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A NEW CHARACTERIZATION OF RULED REAL HYPERSURFACES IN COMPLEX SPACE FORMS

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ABSTRACT. The purpose of this paper is to give another new characterization of ruled real hypersurfaces in a complex space form $M_n(c)$, $c \neq 0$ in terms of the covariant derivative of its Weingarten map in the direction of the structure vector ξ.

1. Introduction

A complex $n(\geq 2)$-dimensional Kaehlerian manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by $M_n(c)$. A complete and simply connected complex space form is a complex projective space P_nC, a complex Euclidean space C^n or a complex hyperbolic space H_nC, according as $c > 0$, $c = 0$ or $c < 0$.

Now, there exist many studies about real hypersurfaces of $M_n(c)$. One of the first researches is the classification of homogeneous real hypersurfaces of a complex projective space P_nC by Takagi [13], who showed that these hypersurfaces of P_nC could be divided into six types which are said to be of type A_1, A_2, B, C, D, and E, and in [3] Cecil-Ryan and [5] Kimura proved that they were realized as the tubes of constant radius over compact Hermitian symmetric spaces of rank 1 or rank 2 if the structure vector field ξ is principal. Also Berndt [2] showed recently that all real hypersurfaces with constant principal curvatures of a complex hyperbolic space H_nC are realized as the tubes

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of constant radius over certain submanifolds if ξ is principal. According to Takagi's classification theorem and Berndt's one, the principal curvatures and their multiplicities of homogeneous real hypersurfaces of $M_n(c)$ are given.

As an example of special real hypersurfaces of P_nC different from the above ones, firstly Kimura [6] introduced the notion of ruled real hypersurfaces in P_nC, which is not complete and not principal. Also Kimura [6] obtained some properties about a ruled real hypersurface M in P_nC, $n\geq 3$. In particular, an example of minimal ruled hypersurfaces of P_nC was constructed. Let T_0 be a distribution defined by a subspace $T_0(x) = \{u \in T_xM : u \perp \xi(x)\}$ of the tangent space $T_x(M)$, which is called the *holomorphic distribution.* The following was proved by Kimura and Maeda [7].

THEOREM A. *Let M be a real hypersurface of P_nC, $n\geq 3$. Then the second fundamental form is η-parallel and the holomorphic distribution T_0 is integrable if and only if M is locally congruent to a ruled real hypersurface.*

In [1] we also introduced the notion of ruled real hypersurfaces in a complex hyperbolic space H_nC and constructed an example of minimal ruled real hypersurfaces of H_nC by using the submersion compatible with the fibration $\pi : H_1^{2n+1} \to H_nC$. From this, together with Kimura's one, in the paper [12] the third author has given a characterization of ruled real hypersurfaces M in $M_n(c)$ in such a way that its shape operator A satisfies

$$(1.1) \qquad (\nabla_X A)Y = f(X,Y)\xi, \quad X, Y \in T_0,$$

where we put

$$f(X,Y) = \beta^2\{g(X,U)g(Y,\phi U) + g(X,\phi U)g(Y,U)\} - \frac{c}{4}g(\phi X, Y)$$

for any vector field X and Y in the distribution T_0 except for the case where the function β identically vanishes. Moreover, this expression of the covariant derivative of the shape operator A will be shown concretely in section 3.

Now the purpose of this paper is to give another new characterization of ruled real hypersurfaces in complex space forms $M_n(c)$ as the covariant derivative of the shape operator A along the direction of ξ. Namely, we assert the following

THEOREM. *Let M be a real hypersurface of $M_n(c)$, $c\neq 0$, $n\geq 3$. If it satisfies*

$$(\nabla_\xi A)X = \beta^2\{g(X,\phi U)U + g(X,U)\phi U\}, \tag{1.2}$$

provided that $d\alpha(\xi)\neq 0$ for any vector field X in the distribution T_0, where A denotes the shape operator, then M is locally congruent to a ruled real hypersurface in $M_n(c)$, on which its mean curvature h is constant along the distribution T_0.

In section 3 some fundamental properties about ruled real hypersurfaces in $M_n(c)$, $c\neq 0$ will be recalled and the covariant derivative of the shape operator A in the direction of the structure vector field ξ, which is given in (1.2), will be explicitly expressed. By paying attention to this formula another new characterization of ruled real hypersurfaces in $M_n(c)$ will be given in section 4.

2. Preliminaries

First of all, we recall basic properties of real hypersurfaces of a complex space form. Let M be a real hypersurface of $n(\geq 2)$-dimensional complex space form $M_n(c)$ of constant holomorphic sectional curvature $c(\neq 0)$ and let C be a unit normal field on a neighborhood of a point x in M. We denote by J an almost complex structure of $M_n(c)$. For a local vector field X on a neighborhood of x in M, the transformation of X and C under J can be represented as

$$JX = \phi X + \eta(X)C, \qquad JC = -\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M, while η and ξ denote a 1-form and a vector field on a neighborhood of x in M, respectively. Moreover, it is seen that $g(\xi, X) = \eta(X)$,where g denotes the induced Riemannian metric on

M. By properties of the almost complex structure J, the set (ϕ, ξ, η, g) of tensors satisfies

$$\phi^2 = -I + \eta\otimes\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1,$$

where I denotes the identity transformation. Accordingly, the set is so called an *almost contact metric structure*. Furthermore the covariant derivative of the structure tensors are given by

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X\xi = \phi AX, \tag{2.1}$$

where ∇ is the Riemannian connection of g and A denotes the shape operator with respect to the unit normal C on M.

Since the ambient space is of constant holomorphic sectional curvature c, the equation of Gauss and Codazzi are respectively given as follows

(2.2)

$$\begin{aligned} R(X,Y)Z =& \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y \\ & - 2g(\phi X,Y)\phi Z\} + g(AY,Z)AX - g(AX,Z)AY, \end{aligned}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}, \tag{2.3}$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X.

The second fundamental form is said to be η-parallel if the shape operator A satisfies $g((\nabla_X A)Y, Z) = 0$ for any vector fields X,Y and Z in T_0.

Next we suppose that the structure vector field ξ is principal with corresponding principal curvature α. Then it is seen in [4] and [8] that α is constant on M and it satisfies

$$A\phi A = \frac{c}{4}\phi + \frac{1}{2}\alpha(a\phi + \phi A). \tag{2.4}$$

3. Ruled Real Hypersurfaces

This section is concerned with necessary properties about ruled real hypersurfaces. First of all, we define a ruled real hypersurface M of $M_n(c)$, $c{\neq}0$. Let $\gamma : I{\rightarrow}M_n(c)$ be any regular curve. For any $t({\in}I)$ let $M^{(t)}_{n-1}(c)$ be a totally geodesic complex hypersurface through the point $\gamma(t)$ of $M_n(c)$ which is orthogonal to a holomorphic plane spanned by $\gamma'(t)$ and $J\gamma'(t)$. Set $M = \{x{\in}M^{(t)}_{n-1}(c) : t{\in}I\}$. Then the construction of M asserts that M is a real hypersurface of $M_n(c)$. Under this construction that the ruled real hypersurface M of $M_n(c)$, $c{\neq}0$, has some fundamental properties.

Let us put $A\xi = \alpha\xi + \beta U$, where U is a unit vector orthogonal to ξ and α and $\beta(\beta{\neq}0)$ are smooth functions on M. As is seen in [7], the shape operator A satisfies

$$AU = \beta\xi, \quad AX = 0 \tag{3.1}$$

for any vector field X orthogonal to ξ and U. It turns out to be

$$A\phi X = -\beta g(X, \phi U)\xi,\ \phi AX = 0,\ X{\in}T_0, \tag{3.2}$$

which implies that

$$g((A\phi - \phi A)X, Y) = 0, X, Y{\in}T_0. \tag{3.3}$$

Because of

$$\begin{aligned}(\mathcal{L}_\xi g)(X, Y) &= \mathcal{L}_\xi(g(X, Y)) - g(\mathcal{L}_\xi X, Y) - g(X, \mathcal{L}_\xi Y)\\ &= g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi),\end{aligned}$$

the above equation is equivalent to

$$(\mathcal{L}_\xi g)(X, Y) = 0, X, Y{\in}T_0. \tag{3.4}$$

Next the covariant derivative $\nabla_X A$ with respect to X in T_0 is explicitly expressed. It is seen in [6] and [7] that the second fundamental form is η-parallel. Also the equation (2.3) of Codazzi gives us to

$$(\nabla_X A)\xi - (\nabla_\xi A)X = -\frac{c}{4}\phi X.$$

By the direct calculation of the left hand side of the above relation and using the property $\nabla_X \xi = \phi AX = 0$ in (3.2), we get

$$\text{(3.5)} \quad d\alpha(X)\xi + d\beta(X)U + \frac{c}{4}\phi X + \beta\nabla_X U - \nabla_\xi(AX) + A\nabla_\xi X = 0,$$

for any X in T_0. Let T_1 be a distribution defined by a subspace $T_1(x) = \{u \in T_0(x) : g(u, U(x)) = g(u, \phi U(x)) = 0\}$. Since AX is expressed as the linear combination of ξ and U by (3.1), we can derive from (3.1),(3.2) and the above equation the following relations:

$$\text{(3.6)} \qquad \beta\nabla_X U = \begin{cases} (\beta^2 - \frac{c}{4})\phi X, & X = U; \\ 0, & X = \phi U; \\ -\frac{c}{4}\phi X, & X \in T_1, \end{cases}$$

$$\text{(3.7)} \qquad d\beta(X) = \begin{cases} 0, & X = U; \\ \beta^2 + \frac{c}{4}, & X = \phi U; \\ 0, & X \in T_1. \end{cases}$$

Using these relations we can obtain the components of $(\nabla_X A)Y$ in the direction of ξ. In fact, we have

$$\begin{aligned} g((\nabla_X A)Y, \xi) &= g((\nabla_X A)\xi, Y) = g(\nabla_X(A\xi) - A\nabla_X \xi, Y) \\ &= d\beta(X)g(Y, U) + \beta g(\nabla_X U, Y), \end{aligned}$$

which yields combining with the above equation that

$$\text{(3.8)} \qquad (\nabla_X A)Y = f(X, Y)\xi, \quad X, Y \in T_0,$$

where we put

$$\text{(3.9)} \qquad \begin{aligned} &f(X, Y) \\ &= \beta^2\{g(X, U)g(Y, \phi U) + g(X, \phi U)g(Y, U)\} - \frac{c}{4}g(\phi X, Y). \end{aligned}$$

which means that A is η-parallel.

Accordingly, by the equation of Codazzi (2.3) and the above equations it can be easily seen that the shape operator of M satisfies

$$\text{(3.10)} \qquad (\nabla_\xi A)X = \beta^2\{g(X, \phi U)U + g(X, U)\phi U\},$$

when its mean curvature $h = \alpha$ is constant along the distribution T_0.

4. Proof of the Theorem

In this section we are only concerned with the proof of Theorem. Let M be a real hypersurface of $M_n(c)$, $c{\neq}0$, $n{\geq}3$. Throughout this section we assume that the structure vector field ξ is not principal. Then we can put

$$A\xi = \alpha\xi + \beta U,$$

where U is a unit vector in the holomorphic distribution T_0 and α and β are smooth functions on M. We may consider that the function β does not vanish identically on M. Let M_0 be an open set of M consisting of points x at which $\beta(x){\neq}0$. In other words, the subset M_0 is not empty. Furthermore we assume that the following condition:

$$(\nabla_\xi A)Y = \beta^2\{g(Y,\phi U)U + g(Y,U)\phi U\}, \quad Y{\in}T_0. \tag{4.1}$$

First of all, from (4.1) we derive the relation in which the derivative of the shape operator is not contained.

LEMMA 4.1. *On the subset M_0 we have*

$$\begin{aligned} &d\alpha(\xi)(A\phi + \phi A)X \\ &=2\beta^2\{g(X,\phi U)A\phi U - g(AX,\phi U)\phi U\} - \beta d\alpha(\xi)g(X,\phi U)\xi \end{aligned} \tag{4.2}$$

for any vector field X in T_0.

Proof. Under the assumption (4.1) and by the assistance of (2.3) it turns out to be

$$(\nabla_Y A)\xi = \beta^2\{g(Y,\phi U)U + g(Y,U)\phi U\} - \frac{c}{4}\phi Y \tag{4.3}$$

for any vector field Y in T_0. Differentiating this equation with respect to X covariantly and taking account of (2.1), we get

$$\begin{aligned} &(\nabla_X\nabla_Y A)\xi + (\nabla_{\nabla_X Y}A)\xi + (\nabla_Y A)\phi AX \\ &=d\beta^2(X)\{g(Y,\phi U)U + g(Y,U)\phi U\} \\ &\quad + \beta^2\big[\{g(\nabla_X Y,\phi U) + g(Y,\phi\nabla_X U)\}U + g(Y,\phi U)\nabla_X U \\ &\quad + \{g(\nabla_X Y,U) + g(Y,\nabla_X U)\}\phi U \\ &\quad - g(Y,U)g(AX,U)\xi + g(Y,U)\phi\nabla_X U\big] \\ &\quad + \frac{c}{4}\{g(AX,Y)\xi - \phi\nabla_X Y\} \end{aligned}$$

for any vector fields X and Y in T_0. For any vector field Z the orthogonal decomposition in the direction of ξ is expressed as

$$Z = (Z)_0 + g(Z, \xi)\xi,$$

where $(Z)_0$ denotes the T_0-component of Z. Since the component of the vector $\nabla_X Y$ in the direction of ξ is given by $-g(\phi AX, Y)$ by the first equation of (2.1), we have the following orthogonal decomposition

$$\nabla_X Y = (\nabla_X Y)_0 - g(\phi AX, Y)\xi.$$

Using the above orthogonal decomposition and taking account of (4.3) itself, we get directly

(4.4)
$$\begin{aligned}(\nabla_X \nabla_Y A)\xi =& g(\phi AX, Y)(\nabla_\xi A)\xi - (\nabla_Y A)\phi AX \\ &+ d\beta^2(X)\{g(Y, \phi U)U + g(Y, U)\phi U\} \\ &+ \beta^2\{g(Y, \phi\nabla_X U)U + g(Y, \phi U)\nabla_X U + g(Y, \nabla_X U)\phi U \\ &- g(Y, U)g(AX, U)\xi + g(Y, U)\phi\nabla_X U\} + \frac{c}{4}g(AX, Y)\xi\end{aligned}$$

for any vector fields X and Y in T_0.

On the other hand, it is well known that the Ricci formula for the shape operator A is given by

$$(\nabla_X \nabla_Y A)Z - (\nabla_Y \nabla_X A)Z = R(X, Y)(AZ) - A(R(X, Y)Z)$$

for any vector fields X, Y and Z. Accordingly, putting $Z = \xi$ in the above Ricci formula, taking X and Y in the distribution T_0 and taking account of the Gauss equation (2.2) and (4.3) imply

(4.5)
$$\begin{aligned}&g((A\phi + \phi A)X, Y)(\nabla_\xi A)\xi + (\nabla_X A)\phi AY - (\nabla_Y A)\phi AX \\ &= \frac{c}{4}\{g(Y, A\xi)X - g(X, A\xi)Y \\ &\quad + g(\phi Y, A\xi)\phi X - g(\phi X, A\xi)\phi Y - 2g(\phi X, Y)\phi A\xi\} \\ &\quad - g(Y, A\xi)A^2X + g(X, A\xi)A^2Y + g(Y, A^2\xi)AX - g(X, A^2\xi)AY\end{aligned}$$

$$
\begin{aligned}
&- d\beta^2(X)\{g(Y,\phi U)U + g(Y,U)\phi U\}\\
&+ d\beta^2(Y)\{g(X,\phi U)U + g(X,U)\phi U\}\\
&+ \beta^2\big[g(X,\phi\nabla_Y U)U + g(X,\phi U)\nabla_Y U\\
&+ g(X,\nabla_Y U)\phi U + g(X,U)\phi\nabla_Y U\\
&- g(Y,\phi\nabla_X U)U - g(Y,\phi U)\nabla_X U\\
&- g(Y,\nabla_X U)\phi U - g(Y,U)\phi\nabla_X U\\
&+ \{g(Y,U)g(AX,U) - g(X,U)g(AY,U)\}\xi\big]
\end{aligned}
$$

for any vector fields X and Y in T_0.

Now, in order to prove Lemma 4.1, we shall express (4.5) with the simpler form. From now on we shall discuss on the open set $M_0 = \{x \in M : \beta(x) \neq 0\}$. By the form $A\xi = \alpha\xi + \beta U$ we have

$$
A^2\xi = \alpha^2\xi + \alpha\beta U + \beta AU.
$$

Accordingly, by substituting the above equation into the equation (4.5), it can be reformed as

(4.6)

$$
\begin{aligned}
&g((A\phi + \phi A)X,Y)(\nabla_\xi A)\xi + (\nabla_X A)\phi AY - (\nabla_Y A)\phi AX\\
&= \frac{c}{4}\beta\{g(Y,U)X - g(X,U)Y - g(Y,\phi U)\phi X\\
&+ g(X,\phi U)\phi Y - 2g(\phi X,Y)\phi U\}\\
&+ \beta\big[-g(Y,U)A^2X + g(X,U)A^2Y + \{\alpha g(Y,U) + g(Y,AU)\}AX\\
&- \{\alpha g(X,U) + g(X,AU)\}AY\big]\\
&- d\beta^2(X)\{g(Y,\phi U)U + g(Y,U)\phi U\}\\
&+ d\beta^2(Y)\{g(X,\phi U)U + g(X,U)\phi U\}\\
&+ \beta^2\big[g(X,\phi\nabla_Y U)U + g(X,\phi U)\nabla_Y U\\
&+ g(X,\nabla_Y U)\phi U + g(X,U)\phi\nabla_X U - g(Y,\phi\nabla_X U)U\\
&- g(Y,\phi U)\nabla_X U - g(Y,\nabla_X U)\phi U - g(Y,U)\phi\nabla_X U\\
&+ \{g(Y,U)g(AX,U) - g(X,U)g(AY,U)\}\xi\big]
\end{aligned}
$$

for any vector fields X and Y in T_0.

Next we want to calculate the inner product of (4.6) and ξ. For the reason, we differentiate $A\xi = \alpha\xi + \beta U$ with respect to ξ covariantly. Then by (2.1) we have

$$(\nabla_\xi A)\xi = d\alpha(\xi)\xi + d\beta(\xi)U + \alpha\beta\phi U - \beta A\phi U + \beta\nabla_\xi U. \tag{4.7}$$

Since it is easily seen by (2.2) and by the choice of the vector field U that the vectors $A\phi U$ and $\nabla_\xi U$ are both orthogonal to ξ, we see

$$g((\nabla_\xi A)\xi, \xi) = d\alpha(\xi). \tag{4.8}$$

On the other hand, (4.3) implies

$$\begin{aligned} &g((\nabla_X A)\phi AY, \xi) \\ &\quad = \beta^2\{g(X,U)g(Y,AU) - g(X,\phi U)g(AY,\phi U)\} - \frac{c}{4}g(AX,Y), \end{aligned} \tag{4.9}$$

where the formulas (2.1) and (4.3) have been used. By taking account of these properties the inner product of (4.6) with ξ gives us the similar equation (4.5). Since Y belongs to the distribution T_0, we find that (4.2) holds on M_0 by the above equation. It completes the proof. □

Now let $L(\xi, U, \phi U)$ be a distribution defined by a subspace $L_x(\xi, U, \phi U)$ in the tangent space T_xM spanned by the vectors $\xi(x), U(x)$ and $\phi U(x)$ at any point x in M_0.

LEMMA 4.2. *The subbundle $L(\xi, U, \phi U)$ is A-invariant and ϕ-invariant on M_0.*

Proof. Suppose that there is a vector field V in the holomorphic distribution T_0 in such a way that AU is expressed as a linear combination of the vector fields ξ, U and V, where U and V are orthonormal. Namely, since the shape operator A is symmetric, we may put

$$AU = \beta\xi + \gamma U + \delta V, \tag{4.10}$$

where γ and δ are smooth functions on M_0. Putting U in place of X in (4.2) and using the expression of AU, we get

$$d\alpha(\xi)A\phi U = -\{2\beta^2\delta g(\phi U, V) + \gamma d\alpha(\xi)\}\phi U - \delta d\alpha(\xi)\phi V. \tag{4.11}$$

Consequently, acting the linear transformation ϕ to the above equation, we have

$$(4.12) \qquad d\alpha(\xi)\phi A\phi U = \{2\beta^2\delta g(\phi U, V) + \gamma d\alpha(\xi)\}U + \delta d\alpha(\xi)V.$$

Putting $X = \phi U$ in (4.2) again and making use of the decomposition of AU and $d\alpha(\xi)\neq 0$, we get

$$d\alpha(\xi)\phi A\phi U = d\alpha(\xi)(\gamma U + \delta V) - 2\beta^2\delta\phi V,$$

from which together with (4.12) it follows that

$$2\beta^2\delta\{g(\phi U, V)U + \phi V\} = 0.$$

Let M_1 be an open subset M_0 consisting of points x at which $\delta(x)\neq 0$. Suppose that M_1 is not empty. Without loss of generality, we may put $V = \phi U$ on M_1 by the above equation. Thus it implies that AU is contained in the subspace $L(\xi, U, \phi U)$. Furthermore by (4.11) we have

$$(4.13) \qquad d\alpha(\xi)A\phi U = \delta d\alpha(\xi)U - \{2\beta^2\delta + \gamma d\alpha(\xi)\}\phi U$$

on M_1.

On the other hand, by (4.11) we have $d\alpha(\xi)A\phi U = -\gamma d\alpha(\xi)\phi U$ on $M_0 - M_1$. Consequently, (4.13) holds on M_0. This means that $L(\xi, U, \phi U)$ is A-invariant. It is evident that it is ϕ-invariant. It completes the proof of Lemma 4.2. □

Next, we investigate the mutual relations among the functions α, β, γ and δ. First we differentiate $AU = \beta\xi + \gamma U + \delta\phi U$ with respect to ξ covariantly. Then taking account of (2.1), we get

$$(4.14) \qquad \begin{aligned} A\nabla_\xi U =& \{d\beta(\xi) - \beta\delta\}\xi + d\gamma(\xi)U \\ &+ d\delta(\xi)\phi U + \gamma\nabla_\xi U + \delta\phi\nabla_\xi U. \end{aligned}$$

By the forms $A\xi = \alpha\xi + \beta U$ and $AU = \beta\xi + \gamma U + \delta\phi U$ it is easily seen that the following equations

$$\begin{aligned} &g(A\nabla_\xi U, \xi) = g(\nabla_\xi U, A\xi) = 0, \\ &g(A\nabla_\xi U, U) = \delta g(\nabla_\xi U, \phi U), \\ &d\alpha(\xi)g(A\nabla_\xi U, \phi U) = -\{2\beta^2\delta + \gamma d\alpha(\xi)\}g(\nabla_\xi U, \phi U) \end{aligned}$$

are obtained, where we have used (4.13) to derive the last equation. Then we consider the inner product of (4.14) and ξ, U and ϕU, respectively. Taking account of the above three equations, we have the following mutual relations:

$$d\beta(\xi) = \beta\delta, \tag{4.15}$$

$$d\gamma(\xi) = 2\delta g(\nabla_\xi U, \phi U), \tag{4.16}$$

$$d\alpha(\xi)d\delta(\xi) = -2\{\beta^2\delta + \gamma d\alpha(\xi)\}g(\nabla_\xi U, \phi U). \tag{4.17}$$

Now we take here the inner product of (4.7) with ϕU. Then the inner product with the left hand side vanishes identically by (4.1) and therefore it implies

$$d\alpha(\xi)g(\nabla_\xi U, \phi U) = -2\beta^2\delta - (\alpha + \gamma)d\alpha(\xi), \tag{4.18}$$

where we have used (4.13).

Now let T_1 be an orthogonal complement in the tangent bundle TM of the subbundle $L(\xi, U, \phi U)$. Since the distribution $L_x(\xi, U, \phi U)$ is A- invariant by Lemma 4.2, the orthogonal distribution T_1 is also A-invariant and moreover it is ϕ-invariant, too. Accordingly, by (4.2), we have the following.

LEMMA 4.3. *The holomorphic distribution T_0 is integrable on M_0, namely the equation*

$$(A\phi + \phi A)X = 0, \quad X \in T_1 \tag{4.19}$$

holds on M_0.

By differentiating (4.19) with respect to ξ covariantly and combining with (2.1) and (4.1), it implies that

$$(A\phi + \phi A)\nabla_\xi X = 0, \quad X \in T_1,$$

because T_1 is invariant. Thus the inner product of this equation with ξ yields

$$g(\nabla_\xi U, \phi X) = 0.$$

Since T_1 is ϕ-invariant, we get

$$g(\nabla_\xi U, X) = 0, \quad X \in T_1. \tag{4.20}$$

Evidently we get

$$g(\nabla_\xi U, \xi) = 0, \quad g(\nabla_\xi U, U) = 0. \tag{4.21}$$

Then (4.20) and (4.21) imply that

$$\nabla_\xi U = \epsilon \phi U, \tag{4.22}$$

where ϵ is a smooth function on M_0. Accordingly, the equations (4.16), (4.17) and (4.18) can be rewritten as follows:

$$d\gamma(\xi) = 2\delta\epsilon, \tag{4.16'}$$

$$d\alpha(\xi) d\delta(\xi) = -2\epsilon\{\beta^2\delta + \gamma d\alpha(\xi)\}, \tag{4.17'}$$

$$(\alpha + \gamma + \epsilon) d\alpha(\xi) = -2\beta^2\delta. \tag{4.18'}$$

By (4.17'), (4.18') and the assumption in Theorem we see

$$d\delta(\xi) = \epsilon(\alpha - \gamma + \epsilon). \tag{4.23}$$

By (4.13) and (4.18') we get also

$$A\phi U = \delta U + (\alpha + \epsilon)\phi U. \tag{4.24}$$

On the other hand, differentiating the function $g(A\phi U, \phi U) = \alpha + \epsilon$ with respect to ξ exteriorly and using (2.1), (4.1) and (4.22) to the obtained equation imply

$$d(\alpha + \epsilon)(\xi) = -2\delta\epsilon.$$

From this together with (4.16') it follows

$$d(\alpha + \gamma + \epsilon)(\xi) = 0. \tag{4.25}$$

Using the some mutual relations obtained above, we have the following.

LEMMA 4.4. *The relations*

$$\begin{cases} A\xi = \alpha\xi + \beta U,\ AU = \beta\xi,\ A\phi U = 0, \\ AX = 0,\ X \in T_1 \\ d\beta(Y) = 0,\ \alpha + \epsilon = 0,\ Y \in T_1,\ Y \perp \phi U \end{cases}$$

holds on M_0.

Proof. Since T_1 is A-invariant, there is a principal vector X in T_1 with principal curvature λ, where X is unit. Then by Lemma 4.3 it turns out to be $A\phi X = -\lambda\phi X$. Differentiating $AX = \lambda X$ with respect to ξ covariantly, we have

$$A\nabla_\xi X = d\lambda(\xi)X + \lambda\nabla_\xi X$$

by (4.1), which implies that

$$d\lambda(\xi) = 0. \tag{4.26}$$

On the other hand, differentiating $A\xi = \alpha\xi + \beta U$ with respect to X covariantly and applying (2.1) and (4.3), we have

$$\beta\nabla_X U = -(\lambda^2 + \alpha\lambda + \frac{c}{4})\phi X - d\alpha(X)\xi - d\beta(X)U.$$

Since T_1 is A-invariant and then ϕ-invariant by Lemma 4.2, the vector $\nabla_X U$ is orthogonal to ξ and U. Furthermore, because it is orthogonal to U, we get $d\beta(X) = 0$. And therefore we see that the above information implies

$$\begin{cases} d\alpha(X) = 0,\ d\beta(X) = 0, \\ \beta\nabla_X U = -(\lambda^2 + \alpha\lambda + \frac{c}{4})\phi X \end{cases} \tag{4.27}$$

for any vector field X in T_1. Furthermore, differentiating $AU = \beta\xi + \gamma U + \delta\phi U$ with respect to X covariantly and making use of (2.2) and (4.27), we have

$$\begin{aligned} \beta(\nabla_X A)U =& \beta d\gamma(X)U + \beta d\delta(X)\phi U + \delta(\lambda^2 + \alpha\lambda + \frac{c}{4})X \\ &+ \{\beta^2\lambda - (\lambda + \gamma)(\lambda^2 + \alpha\lambda + \frac{c}{4})\}\phi X. \end{aligned}$$

Thus we get

$$(4.28)\qquad \begin{cases} \beta g((\nabla_X A)U, X) = \delta(\lambda^2 + \alpha\lambda + \frac{c}{4}), \\ \beta g((\nabla_X A)U, \phi X) = \beta^2\lambda - (\lambda+\gamma)(\lambda^2 + \alpha\lambda + \frac{c}{4}) \end{cases}$$

for any unit vector field X in T_1. Similarly, by (4.24) the vector field $A\phi U$ is given by $\delta U + (\alpha+\epsilon)\phi U$ and therefore we get

$$(4.29)\qquad \begin{cases} \beta g((\nabla_X A)\phi U, X) &= (-\lambda+\alpha+\epsilon)(\lambda^2 + \alpha\lambda + \frac{c}{4}), \\ \beta g((\nabla_X A)\phi U, \phi X) &= -\delta(\lambda^2 + \alpha\lambda + \frac{c}{4}) \end{cases}$$

for any unit vector field X in T_1.

Next, we shall consider the equation (4.6) for any unit vector field X in T_1. Putting $Y = U$ in it and taking account of (4.18) and (4.27), we have

$$(\nabla_X A)\phi AU - (\nabla_U A)\phi AX = \beta(-2\lambda^2 + \gamma\lambda)X.$$

From this, together with the expressions of AU and $\beta(\nabla_X A)U$ in above, it follows

$$\begin{aligned} \beta\gamma(\nabla_X A)\phi U - \beta\lambda(\nabla_U A)\phi X = {} & \beta\delta d\gamma(X)U \\ & + \beta\delta d\delta(X)\phi U + \delta^2(\lambda^2 + \alpha\lambda + \frac{c}{4})X \\ & + \delta\{\beta^2\lambda - (\lambda+\gamma)(\lambda^2 + \alpha\lambda + \frac{c}{4})\}\phi X \\ & + \beta^2(-2\lambda^2 + \gamma\lambda)X. \end{aligned}$$

Then, if we take account of (2.3), (4.28) and (4.29) and by the direct calculation, we see that any principal curvature in T_1 satisfies the following equation:

$$(4.30)\qquad \begin{aligned} & x^4 + \alpha x^3 + \{\beta^2 - \delta^2 + (\alpha+\epsilon)\gamma + \frac{c}{4}\}x^2 \\ & + \{-\alpha\delta^2 - \beta^2\gamma + \alpha\gamma(\alpha+\epsilon)\}x + \frac{c}{4}\{\gamma(\alpha+\epsilon) - \delta^2\} = 0. \end{aligned}$$

Now, we want to prove the fact that all principal curvatures in the direction of T_1 vanish identically on the subset M_0. First suppose that there are a principal curvature λ and a point x in $M_0 = \{x \in M : \beta(x) \neq 0\}$ at which $\lambda(x) \neq 0$. For the principal curvature λ, there is a neighborhood $\mathcal{U}$ of x in M_0 on which λ has no zero points. Then we lead a contradiction. Since by Lemma 4.3 $-\lambda \neq 0$ is also principal on the neighborhood $\mathcal{U}$ and also a root of (4.30), it follows

$$\alpha\lambda^2 + \{-\alpha\delta^2 - \beta^2\gamma + \alpha\gamma(\alpha + \epsilon)\} = 0. \tag{4.31}$$

Differentiating this equation with respect to ξ covariantly and taking account of (4.15), (4.16')$\sim$(4.18'), (4.23) and (4.26), we have

$$\{\lambda^2 + \gamma(\alpha + \epsilon) - \delta^2\}d\alpha(\xi) - 2\beta^2\delta(\gamma + \epsilon) = 0.$$

Accordingly we have by (4.31)

$$\beta^2\{\gamma d\alpha(\xi) - 2\alpha\delta(\gamma + \epsilon)\} = 0 \tag{4.32}$$

on M_0.

We first consider the subset $M_1 \cap \mathcal{U}$, where $M_1 = \{x \in M_0 : \delta(x) \neq 0\}$. From the above equation together with (4.18') it follows that

$$\beta^2\gamma + \alpha(\gamma + \epsilon)(\alpha + \gamma + \epsilon) = 0.$$

Again, differentiating this with respect to ξ covariantly and taking account of (4.15), (4.16'), (4.18') and (4.25), we have

$$\alpha\beta^2\delta = 0$$

on $M_1 \cap \mathcal{U}$. This shows that $\alpha = 0$ on $M_1 \cap \mathcal{U}$. It contradicts to the assumption of the Theorem.

Next, suppose that the interior of $M_0 - M_1$ is not empty. On the subset we see $\beta \neq 0$ and $\delta = 0$. By (4.32) we have $\gamma = 0$. Then from (4.31) we have $\alpha = 0$ on $Int(M_0 - M_1) \cap \mathcal{U}$. Thus we leads a contradiction.

Therefore it means that for the holomorphic distribution T_1

$$AX = 0$$

for any vector field X in T_1 on M_0. Then we have

$$-\delta^2 + \gamma(\alpha + \epsilon) = 0$$

by (4.30). Since the vector field $\nabla_X Y$ for any X and Y in T_1 is expressed as

$$\beta \nabla_X Y = \beta(\nabla_X Y)_1 + \frac{c}{4}\{g(\phi X, Y)U - g(X, Y)\phi U\}$$

by (4.27), we have

$$\beta(\nabla_X A)Y + \frac{c}{4}\{g(\phi X, Y)AU - g(X, Y)A\phi U\} = 0$$

where $(Z)_1$ denotes the T_1-component of the vector field Z. From this combined with (2.3) it follows that

$$g(\phi X, Y)(\gamma U + \delta \phi U) = 0, \quad X, Y \in T_1.$$

Thus we have

$$\gamma = \delta = 0$$

on M_0. By virtue of (4.18'),(4.24) and (4.27), the relations in above lemma are derived. It completes the proof of Lemma 4.4. □

Next, after the above preparation we are in position to prove the main Theorem. Suppose that the interior of $M - M_0$ is not empty. On the subset the function β vanishes identically and therefore ξ is principal. It is seen in [4] and [8] that the principal curvature α is constant on the interior of $M - M_0$, because this is a local property. Thus we have

$$(\nabla_X A)\xi + A\nabla_X \xi = \alpha \nabla_X \xi.$$

Since it can be easily seen that Codazzi equation (2.3) and the assumption of the Theorem imply

$$(\nabla_X A)\xi = -\frac{1}{4}c\phi X.$$

Accordingly, by (2.1), we know that the above equation is equivalent to

$$A\phi AX = \alpha \phi AX + \frac{1}{4}c\phi X.$$

On the other hand, on the subset $M-M_0$ it is seen that the equation (2.4) holds, namely we see

$$A\phi A = \frac{1}{4}c\phi + \frac{1}{2}c\alpha(A\phi + \phi A),$$

and therefore from the above two equations it follows that

$$\alpha(A\phi - \phi A)X = 0 \tag{4.33}$$

on the interior of $M-M_0$. Suppose that α is not zero. For any principal vector X in T_0 with principal curvature λ, we have

$$(2\lambda - \alpha)A\phi X = \left(\frac{1}{2}c + \alpha\lambda\right)\phi X.$$

Using (4.33) and the above equation, we get

$$4\lambda^2 - 4\alpha\lambda - c = 0, \tag{4.34}$$

from which it follows that all principal curvatures are non-zero constant on the interior of $M-M_0$. In the case where $\alpha = 0$, we have $\lambda^2 = \frac{c}{4}$, which means that all principal curvatures are non-zero constant on the interior of the subset $M-M_0$. Since we have supposed that the set M_0 is not empty, the equations in Lemma 4.4 means that

$$AX = 0, \quad X \in T_1$$

on M_0. So, by means of the continuity of principal curvatures, (4.34) and the above equation lead a contradiction.

It shows that the interior of $M-M_0$ must be empty. Thus the open set M_0 is dense. By the continuity of principal curvatures again, we see that the shape operator satisfies the relations in Lemma 4.4 on the whole M. Accordingly we get $g(\nabla_X Y, \xi) = -g(\nabla_X \xi, Y) = -g(\phi AX, \xi) = 0$ by (2.1), which means that $\nabla_X Y - \nabla_Y X$ is also contained in T_0. Hence the distribution T_0 is integrable on M. Moreover the integral manifold of T_0 can be regarded as the submanifold of codimension 2 in $M_n(c)$ whose normal vectors are ξ and C. Since we have

$$\bar{g}(\bar{\nabla}_X Y, \xi) = g(\nabla_X Y, \xi) = 0$$

and

$$\bar{g}(\bar{\nabla}_X Y, C) = -\bar{g}(\bar{\nabla}_X C, Y) = g(AX, Y) = 0$$

for any vector fields X and Y in T_0 by (2.1) and Lemma 4.4, where $\bar{\nabla}$ denotes the Riemannian connection of $M_n(c)$, it is seen that the submanifold is totally geodesic in $M_n(c)$. Since T_0 is also J-invariant, its integral manifold is a complex submanifold and therefore it is a complex space form $M_{n-1}(c)$. Thus M is a ruled real hypersurface. However, it satisfies the last two equations in Lemma 4.4. So the meaning of these equations is that the mean curvature h of M is equal to α. Then we know from (3.9), (3.10) and the equation of Codazzi (2.3) that the mean curvature h is constant along the distribution T_0.

Conversely, it was shown in section 3 that ruled hypersurfaces of a complex space form $M_n(c), c\neq 0$, whose mean curvature $h = \alpha$ is constant along the distribution T_0 satisfy the condition (1.2) of the Theorem. It completes the proof of our Theorem. □

REMARK 1. Recently an example of minimal ruled real hypersurfaces of H_nC was constructed by Ahn, Lee and Suh [1]. It satisfies not only the equations of (4.20), but also $d\alpha(\xi) = 0$.

REMARK 2. We do not know an example of ruled real hypersurfaces in $M_n(c)$ which satisfies $d\alpha(\xi)\neq 0$ and (1.2).

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