A NEW CHARACTERIZATION OF RULED REAL HYPERSURFACES IN COMPLEX SPACE FORMS

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ABSTRACT. The purpose of this paper is to give another new characterization of ruled real hypersurfaces in a complex space form $M_n(c)$, $c \neq 0$ in terms of the covariant derivative of its Weingarten map in the direction of the structure vector $\xi$.

1. Introduction

A complex $n(\geq 2)$-dimensional Kaehlerian manifold of constant holomorphic sectional curvature $c$ is called a complex space form, which is denoted by $M_n(c)$. A complete and simply connected complex space form is a complex projective space $P_nC$, a complex Euclidean space $C^n$ or a complex hyperbolic space $H_nC$, according as $c > 0$, $c = 0$ or $c < 0$.

Now, there exist many studies about real hypersurfaces of $M_n(c)$. One of the first researches is the classification of homogeneous real hypersurfaces of a complex projective space $P_nC$ by Takagi [13], who showed that these hypersurfaces of $P_nC$ could be divided into six types which are said to be of type $A_1, A_2, B, C, D,$ and $E$, and in [3] Cecil-Ryan and [5] Kimura proved that they were realized as the tubes of constant radius over compact Hermitian symmetric spaces of rank 1 or rank 2 if the structure vector field $\xi$ is principal. Also Berndt [2] showed recently that all real hypersurfaces with constant principal curvatures of a complex hyperbolic space $H_nC$ are realized as the tubes.

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of constant radius over certain submanifolds if $\xi$ is principal. According to Takagi's classification theorem and Berndt's one, the principal curvatures and their multiplicities of homogeneous real hypersurfaces of $M_n(c)$ are given.

As an example of special real hypersurfaces of $P_nC$ different from the above ones, firstly Kimura [6] introduced the notion of ruled real hypersurfaces in $P_nC$, which is not complete and not principal. Also Kimura [6] obtained some properties about a ruled real hypersurface $M$ in $P_nC$, $n \geq 3$. In particular, an example of minimal ruled hypersurfaces of $P_nC$ was constructed. Let $T_0$ be a distribution defined by a subspace $T_0(x) = \{u \in T_xM : u \perp \xi(x)\}$ of the tangent space $T_x(M)$, which is called the holomorphic distribution. The following was proved by Kimura and Maeda [7].

**THEOREM A.** Let $M$ be a real hypersurface of $P_nC$, $n \geq 3$. Then the second fundamental form is $\eta$-parallel and the holomorphic distribution $T_0$ is integrable if and only if $M$ is locally congruent to a ruled real hypersurface.

In [1] we also introduced the notion of ruled real hypersurfaces in a complex hyperbolic space $H_nC$ and constructed an example of minimal ruled real hypersurfaces of $H_nC$ by using the submersion compatible with the fibration $\pi : H_{2n+1}^1 \to H_nC$. From this, together with Kimura's one, in the paper [12] the third author has given a characterization of ruled real hypersurfaces $M$ in $M_n(c)$ in such a way that its shape operator $A$ satisfies

\[(\nabla_XA)Y = f(X,Y)\xi, \quad X,Y \in T_0,\]

where we put

\[f(X,Y) = \beta^2\{g(X,\phi U)g(Y,\phi U) + g(X,\phi U)g(Y,U)\} - \frac{c}{4}g(\phi X,Y)\]

for any vector field $X$ and $Y$ in the distribution $T_0$ except for the case where the function $\beta$ identically vanishes. Moreover, this expression of the covariant derivative of the shape operator $A$ will be shown concretely in section 3.
Ruled real hypersurfaces

Now the purpose of this paper is to give another new characterization of ruled real hypersurfaces in complex space forms $M_n(c)$ as the covariant derivative of the shape operator $A$ along the direction of $\xi$. Namely, we assert the following

THEOREM. Let $M$ be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 3$. If it satisfies

\begin{equation}
(\nabla_\xi A)X = \beta^2 \{g(X, \phi U)U + g(X, U)\phi U\},
\end{equation}

provided that $d\alpha(\xi) \neq 0$ for any vector field $X$ in the distribution $T_0$, where $A$ denotes the shape operator, then $M$ is locally congruent to a ruled real hypersurface in $M_n(c)$, on which its mean curvature $h$ is constant along the distribution $T_0$.

In section 3 some fundamental properties about ruled real hypersurfaces in $M_n(c)$, $c \neq 0$ will be recalled and the covariant derivative of the shape operator $A$ in the direction of the structure vector field $\xi$, which is given in (1.2), will be explicitly expressed. By paying attention to this formula another new characterization of ruled real hypersurfaces in $M_n(c)$ will be given in section 4.

2. Preliminaries

First of all, we recall basic properties of real hypersurfaces of a complex space form. Let $M$ be a real hypersurface of $n(\geq 2)$-dimensional complex space form $M_n(c)$ of constant holomorphic sectional curvature $c(\neq 0)$ and let $C$ be a unit normal field on a neighborhood of a point $x$ in $M$. We denote by $J$ an almost complex structure of $M_n(c)$. For a local vector field $X$ on a neighborhood of $x$ in $M$, the transformation of $X$ and $C$ under $J$ can be represented as

\[ JX = \phi X + \eta(X)C, \quad JC = -\xi, \]

where $\phi$ defines a skew-symmetric transformation on the tangent bundle $TM$ of $M$, while $\eta$ and $\xi$ denote a 1-form and a vector field on a neighborhood of $x$ in $M$, respectively. Moreover, it is seen that $g(\xi, X) = \eta(X)$, where $g$ denotes the induced Riemannian metric on
\[ M. \text{ By properties of the almost complex structure } J, \text{ the set } (\phi, \xi, \eta, g) \]

of tensors satisfies

\[ \phi^2 = -I + \eta \otimes \xi, \quad \phi \xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \]

where \( I \) denotes the identity transformation. Accordingly, the set is so called an \emph{almost contact metric structure}. Furthermore the covariant derivative of the structure tensors are given by

\[ (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX, \quad (2.1) \]

where \( \nabla \) is the Riemannian connection of \( g \) and \( A \) denotes the shape operator with respect to the unit normal \( C \) on \( M \).

Since the ambient space is of constant holomorphic sectional curvature \( c \), the equation of Gauss and Codazzi are respectively given as follows

\[ R(X, Y)Z = \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z) \phi X - g(\phi X, Z) \phi Y - 2g(\phi X, Y) \phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \quad (2.2) \]

\[ (\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X) \phi Y - \eta(Y) \phi X - 2g(\phi X, Y) \xi\}, \quad (2.3) \]

where \( R \) denotes the Riemannian curvature tensor of \( M \) and \( \nabla_X A \) denotes the covariant derivative of the shape operator \( A \) with respect to \( X \).

The second fundamental form is said to be \( \eta \)-parallel if the shape operator \( A \) satisfies \( g((\nabla_X A)Y, Z) = 0 \) for any vector fields \( X, Y \) and \( Z \) in \( T_0 \).

Next we suppose that the structure vector field \( \xi \) is principal with corresponding principal curvature \( \alpha \). Then it is seen in [4] and [8] that \( \alpha \) is constant on \( M \) and it satisfies

\[ A\phi A = \frac{c}{4}\phi + \frac{1}{2}\alpha(\alpha \phi + \phi A), \quad (2.4) \]
3. Ruled Real Hypersurfaces

This section is concerned with necessary properties about ruled real hypersurfaces. First of all, we define a ruled real hypersurface \( M \) of \( M_n(c), \; c \neq 0 \). Let \( \gamma : I \rightarrow M_n(c) \) be any regular curve. For any \( t(\in I) \) let \( M_n(t)_{n-1}(c) \) be a totally geodesic complex hypersurface through the point \( \gamma(t) \) of \( M_n(c) \) which is orthogonal to a holomorphic plane spanned by \( \gamma'(t) \) and \( J\gamma'(t) \). Set \( M = \{ x \in M_n(t)_{n-1}(c) : t \in I \} \). Then the construction of \( M \) asserts that \( M \) is a real hypersurface of \( M_n(c) \). Under this construction that the ruled real hypersurface \( M \) of \( M_n(c), \; c \neq 0 \), has some fundamental properties.

Let us put \( A\xi = \alpha\xi + \beta U \), where \( U \) is a unit vector orthogonal to \( \xi \) and \( \alpha \) and \( \beta(\beta \neq 0) \) are smooth functions on \( M \). As is seen in [7], the shape operator \( A \) satisfies

\[
(3.1) \quad AU = \beta\xi, \quad AX = 0
\]

for any vector field \( X \) orthogonal to \( \xi \) and \( U \). It turns out to be

\[
(3.2) \quad A\phi X = -\beta g(X, \phi U)\xi, \quad \phi AX = 0, \; X \in T_0,
\]

which implies that

\[
(3.3) \quad g((A\phi - \phi A)X, Y) = 0, X, Y \in T_0.
\]

Because of

\[
(\mathcal{L}_\xi g)(X, Y) = \mathcal{L}_\xi(g(X, Y)) - g(\mathcal{L}_\xi X, Y) - g(X, \mathcal{L}_\xi Y)
\]

\[
= g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi),
\]

the above equation is equivalent to

\[
(3.4) \quad (\mathcal{L}_\xi g)(X, Y) = 0, X, Y \in T_0.
\]

Next the covariant derivative \( \nabla_X A \) with respect to \( X \) in \( T_0 \) is explicitly expressed. It is seen in [6] and [7] that the second fundamental form is \( \eta \)-parallel. Also the equation (2.3) of Codazzi gives us to

\[
(\nabla_X A)\xi - (\nabla_\xi A)X = -\frac{c}{4}\phi X.
\]
By the direct calculation of the left hand side of the above relation and using the property \( \nabla_X \xi = \phi AX = 0 \) in (3.2), we get

\[
(3.5) \quad d\alpha(X)\xi + d\beta(X)U + \frac{c}{4}\phi X + \beta \nabla_X U - \nabla_x(AX) + A \nabla_x X = 0,
\]

for any \( X \) in \( T_0 \). Let \( T_1 \) be a distribution defined by a subspace \( T_1(x) = \{ u \in T_0(x) : g(u, U(x)) = g(u, \phi U(x)) = 0 \} \). Since \( AX \) is expressed as the linear combination of \( \xi \) and \( U \) by (3.1), we can derive from (3.1),(3.2) and the above equation the following relations:

\[
(3.6) \quad \beta \nabla_X U = \begin{cases} 
(\beta^2 - \frac{c}{4})\phi X, & X = U; \\
0, & X = \phi U; \\
-\frac{c}{4}\phi X, & X \in T_1,
\end{cases}
\]

\[
(3.7) \quad d\beta(X) = \begin{cases} 
0, & X = U; \\
\beta^2 + \frac{c}{4}, & X = \phi U; \\
0, & X \in T_1.
\end{cases}
\]

Using these relations we can obtain the components of \((\nabla_X A)Y\) in the direction of \( \xi \). In fact, we have

\[
g((\nabla_X A)Y, \xi) = g((\nabla_X A)\xi, Y) = g(\nabla_X (A\xi) - A \nabla_X \xi, Y) \\
= d\beta(X)g(Y, U) + \beta g(\nabla_X U, Y),
\]

which yields combining with the above equation that

\[
(3.8) \quad (\nabla_X A)Y = f(X, Y)\xi, \quad X, Y \in T_0,
\]

where we put

\[
(3.9) \quad f(X, Y) = \beta^2\{g(X, U)g(Y, \phi U) + g(X, \phi U)g(Y, U)\} - \frac{c}{4}g(\phi X, Y).
\]

which means that \( A \) is \( \eta \)-parallel.

Accordingly, by the equation of Codazzi (2.3) and the above equations it can be easily seen that the shape operator of \( M \) satisfies

\[
(3.10) \quad (\nabla_\xi A)X = \beta^2\{g(X, \phi U)U + g(X, U)\phi U\},
\]

when its mean curvature \( h = \alpha \) is constant along the distribution \( T_0 \).
4. Proof of the Theorem

In this section we are only concerned with the proof of Theorem. Let $M$ be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 3$. Throughout this section we assume that the structure vector field $\xi$ is not principal. Then we can put

$$A\xi = \alpha \xi + \beta U,$$

where $U$ is a unit vector in the holomorphic distribution $T_0$ and $\alpha$ and $\beta$ are smooth functions on $M$. We may consider that the function $\beta$ does not vanish identically on $M$. Let $M_0$ be an open set of $M$ consisting of points $x$ at which $\beta(x) \neq 0$. In other words, the subset $M_0$ is not empty. Furthermore we assume that the following condition:

$$(4.1) \quad (\nabla_\xi A)Y = \beta^2 \{g(Y, \phi U)U + g(Y, U)\phi U\}, \quad Y \in T_0.$$

First of all, from (4.1) we derive the relation in which the derivative of the shape operator is not contained.

**Lemma 4.1.** On the subset $M_0$ we have

$$(4.2) \quad d\alpha(\xi)(A\phi + \phi A)X = 2\beta^2 \{g(X, \phi U)A\phi U - g(AX, \phi U)\phi U\} - \beta d\alpha(\xi)g(X, \phi U)\xi$$

for any vector field $X$ in $T_0$.

**Proof.** Under the assumption (4.1) and by the assistance of (2.3) it turns out to be

$$(4.3) \quad (\nabla_Y A)\xi = \beta^2 \{g(Y, \phi U)U + g(Y, U)\phi U\} - \frac{c}{4} \phi Y$$

for any vector field $Y$ in $T_0$. Differentiating this equation with respect to $X$ covariantly and taking account of (2.1), we get

$$\begin{align*}
(\nabla_X \nabla_Y A)\xi + (\nabla_{\nabla_X Y} A)\xi + (\nabla_Y A)\phi AX &= d\beta^2(X)\{g(Y, \phi U)U + g(Y, U)\phi U\} \\
&\quad + \beta^2 \{g(\nabla_X Y, \phi U) + g(Y, \phi \nabla_X U)\}U + g(Y, \phi U)\nabla_X U \\
&\quad + \{g(\nabla_X Y, U) + g(Y, \nabla_X U)\}\phi U \\
&\quad - g(Y, U)g(AX, U)\xi + g(Y, U)\phi \nabla_X U \\
&\quad + \frac{c}{4} \{g(AX, Y)\xi - \phi \nabla_X Y\}
\end{align*}$$

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for any vector fields $X$ and $Y$ in $T_0$. For any vector field $Z$ the orthogonal decomposition in the direction of $\xi$ is expressed as

$$Z = (Z)_0 + g(Z, \xi)\xi,$$

where $(Z)_0$ denotes the $T_0$-component of $Z$. Since the component of the vector $\nabla_XY$ in the direction of $\xi$ is given by $-g(\phi AX, Y)$ by the first equation of (2.1), we have the following orthogonal decomposition

$$\nabla_XY = (\nabla_XY)_0 - g(\phi AX, Y)\xi.$$

Using the above orthogonal decomposition and taking account of (4.3) itself, we get directly

(4.4)

$$\nabla_X(\nabla_Y A)\xi = g(\phi AX, Y)(\nabla_\xi A)\xi - (\nabla_Y A)\phi AX + d\beta^2(X)\{g(Y, \phi U)U + g(Y, U)\phi U\}
+ \beta^2\{g(Y, \phi \nabla X U)U + g(Y, \phi U)\nabla X U + g(Y, \nabla X U)\phi U
- g(Y, U)g(AX, U)\xi + g(Y, U)\phi \nabla X U\} + \frac{c}{4}g(AX, Y)\xi$$

for any vector fields $X$ and $Y$ in $T_0$.

On the other hand, it is well known that the Ricci formula for the shape operator $A$ is given by

$$(\nabla_Y \nabla_X A)Z - (\nabla_X \nabla_Y A)Z = R(X, Y)(AZ) - A(R(X, Y)Z)$$

for any vector fields $X, Y$ and $Z$. Accordingly, putting $Z = \xi$ in the above Ricci formula, taking $X$ and $Y$ in the distribution $T_0$ and taking account of the Gauss equation (2.2) and (4.3) imply

(4.5)

$$g((A\phi + \phi A)X, Y)(\nabla_\xi A)\xi + (\nabla_X A)\phi AY - (\nabla_Y A)\phi AX
= \frac{c}{4}(g(Y, A\xi)X - g(X, A\xi)Y
+ g(\phi Y, A\xi)\phi X - g(\phi X, A\xi)\phi Y - 2g(\phi X, Y)\phi A\xi
- g(Y, A\xi)A^2 X + g(X, A\xi)A^2 Y + g(Y, A^2 \xi)AX - g(X, A^2 \xi)AY)$$

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\[- d\beta^2(X)\{g(Y, \phi U)U + g(Y, U)\phi U\} + d\beta^2(Y)\{g(X, \phi U)U + g(X, U)\phi U\} + \beta^2 [g(X, \phi \nabla_Y U)U + g(X, \phi U)\nabla_Y U + g(X, \nabla_Y U)\phi U + g(X, U)\phi \nabla_Y U - g(Y, \phi \nabla_X U)U - g(Y, \phi U)\nabla_X U - g(Y, \nabla_X U)\phi U - g(Y, U)\phi \nabla_X U + \{g(Y, U)g(AX, U) - g(X, U)g(AY, U)\}e] \]

for any vector fields \(X\) and \(Y\) in \(T_0\).

Now, in order to prove Lemma 4.1, we shall express (4.5) with the simpler form. From now on we shall discuss on the open set \(M_0 = \{x \in M : \beta(x) \neq 0\}\). By the form \(A\xi = \alpha\xi + \beta U\) we have

\[A^2\xi = \alpha^2\xi + \alpha\beta U + \beta AU.\]

Accordingly, by substituting the above equation into the equation (4.5), it can be reformed as

(4.6)

\[g((A\phi + \phi A)X, Y)(\nabla_\xi A)\xi + (\nabla_X A)\phi AY - (\nabla_Y A)\phi AX = \frac{c}{4}\beta\{g(Y, U)X - g(X, U)Y - g(Y, \phi U)\phi X + g(X, \phi U)\phi Y - 2g(\phi X, Y)\phi U\} + \beta \{ - g(Y, U)A^2X + g(X, U)A^2Y + \{\alpha g(Y, U) + g(Y, AU)\}AX - \{\alpha g(X, U) + g(X, AU)\}AY \}

- d\beta^2(X)\{g(Y, \phi U)U + g(Y, U)\phi U\} + d\beta^2(Y)\{g(X, \phi U)U + g(X, U)\phi U\} + \beta^2 [g(X, \phi \nabla_Y U)U + g(X, \phi U)\nabla_Y U + g(X, \nabla_Y U)\phi U + g(X, U)\phi \nabla_Y U - g(Y, \phi \nabla_X U)U - g(Y, \phi U)\nabla_X U - g(Y, \nabla_X U)\phi U - g(Y, U)\phi \nabla_X U + \{g(Y, U)g(AX, U) - g(X, U)g(AY, U)\}e] \]

for any vector fields \(X\) and \(Y\) in \(T_0\).
Next we want to calculate the inner product of (4.6) and $\xi$. For the reason, we differentiate $A\xi = \alpha\xi + \beta U$ with respect to $\xi$ covariantly. Then by (2.1) we have

$$\nabla_\xi A)(\xi) = d\alpha(\xi)\xi + d\beta(\xi)U + \alpha\beta\phi U - \beta A\phi U + \beta \nabla_\xi U.$$  

(4.7)

Since it is easily seen by (2.2) and by the choice of the vector field $U$ that the vectors $A\phi U$ and $\nabla_\xi U$ are both orthogonal to $\xi$, we see

$$g((\nabla_\xi A)(\xi), \xi) = d\alpha(\xi).$$

(4.8)

On the other hand, (4.3) implies

$$g((\nabla X A)\phi Y, \xi)$$

$$= \beta^2\{g(X, U)g(Y, AU) - g(X, \phi U)g(AY, \phi U)\} - \frac{c}{4}g(AX, Y),$$

(4.9)

where the formulas (2.1) and (4.3) have been used. By taking account of these properties the inner product of (4.6) with $\xi$ gives us the similar equation (4.5). Since $Y$ belongs to the distribution $T_0$, we find that (4.2) holds on $M_0$ by the above equation. It completes the proof.  

Now let $L(\xi, U, \phi U)$ be a distribution defined by a subspace $L_x(\xi, U, \phi U)$ in the tangent space $T_x M$ spanned by the vectors $\xi(x), U(x)$ and $\phi U(x)$ at any point $x$ in $M_0$.

**Lemma 4.2.** The subbundle $L(\xi, U, \phi U)$ is $A$-invariant and $\phi$-invariant on $M_0$.

**Proof.** Suppose that there is a vector field $V$ in the holomorphic distribution $T_0$ in such a way that $AU$ is expressed as a linear combination of the vector fields $\xi$, $U$ and $V$, where $U$ and $V$ are orthonormal. Namely, since the shape operator $A$ is symmetric, we may put

$$AU = \beta\xi + \gamma U + \delta V,$$

(4.10)

where $\gamma$ and $\delta$ are smooth functions on $M_0$. Putting $U$ in place of $X$ in (4.2) and using the expression of $AU$, we get

$$d\alpha(\xi)A\phi U = -\{2\beta^2\delta g(\phi U, V) + \gamma d\alpha(\xi)\}\phi U - \delta d\alpha(\xi)\phi V.$$

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Consequently, acting the linear transformation $\phi$ to the above equation, we have

$$d\alpha(\xi)\phi A\phi U = \{2\beta^2 \delta g(\phi U, V) + \gamma da(\xi)\}U + \delta d\alpha(\xi)V.$$  \hfill (4.12)

Putting $X = \phi U$ in (4.2) again and making use of the decomposition of $AU$ and $d\alpha(\xi) \neq 0$, we get

$$d\alpha(\xi)\phi A\phi U = d\alpha(\xi)(\gamma U + \delta V) - 2\beta^2 \delta \phi V,$$

from which together with (4.12) it follows that

$$2\beta^2 \delta \{g(\phi U, V)U + \phi V\} = 0.$$

Let $M_1$ be an open subset $M_0$ consisting of points $x$ at which $\delta(x) \neq 0$. Suppose that $M_1$ is not empty. Without loss of generality, we may put $V = \phi U$ on $M_1$ by the above equation. Thus it implies that $AU$ is contained in the subspace $L(\xi, U, \phi U)$. Furthermore by (4.11) we have

$$d\alpha(\xi)A\phi U = \delta d\alpha(\xi)U - \{2\beta^2 \delta + \gamma da(\xi)\}\phi U$$  \hfill (4.13)

on $M_1$.

On the other hand, by (4.11) we have $d\alpha(\xi)A\phi U = -\gamma da(\xi)\phi U$ on $M_0 - M_1$. Consequently, (4.13) holds on $M_0$. This means that $L(\xi, U, \phi U)$ is $A$-invariant. It is evident that it is $\phi$-invariant. It completes the proof of Lemma 4.2.

Next, we investigate the mutual relations among the functions $\alpha$, $\beta$, $\gamma$ and $\delta$. First we differentiate $AU = \beta\xi + \gamma U + \delta \phi U$ with respect to $\xi$ covariantly. Then taking account of (2.1), we get

$$A\nabla_\xi U = (d\beta(\xi) - \beta \delta)\xi + d\gamma(\xi)U + d\phi(\xi)\phi U + \gamma \nabla_\xi U + \delta \phi \nabla_\xi U.$$  \hfill (4.14)

By the forms $A\xi = \alpha \xi + \beta U$ and $AU = \beta\xi + \gamma U + \delta \phi U$ it is easily seen that the following equations

$$g(A\nabla_\xi U, \xi) = g(\nabla_\xi U, A\xi) = 0,$$
$$g(A\nabla_\xi U, U) = \delta g(\nabla_\xi U, \phi U),$$
$$d\alpha(\xi)g(A\nabla_\xi U, \phi U) = -\{2\beta^2 \delta + \gamma da(\xi)\}g(\nabla_\xi U, \phi U)$$

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are obtained, where we have used (4.13) to derive the last equation. Then we consider the inner product of (4.14) and $\xi$, $U$ and $\phi U$, respectively. Taking account of the above three equations, we have the following mutual relations:

\begin{equation}
\tag{4.15} d\beta(\xi) = \beta \delta, \end{equation}

\begin{equation}
\tag{4.16} d\gamma(\xi) = 2\delta g(\nabla_\xi U, \phi U), \end{equation}

\begin{equation}
\tag{4.17} d\alpha(\xi) d\delta(\xi) = -2(\beta^2 \delta + \gamma d\alpha(\xi)) g(\nabla_\xi U, \phi U). \end{equation}

Now we take here the inner product of (4.7) with $\phi U$. Then the inner product with the left hand side vanishes identically by (4.1) and therefore it implies

\begin{equation}
\tag{4.18} d\alpha(\xi) g(\nabla_\xi U, \phi U) = -2\beta^2 \delta - (\alpha + \gamma) d\alpha(\xi), \end{equation}

where we have used (4.13).

Now let $T_1$ be an orthogonal complement in the tangent bundle $TM$ of the subbundle $L(\xi, U, \phi U)$. Since the distribution $L_\xi(\xi, U, \phi U)$ is $A$-invariant by Lemma 4.2, the orthogonal distribution $T_1$ is also $A$-invariant and moreover it is $\phi$-invariant, too. Accordingly, by (4.2), we have the following.

**Lemma 4.3.** The holomorphic distribution $T_0$ is integrable on $M_0$, namely the equation

\begin{equation}
\tag{4.19} (A\phi + \phi A)X = 0, \quad X \in T_1 \end{equation}

holds on $M_0$.

By differentiating (4.19) with respect to $\xi$ covariantly and combining with (2.1) and (4.1), it implies that

\[(A\phi + \phi A)\nabla_\xi X = 0, \quad X \in T_1,\]
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because $T_1$ is invariant. Thus the inner product of this equation with $\xi$ yields

$$g(\nabla_\xi U, \phi X) = 0.$$  

Since $T_1$ is $\phi$-invariant, we get

$$(4.20) \quad g(\nabla_\xi U, X) = 0, \quad X \in T_1.$$  

Evidently we get

$$(4.21) \quad g(\nabla_\xi U, \xi) = 0, \quad g(\nabla_\xi U, U) = 0.$$  

Then (4.20) and (4.21) imply that

$$(4.22) \quad \nabla_\xi U = \epsilon \phi U,$$

where $\epsilon$ is a smooth function on $M_0$. Accordingly, the equations (4.16), (4.17) and (4.18) can be rewritten as follows:

$$(4.16') \quad d\gamma(\xi) = 2\delta \epsilon,$$

$$(4.17') \quad d\alpha(\xi)d\delta(\xi) = -2\epsilon\{\beta^2 \delta + \gamma d\alpha(\xi)\},$$

$$(4.18') \quad (\alpha + \gamma + \epsilon)d\alpha(\xi) = -2\beta^2 \delta.$$  

By (4.17'), (4.18') and the assumption in Theorem we see

$$(4.23) \quad d\delta(\xi) = \epsilon(\alpha - \gamma + \epsilon).$$  

By (4.13) and (4.18') we get also

$$(4.24) \quad A\phi U = \delta U + (\alpha + \epsilon)\phi U.$$  

On the other hand, differentiating the function $g(A\phi U, \phi U) = \alpha + \epsilon$ with respect to $\xi$ exteriorly and using (2.1), (4.1) and (4.22) to the obtained equation imply

$$d(\alpha + \epsilon)(\xi) = -2\delta \epsilon.$$  

From this together with (4.16') it follows

$$(4.25) \quad d(\alpha + \gamma + \epsilon)(\xi) = 0.$$  

Using the some mutual relations obtained above, we have the following.

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\textbf{Lemma 4.4.} The relations

\[
\begin{aligned}
& A\xi = \alpha \xi + \beta U, \; AU = \beta \xi, \; A\phi U = 0, \\
& AX = 0, \; X \in T_1 \\
& d\beta(Y) = 0, \; \alpha + \epsilon = 0, \; Y \in T_1, \; Y \perp \phi U
\end{aligned}
\]

holds on \( M_0 \).

\textit{Proof.} Since \( T_1 \) is \( A \)-invariant, there is a principal vector \( X \) in \( T_1 \) with principal curvature \( \lambda \), where \( X \) is unit. Then by Lemma 4.3 it turns out to be \( A\phi X = -\lambda \phi X \). Differentiating \( AX = \lambda X \) with respect to \( \xi \) covariantly, we have

\[
A \nabla_\xi X = d\lambda(\xi)X + \lambda \nabla_\xi X
\]

by (4.1), which implies that

\begin{equation}
(4.26) \quad d\lambda(\xi) = 0.
\end{equation}

On the other hand, differentiating \( A\xi = \alpha \xi + \beta U \) with respect to \( X \) covariantly and applying (2.1) and (4.3), we have

\[
\beta \nabla_X U = -(\lambda^2 + \alpha \lambda + \frac{c}{4})\phi X - d\alpha(X)\xi - d\beta(X)U.
\]

Since \( T_1 \) is \( A \)-invariant and then \( \phi \)-invariant by Lemma 4.2, the vector \( \nabla_X U \) is orthogonal to \( \xi \) and \( U \). Furthermore, because it is orthogonal to \( U \), we get \( d\beta(X) = 0 \). And therefore we see that the above information implies

\begin{equation}
(4.27) \quad \begin{cases} 
 d\alpha(X) = 0, \\
 d\beta(X) = 0, \\
 \beta \nabla_X U = -(\lambda^2 + \alpha \lambda + \frac{c}{4})\phi X
\end{cases}
\end{equation}

for any vector field \( X \) in \( T_1 \). Furthermore, differentiating \( AU = \beta \xi + \gamma U + \delta \phi U \) with respect to \( X \) covariantly and making use of (2.2) and (4.27), we have

\[
\beta(\nabla_X A)U = \beta d\gamma(X)U + \beta d\delta(X)\phi U + \delta(\lambda^2 + \alpha \lambda + \frac{c}{4})X \\
+ \{\beta^2 \lambda - (\lambda + \gamma)(\lambda^2 + \alpha \lambda + \frac{c}{4})\}\phi X.
\]
Thus we get

\begin{equation}
\begin{aligned}
\beta g((\nabla X A)U, X) &= \delta(\lambda^2 + \alpha \lambda + \frac{c}{4}), \\
\beta g((\nabla X A)U, \phi X) &= \beta^2 \lambda - (\lambda + \gamma)(\lambda^2 + \alpha \lambda + \frac{c}{4})
\end{aligned}
\end{equation}

for any unit vector field \(X\) in \(T_1\). Similarly, by (4.24) the vector field \(A\phi U\) is given by \(\delta U + (\alpha + \epsilon)\phi U\) and therefore we get

\begin{equation}
\begin{aligned}
\beta g((\nabla X A)\phi U, X) &= (-\lambda + \alpha + \epsilon)(\lambda^2 + \alpha \lambda + \frac{c}{4}), \\
\beta g((\nabla X A)\phi U, \phi X) &= -\delta(\lambda^2 + \alpha \lambda + \frac{c}{4})
\end{aligned}
\end{equation}

for any unit vector field \(X\) in \(T_1\).

Next, we shall consider the equation (4.6) for any unit vector field \(X\) in \(T_1\). Putting \(Y = U\) in it and taking account of (4.18) and (4.27), we have

\[(\nabla X A)\phi AU - (\nabla U A)\phi AX = \beta(-2\lambda^2 + \gamma \lambda)X.\]

From this, together with the expressions of \(AU\) and \(\beta(\nabla X A)U\) in above, it follows

\[\beta \gamma(\nabla X A)\phi U - \beta \lambda(\nabla U A)\phi X = \beta \delta \delta \gamma(X)U + \beta \delta \delta \delta(X)\phi U + \delta^2(\lambda^2 + \alpha \lambda + \frac{c}{4})X + \delta(\beta^2 \lambda - (\lambda + \gamma)(\lambda^2 + \alpha \lambda + \frac{c}{4}))\phi X + \beta^2(-2\lambda^2 + \gamma \lambda)X.\]

Then, if we take account of (2.3), (4.28) and (4.29) and by the direct calculation, we see that any principal curvature in \(T_1\) satisfies the following equation:

\begin{equation}
\begin{aligned}
x^4 + \alpha x^3 + \{\beta^2 - \delta^2 + (\alpha + \epsilon)\gamma + \frac{c}{4}\}x^2 \\
+ \{-\alpha \delta^2 - \beta^2 \gamma + \alpha \gamma(\alpha + \epsilon)\}x + \frac{c}{4}(\gamma(\alpha + \epsilon) - \delta^2) = 0.
\end{aligned}
\end{equation}

Now, we want to prove the fact that all principal curvatures in the direction of \(T_1\) vanish identically on the subset \(M_0\). First suppose that
there are a principal curvature $\lambda$ and a point $x$ in $M_0 = \{x \in M : \beta(x) \neq 0\}$ at which $\lambda(x) \neq 0$. For the principal curvature $\lambda$, there is a neighborhood $U$ of $x$ in $M_0$ on which $\lambda$ has no zero points. Then we lead a contradiction. Since by Lemma 4.3 $-\lambda \neq 0$ is also principal on the neighborhood $U$ and also a root of (4.30), it follows

(4.31) $\alpha \lambda^2 + \{-\alpha \delta^2 - \beta^2 \gamma + \alpha \gamma (\alpha + \epsilon)\} = 0$.

Differentiating this equation with respect to $\xi$ covariantly and taking account of (4.15), (4.16')~(4.18'), (4.23) and (4.26), we have

$\{\lambda^2 + \gamma (\alpha + \epsilon) - \delta^2\} d\alpha(\xi) - 2\beta^2 \delta (\gamma + \epsilon) = 0$.

Accordingly we have by (4.31)

(4.32) $\beta^2 \{\gamma d\alpha(\xi) - 2\alpha \delta (\gamma + \epsilon)\} = 0$

on $M_0$.

We first consider the subset $M_1 \cap U$, where $M_1 = \{x \in M_0 : \delta(x) \neq 0\}$. From the above equation together with (4.18') it follows that

$\beta^2 \gamma + \alpha (\gamma + \epsilon)(\alpha + \gamma + \epsilon) = 0$.

Again, differentiating this with respect to $\xi$ covariantly and taking account of (4.15), (4.16'), (4.18') and (4.25), we have

$\alpha \beta^2 \delta = 0$

on $M_1 \cap U$. This shows that $\alpha = 0$ on $M_1 \cap U$. It contradicts to the assumption of the Theorem.

Next, suppose that the interior of $M_0 - M_1$ is not empty. On the subset we see $\beta \neq 0$ and $\delta = 0$. By (4.32) we have $\gamma = 0$. Then from (4.31) we have $\alpha = 0$ on $Int(M_0 - M_1) \cap U$. Thus we leads a contradiction.

Therefore it means that for the holomorphic distribution $T_1$

$AX = 0$

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for any vector field $X$ in $T_1$ on $M_0$. Then we have

$$-\delta^2 + \gamma(\alpha + \epsilon) = 0$$

by (4.30). Since the vector field $\nabla_X Y$ for any $X$ and $Y$ in $T_1$ is expressed as

$$\beta \nabla_X Y = \beta(\nabla_X Y)_1 + \frac{c}{4}\{g(\phi X, Y)U - g(X, Y)\phi U\}$$

by (4.27), we have

$$\beta(\nabla_X A)Y + \frac{c}{4}\{g(\phi X, Y)AU - g(X, Y)A\phi U\} = 0$$

where $(Z)_1$ denotes the $T_1$-component of the vector field $Z$. From this combined with (2.3) it follows that

$$g(\phi X, Y)(\gamma U + \delta \phi U) = 0, \quad X, Y \in T_1.$$ 

Thus we have

$$\gamma = \delta = 0$$

on $M_0$. By virtue of (4.18'), (4.24) and (4.27), the relations in above lemma are derived. It completes the proof of Lemma 4.4. \(\Box\)

Next, after the above preparation we are in position to prove the main Theorem. Suppose that the interior of $M - M_0$ is not empty. On the subset the function $\beta$ vanishes identically and therefore $\xi$ is principal. It is seen in [4] and [8] that the principal curvature $\alpha$ is constant on the interior of $M - M_0$, because this is a local property. Thus we have

$$(\nabla_X A)\xi + A\nabla_X \xi = \alpha \nabla_X \xi.$$ 

Since it can be easily seen that Codazzi equation (2.3) and the assumption of the Theorem imply

$$(\nabla_X A)\xi = -\frac{1}{4}c\phi X.$$ 

Accordingly, by (2.1), we know that the above equation is equivalent to

$$A\phi AX = \alpha \phi AX + \frac{1}{4}c\phi X.$$ 

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On the other hand, on the subset $M - M_0$ it is seen that the equation (2.4) holds, namely we see

$$A\phi A = \frac{1}{4} c\phi + \frac{1}{2} c\alpha (A\phi + \phi A),$$

and therefore from the above two equations it follows that

$$(4.33) \quad \alpha (A\phi - \phi A) X = 0$$

on the interior of $M - M_0$. Suppose that $\alpha$ is not zero. For any principal vector $X$ in $T_0$ with principal curvature $\lambda$, we have

$$(2\lambda - \alpha) A\phi X = \left( \frac{1}{2} c + \alpha\lambda \right) \phi X.$$

Using (4.33) and the above equation, we get

$$(4.34) \quad 4\lambda^2 - 4\alpha\lambda - c = 0,$$

from which it follows that all principal curvatures are non-zero constant on the interior of $M - M_0$. In the case where $\alpha = 0$, we have $\lambda^2 = \frac{c}{4}$, which means that all principal curvatures are non-zero constant on the interior of the subset $M - M_0$. Since we have supposed that the set $M_0$ is not empty, the equations in Lemma 4.4 means that

$$AX = 0, \quad X \in T_1$$

on $M_0$. So, by means of the continuity of principal curvatures, (4.34) and the above equation lead a contradiction.

It shows that the interior of $M - M_0$ must be empty. Thus the open set $M_0$ is dense. By the continuity of principal curvatures again, we see that the shape operator satisfies the relations in Lemma 4.4 on the whole $M$. Accordingly we get $g(\nabla_X Y, \xi) = -g(\nabla_X \xi, Y) = -g(\phi AX, \xi) = 0$ by (2.1), which means that $\nabla_X Y - \nabla_Y X$ is also contained in $T_0$. Hence the distribution $T_0$ is integrable on $M$. Moreover
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the integral manifold of $T_0$ can be regarded as the submanifold of codimension 2 in $M_n(c)$ whose normal vectors are $\xi$ and $C$. Since we have

$$\bar{g}(\bar{\nabla}_XY, \xi) = g(\nabla XY, \xi) = 0$$

and

$$\bar{g}(\bar{\nabla}_XY, C) = -\bar{g}(\bar{\nabla}_XC, Y) = g(AX, Y) = 0$$

for any vector fields $X$ and $Y$ in $T_0$ by (2.1) and Lemma 4.4, where $\bar{\nabla}$ denotes the Riemannian connection of $M_n(c)$, it is seen that the submanifold is totally geodesic in $M_n(c)$. Since $T_0$ is also $J$-invariant, its integral manifold is a complex submanifold and therefore it is a complex space form $M_{n-1}(c)$. Thus $M$ is a ruled real hypersurface. However, it satisfies the last two equations in Lemma 4.4. So the meaning of these equations is that the mean curvature $h$ of $M$ is equal to $\alpha$. Then we know from (3.9), (3.10) and the equation of Codazzi (2.3) that the mean curvature $h$ is constant along the distribution $T_0$.

Conversely, it was shown in section 3 that ruled hypersurfaces of a complex space form $M_n(c)$, $c\neq 0$, whose mean curvature $h = \alpha$ is constant along the distribution $T_0$ satisfy the condition (1.2) of the Theorem. It completes the proof of our Theorem. □

REMARK 1. Recently an example of minimal ruled real hypersurfaces of $H_nC$ was constructed by Ahn, Lee and Suh [1]. It satisfies not only the equations of (4.20), but also $d\alpha(\xi) = 0$.

REMARK 2. We do not know an example of ruled real hypersurfaces in $M_n(c)$ which satisfies $d\alpha(\xi) \neq 0$ and (1.2).

References


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