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SOME RESULTS ON THE COMMUTATIVE PRODUCT OF DISTRIBUTIONS

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ABSTRACT. The commutative product of the distributions $x^r \ln x$ and x^{-r-1} is evaluated for $r = 0, 1, 2, \dots$. The commutative product of the distributions $x^r \ln(x + i0)$ and $(x + i0)^{-r-1}$ is also evaluated for $r = 1, 2, \dots$. Further products are deduced.

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. The distributions x^{-r}, x_+^{-r} and x_-^{-r} are defined by the equations

$$x^{-r} = \frac{(-1)^{r-1}(\ln|x|)^{(r)}}{(r-1)!}, \qquad x_+^{-r} = \frac{(-1)^{r-1}(\ln x_+)^{(r)}}{(r-1)!}, \qquad x_-^{-r} = -\frac{(\ln x_-)^{(r)}}{(r-1)!}$$

for $r = 1, 2, \dots$. Note that Gel'fand and Shilov [5] give a different definition of x_+^{-r} and x_-^{-r}. They define the distribution x_+^{-r}, which we here denote by $F(x_+, -r)$, by the equation

$$\begin{aligned}&\langle F(x_+, -r), \varphi(x)\rangle \\ &= \int_0^\infty x^{-r}\Big[\varphi(x) - \sum_{i=0}^{r-2} \frac{\varphi^{(i)}(0)}{i!}x^i - \frac{\varphi^{(r-1)}(0)}{(r-1)!}H(1-x)x^{r-1}\Big]\,dx\end{aligned}$$

for $r = 1, 2, \dots$, where H denotes Heaviside's function.

It was proved in [3] that

$$x_+^{-r} = F(x_+, -r) + \frac{(-1)^r \phi(r-1)}{(r-1)!}\delta^{(r-1)}(x),$$

where

$$\phi(r) = \begin{cases} \sum_{i=1}^r i^{-1}, & r = 1, 2, \dots, \\ 0, & r = 0. \end{cases}$$

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Since it is easily proved that

$$x^s F(x_+, -r) = F(x_+, -r+s), \quad x^s \delta^{(r)}(x) = \frac{(-1)^s r!}{(r-s)!}\delta^{(r-s)}(x),$$

for $s = 0, 1, \ldots, r$, it follows that

$$x^s x_+^{-r} = x_+^{-r+s} + \frac{(-1)^{r+s}\phi(r-1)}{(r-s-1)!}\delta^{(r-s-1)}(x), \tag{1}$$

for $s = 0, 1, \ldots, r-1$ and $r = 1, 2, \ldots$.

Further, the distributions $x^{-1}\ln|x|$, $x_+^{-1}\ln x_+$ and $x_-^{-1}\ln x_-$ are defined by the equations

$$x^{-1}\ln|x| = \tfrac{1}{2}(\ln^2|x|)', \quad x_+^{-1}\ln x_+ = \tfrac{1}{2}(\ln^2 x_+)', \quad x_-^{-1}\ln x_- = -\tfrac{1}{2}(\ln^2 x_-)'.$$

Now let $\rho(x)$ be a function in $\mathcal{D}$ having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\displaystyle\int_{-1}^{1} \rho(x)\,dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

If now f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t)\rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

A first extension of the product of a distribution and an infinitely differentiable function is the following, see for example [2].

Definition 1. Let f and g be distributions in $\mathcal{D}'$ for which on the interval (a, b), f is the k-th derivative of a locally summable function F in $L^p(a, b)$ and $g^{(k)}$ is a locally summable function in $L^q(a, b)$ with $1/p + 1/q = 1$. Then the product $fg = gf$ of f and g is defined on the interval (a, b) by

$$fg = \sum_{i=0}^{k} \binom{k}{i} (-1)^i [F g^{(i)}]^{(k-i)}.$$

The following definition was suggested by Mikusiński [6].

DEFINITION 2. Let f_1, f_2, g_1 and g_2 be distributions in $\mathcal{D}'$ and suppose that the products $f_1.g_1$ and $f_2.g_2$ do not exist. Let $f_{1n}(x) = (f_1 * \delta_n)(x)$, $f_{2n}(x) = (f_2 * \delta_n)(x)$, $g_{1n}(x) = (g_1 * \delta_n)(x)$, $g_{2n}(x) = (g_2 * \delta_n)(x)$. We say that the sum $f_1.g_1 + f_2.g_2$ exists as a single entity and is equal to the distribution h on the interval (a,b) if

$$\mathrm{N}-\lim_{n\to\infty}\langle f_{1n}(x)g_{1n}(x) + f_{2n}(x)g_{2n}(x), \varphi(x)\rangle = \langle h(x), \varphi(x)\rangle$$

for all functions φ in $\mathcal{D}$ with support contained in the interval (a,b).

We first of all prove the following theorem.

THEOREM 1. *Let f and g be distributions and suppose that the products $f.g$ and $f.g'$ exist. Then the product $f'.g$ exists and*

$$f'.g = (f.g)' - f.g'.$$

Proof. Since f_n and g_n are infinitely differentiable functions, we have

$$f_n' g_n = (f_n g_n)' - f_n g_n'$$

and so for arbitrary function φ in $\mathcal{D}$

$$\langle f_n'.g_n, \varphi\rangle = \langle (f_n.g_n)' - f_n.g_n', \varphi\rangle.$$

It follows that

$$\begin{aligned}\lim_{n\to\infty}\langle f_n'.g_n, \varphi\rangle &= \lim_{n\to\infty}\langle (f_n.g_n)'\varphi\rangle - \lim_{n\to\infty}\langle f_n.g_n', \varphi\rangle \\ &= \langle (f.g)', \varphi\rangle - \langle f.g', \varphi\rangle\end{aligned}$$

and the result of the theorem follows.

We now prove our main theorem.

THEOREM 2. *The product $(x^r \ln|x|).x^{-r-1}$ exists and*

$$(x^r \ln|x|).x^{-r-1} = x^{-1}\ln|x| \tag{2}$$

for $r = 0, 1, 2, \ldots$.

Proof. We put

$$(\ln|x|)_n = \ln|x| * \delta_n(x) = \int_{-1/n}^{1/n} \ln|x-t|\delta_n(t)\,dt$$

and

$$(x^{-1})_n = x^{-1} * \delta_n(x) = \int_{-1/n}^{1/n} \ln|x-t|\delta_n'(t)\,dt.$$

Since $\ln|x|$ and $\ln^2|x|$ are locally summable functions, it follows that

$$\lim_{n\to\infty} (\ln|x|)_n^2 = \ln^2|x|.$$

Thus, for arbitrary function φ in $\mathcal{D}$, we have

$$\begin{aligned}\lim_{n\to\infty}\langle[(\ln|x|)_n^2]', \varphi(x)\rangle &= 2\lim_{n\to\infty}\langle(\ln|x|)_n(x^{-1})_n, \varphi(x)\rangle \\ &= \langle(\ln^2|x|)', \varphi(x)\rangle \\ &= 2\langle x^{-1}\ln|x|, \varphi(x)\rangle\end{aligned}$$

and equation (2) follows for the case $r=0$.

Now suppose that equation (2) holds for some non-negative integer r. With this r, the product $(x^{r+1}\ln|x|)x^{-r-1}$ clearly exists by Definition 1 and

$$(x^{r+1}\ln|x|)x^{-r-1} = \ln|x|.$$

Further, since $x^r x^{-r-1} = x^{-1}$, it follows from our assumption that

$$[(r+1)(x^r\ln|x|+x^r].x^{-r-1} = (r+1)x^{-1}\ln|x|+x^{-1} = [(x^{r+1}\ln|x|)]'.x^{-r-1}.$$

It now follows from Theorem 1 that $(x^{r+1}\ln|x|).(x^{-r-1})'$ exists and

$$(x^{r+1}\ln|x|).(x^{-r-1})' = [(x^{r+1}\ln|x|)x^{-r-1}]' - [(x^{r+1}\ln|x|)]'.x^{-r-1}$$

or equivalently

$$-(r+1)(x^{r+1}\ln|x|).x^{-r-2} = x^{-1} - [(r+1)x^{-1}\ln|x| + x^{-1}]$$

and so

$$(x^{r+1}\ln|x|).x^{-r-2} = x^{-1}\ln|x|.$$

Equation (2) now follows by induction for $r = 0, 1, 2, \ldots$.

For our next theorem we need the following distributions.

$$\begin{aligned}\ln^2(x+i0) &= \ln^2|x| + 2i\pi\ln x_- - \pi^2 H(-x), \\ (x+i0)^r\ln(x+i0) &= x^r\ln(x+i0) = x^r\ln|x| + i\pi(-1)^r x_-^r\end{aligned}$$

for $r = 0, 1, 2, \dots$ and

$$
\begin{aligned}
(x+i0)^{-r} &= x^{-r} + \frac{i\pi(-1)^r}{(r-1)!}\delta^{(r-1)}(x), \\
(x+i0)^{-r}\ln(x+i0) &= x^{-r}\ln|x| + (-1)^r i\pi x_-^{-r} - \frac{\pi^2(-1)^r}{2(r-1)!}\delta^{(r-1)}(x)
\end{aligned}
$$

for $r = 1, 2, \dots$, see Gel'fand and Shilov [5].

THEOREM 3. *The product* $[x^r\ln(x+i0)].(x+i0)^{-r-1}$ *exists and*

$$
[x^r\ln(x+i0)].(x+i0)^{-r-1} = (x+i0)^{-1}\ln(x+i0). \tag{3}
$$

for $r = 0, 1, 2, \dots$.

Proof. We first of all note that $\ln(x+i0)$ and $\ln^2(x+i0)$ are locally summable functions and that

$$
[\ln(x+i0)]^2 = \ln^2(x+i0).
$$

It follows as above that we can differentiate this equation to give

$$
\ln(x+i0).(x+i0)^{-1} = (x+i0)^{-1}\ln(x+i0),
$$

proving equation (3) when $r = 0$.

Assume equation (3) holds for some r. The product $[x^{r+1}\ln(x+i0)](x+i0)^{-r-1}$ exists by Definition 1 and

$$
[x^{r+1}\ln(x+i0)](x+i0)^{-r-1} = \ln(x+i0).
$$

Since

$$
[\ln(x+i0)]' = (x+i0)^{-1}, \qquad [(x+i0)^{-r-1}]' = -(r+1)(x+i0)^{-r-2},
$$
$$
[x^{r+1}\ln(x+i0)]' = (r+1)x^r\ln(x+i0) + x^r,
$$

it follows by induction, as above, that the product $[x^r\ln(x+i0)].(x+i0)^{-r-1}$ exists and satisfies equation (3).

COROLLARY 3.1 *The products* $x_-^r.\delta^{(r)}(x)$ *and* $x_+^r.\delta^{(r)}(x)$ *exist and*

$$
x_-^r.\delta^{(r)}(x) = \tfrac{1}{2}r!\delta(x), \tag{4}
$$

$$
x_+^r.\delta^{(r)}(x) = \tfrac{1}{2}(-1)^r\delta(x), \tag{5}
$$

$$
x_-^r.x^{-r-1} - \frac{1}{r!}(x^r\ln|x|).\delta^{(r)}(x) = (-1)^{r-1}x_-^{-1}, \tag{6}
$$

$$
x_+^r.x^{-r-1} + \frac{(-1)^r}{r!}(x^r\ln|x|).\delta^{(r)}(x) = x_+^{-1} \tag{7}
$$

for $r = 0, 1, 2, \ldots$.

Proof. It follows from equation (3) that

$$[x^r \ln|x| + i\pi(-1)^r x_-^r].\left[x^{-r-1} - \frac{i\pi(-1)^r}{r!}\delta^{(r)}(x)\right] = x^{-1}\ln|x| - i\pi x_-^{-1} + \tfrac{1}{2}\pi^2\delta(x). \tag{8}$$

Expanding the left hand side of this equation, remembering that $(x^r \ln|x|).x^{-r-1} = x^{-1}\ln|x|)$ and equating the real parts, equation (4) follows. Replacing x by $-x$ in equation (4) gives equation (5).

Equating the imaginary parts in equation (8) gives equation (6). Replacing x by $-x$ in equation (6) gives equation (7). Note that the left hand sides of equations (6) and (7) only exist as single entities, the individual products not existing.

Equations (4) and (5) were originally given in [2].

THEOREM 4. *The products* $[x_+^r \ln x_+ - \phi(r)x_+^r].x_+^{-r-1}$, $[x_-^r \ln x_- - \phi(r)x_-^r].\, x_-^{-r-1}$, $[x_+^r \ln x_+ + (-1)^r\phi(r)x_-^r].x_+^{-r-1}$ *and* $[x_-^r \ln x_- + (-1)^r\phi(r)x_+^r].\, x_-^{-r-1}$ *exist and*

$$[x_+^r \ln x_+ - \phi(r)x_+^r].x_+^{-r-1} = x_+^{-1}\ln x_+ - \phi(r)x_+^{-1} + \sum_{i=1}^{r}\frac{\phi(i)}{i}\delta(x), \tag{9}$$

$$[x_-^r \ln x_- - \phi(r)x_-^r].x_-^{-r-1} = x_-^{-1}\ln x_- - \phi(r)x_-^{-1} + \sum_{i=1}^{r}\frac{\phi(i)}{i}\delta(x), \tag{10}$$

$$[x_+^r \ln x_+ + (-1)^r\phi(r)x_-^r].x_+^{-r-1} = x_-^{-1}\ln x_- + \left[\sum_{i=1}^{r}\frac{\phi(i)}{i} + \phi^2(r)\right]\delta(x), \tag{11}$$

$$[x_-^r \ln x_- + (-1)^r\phi(r)x_+^r].x_-^{-r-1} = x_-^{-1}\ln x_- + \left[\sum_{i=1}^{r}\frac{\phi(i)}{i} + \phi^2(r)\right]\delta(x) \tag{12}$$

for $r = 0, 1, 2, \ldots$, *the sums being empty when* $r = 0$.

Proof. Since $\ln x_+$ and $\ln^2 x_+$ are locally summable functions and $(\ln x_+)^2 = \ln^2 x_+$, it follows as above that

$$\ln x_+.x_+^{-1} = x_+^{-1}\ln x_+,$$

proving equation (9) when $r = 0$.

Now assume that equation (9) holds for some positive integer r. The product $[x_+^{r+1}\ln x_+ - \phi(r+1)x_+^{r+1}]x_+^{-r-1}$ exists by Definition 1 and

$$[x_+^{r+1}\ln x_+ - \phi(r+1)x_+^{r+1}]x_+^{-r-1} = \ln x_+ - \phi(r+1)H(x).$$

Formal differentiation of this equation gives

$$(r+1)[x_+^r\ln x_+ - \phi(r)x_+^r].x_+^{-r-1} - (r+1)[x_+^{r+1}\ln x_+ - \phi(r+1)x_+^{r+1}].x_+^{-r-2}$$
$$= x_+^{-1} - \phi(r+1)\delta(x).$$

Since the product $[x_+^r\ln x_+ - \phi(r)x_+^r].x_+^{-r-1}$ exists by our assumption, it follows from Theorem 1 that the product $[x_+^{r+1}\ln x_+ - \phi(r+1)x_+^{r+1}].x_+^{-r-2}$ exists and

$$\begin{aligned}
&[x_+^{r+1}\ln x_+ - \phi(r+1)x_+^{r+1}].x_+^{-r-2} \\
&= [x_+^r\ln x_+ - \phi(r)x_+^r].x_+^{-r-1} + -(r+1)^{-1}[x_+^{-1} - \phi(r+1)]\delta(x) \\
&= x_+^{-1}\ln x_+ - \phi(r)x_+^{-1} + \sum_{i=1}^{r}\frac{\phi(i)}{i}\delta(x) - (r+1)^{-1}[x_+^{-1} - \phi(r+1)]\delta(x) \\
&= x_+^{-1}\ln x_+ - \phi(r+1)x_+^{-1} + \sum_{i=1}^{r+1}\frac{\phi(i)}{i}\delta(x)
\end{aligned}$$

and equation (9) follows by induction.

Replacing x by $-x$ in equation (9) gives equation (10).

To prove equation (11), we note that

$$x_+^r = x^r - (-1)^r x_-^r$$

and

$$x^r x_+^{-r-1} = x_+^{-1} - \phi(r)\delta(x)$$

from equation (1). Equation (11) now follows from equation (9). Equation (12) follows from equation (11) on replacing x by $-x$.

References

[1] J. G. van der Corput, *Introduction to the neutrix calculus*, J. Analyse Math., **7** (1959-60), 291-398.

[2] B. Fisher, *The product of distributions*, Quart. J. Math. Oxford, **22** (1971), no. 2, 291-298.

[3] ———, *Some notes on distributions*, Math. Student, **48**(1980), 269-281.

[4] A. Kılıçman and B. Fisher, *Some results on the commutative neutrix product of distributions*, Serdica, **20** (1994), 257-268.

[5] I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, Vol. I, Academic Press, 1964.

[6] J. Mikusiński, *On the square of the Dirac delta-distribution*, Bull. Acad. Polon. Sci., sér. sci. math. astr. phys., **14** (1966), 511-513.

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