}(\xi)\big)(x)\big| &\le |x^\alpha|^{-rk} \int_{\mathbb{R}^n} \Big|\partial^{rk\alpha}\big(\psi_k(\xi)\hat{u}(\xi)\xi^\beta\big)\Big|\, d\xi \\
&\le C\,|x^\alpha|^{-rk}(1+e^2 m_{k+1})^{|\beta|} \sum_{\gamma \le rk\alpha} \binom{rk\alpha}{\gamma} \Big(\frac{1}{m_k}\Big)^{|\gamma|} h^{|\gamma|} M_{|\gamma|} \\
&\qquad \times \sqrt{\frac{\alpha^\alpha}{|\alpha|^{|\alpha|}}}\,(R+\epsilon)^{|\alpha|} \exp M(Le^2 m_{k+1})(e^2 m_{k+1})^n
\end{aligned}$$

$$\le C'\Big(\frac{1}{1+\epsilon}\Big)^{rk}\Big(\frac{2\sqrt{n}}{\epsilon}\Big)^{|\beta|}\exp((L'+\delta)k)$$
$$\times\sum_{\gamma\le rk\alpha}\binom{rk\alpha}{\gamma}\Big(\frac{h\sqrt{n}\,H^{r|\alpha|+1}}{\epsilon\, e}\Big)^{|\gamma|}\Big(\frac{M_k}{m_k^k}\Big)^{r|\alpha|}M_{|\beta|}$$
$$\le C''\Big(\frac{2\sqrt{n}}{\epsilon}\Big)^{|\beta|}\Big(e^{L'+\delta}\Big(\frac{(1+h\sqrt{n}\,H^{r|\alpha|+1}/\epsilon)^{|\alpha|}}{1+\epsilon}\Big)^{r}\Big)^{k}M_{|\beta|},$$

where the second inequality is obtained from the inequality (3.3) since the integral is taken over the support of ψ_k and then $M(|\xi|)\ge M(e\,m_k)$ $\ge m(m_k)=k$ for any $\xi\in\operatorname{supp}\psi_k$, where $m(t)$ denotes the number of $m_p=M_p/M_{p-1},\ p=1,2,\dots$, less than or equal to t. Therefore we obtain the required inequality (3.5) if we choose a suitable $h>0$ for sufficiently large r and k. □

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References

[1] G. Björck, *Linear partial differential operators and generalized distributions*, Ark. Mat. **6** (1966), 351–407.

[2] J. Chung, S.-Y. Chung and D. Kim, *Real version of Paley-Wiener theorem for ultradistributions and hyperfunctions*, Preprint.

[3] S.-Y. Chung, D. Kim and S. K. Kim, *Equivalence of the spaces of the ultradifferentiable functions and its applications to Whitney extension theorem*, Rendiconti di Matematica, Serie VII **12** (1992), 365–380.

[4] I. M. Gelfand and G. E. Shilov, *Generalized functions*, vol. II, Academic Press, New York, 1968.

[5] L. Hörmander, *The analysis of linear partial differential operators I*, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1983.

[6] H. Komatsu, *Ultradistributions* I, *Structure theorem and a characterization*, J. Fac. Sci. Tokyo, Sec IA **20** (1973), 25–105.

[7] N. Mandache, *A Paley-Wiener theorem and pseudolocal operators*, Rev. Roumaine Math. Pures Appl. **35** (1990), 321-328.

[8] W. Matsumoto, *Theory of pseudo-differential operators of ultradifferentiable class*, J. Math. Kyoto Univ. **27-3** (1987), 453–500.

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