ORTHOGONAL POLYNOMIALS RELATIVE TO LINEAR PERTURBATIONS OF QUASI-DEFINITE MOMENT FUNCTIONALS


ABSTRACT. Consider a symmetric bilinear form defined on $\Pi \times \Pi$ by

$$\langle f, g \rangle_{\lambda, \mu} = \langle \sigma, fg \rangle + \lambda L[f](a)L[g](a) + \mu M[f](b)M[g](b),$$

where $\sigma$ is a quasi-definite moment functional, $L$ and $M$ are linear operators on $\Pi$, the space of all real polynomials and $a, b, \lambda,$ and $\mu$ are real constants. We find a necessary and sufficient condition for the above bilinear form to be quasi-definite and study various properties of corresponding orthogonal polynomials. This unifies many previous works which treated cases when both $L$ and $M$ are differential or difference operators. Finally, infinite order operator equations having such orthogonal polynomials as eigenfunctions are given when $\mu = 0$.

1. Introduction

All polynomials in this work are assumed to be real polynomials in one variable and we let $\Pi$ be the space of all real polynomials. We let $\deg(\pi)$ be the degree of a polynomial $\pi(x)$ with the convention $\deg(0) = -1$. By a polynomial system (PS), we mean a sequence of polynomials $\{P_n(x)\}_{n=0}^\infty$ with $\deg(P_n) = n$, $n \geq 0$.

Let $L$ and $M$ be any two linear operators defined on $\Pi$. We now consider a symmetric bilinear form on $\Pi \times \Pi$ given by

$$\langle f, g \rangle_{\lambda, \mu} := \langle \sigma, fg \rangle + \lambda L[f](a)L[g](a) + \mu M[f](b)M[g](b), \quad (f, g \in \Pi)$$


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where $\sigma$ is a moment functional (i.e., a linear functional on $\Pi$) and $\lambda$, $\mu$, $a$, and $b$ are real numbers. We say that the bilinear form $\langle \cdot, \cdot \rangle_{\lambda,\mu}$ is quasi-definite (respectively, positive-definite) if

$$
\Delta_n := \begin{vmatrix}
\langle 1, 1 \rangle_{\lambda,\mu} & \langle 1, x \rangle_{\lambda,\mu} & \cdots & \langle 1, x^n \rangle_{\lambda,\mu} \\
\langle x, 1 \rangle_{\lambda,\mu} & \langle x, x \rangle_{\lambda,\mu} & \cdots & \langle x, x^n \rangle_{\lambda,\mu} \\
\vdots & \vdots & \ddots & \vdots \\
\langle x^n, 1 \rangle_{\lambda,\mu} & \langle x^n, x \rangle_{\lambda,\mu} & \cdots & \langle x^n, x^n \rangle_{\lambda,\mu}
\end{vmatrix} \neq 0,
$$

(respectively, $\Delta_n > 0$) $n \geq 0$.

When $\langle \cdot, \cdot \rangle_{\lambda,\mu}$ is quasi-definite (respectively, positive-definite), there is a unique monic PS $\{R_n(x)\}_{n=0}^{\infty}$ such that

$$
\langle R_m, R_n \rangle_{\lambda,\mu} = K_n \delta_{mn}, \quad m \text{ and } n \geq 0,
$$

where $K_n$ is a non-zero (respectively, a positive) constant and vice versa. In this case, we call $\{R_n(x)\}_{n=0}^{\infty}$ a monic orthogonal polynomial system (MOPS) relative to $\langle \cdot, \cdot \rangle_{\lambda,\mu}$ (or a MOPS relative to $\sigma$ when $\lambda = \mu = 0$). When $L = M = Id$, the identity operator,

$$
\langle f, g \rangle_{\lambda,\mu} = \langle \sigma + \lambda \delta(x - a) + \mu \delta(x - b), fg \rangle, \quad f \text{ and } g \in \Pi
$$

is just a point mass perturbation of $\sigma$, which is already handled by many authors ([5, 6, 11, 14, 15]). When $L$ and $M$ are differential or difference operators, $\{R_n(x)\}_{n=0}^{\infty}$ is called a Sobolev-type orthogonal polynomials (see [1, 7, 8] and references therein). Generalizing these examples, we consider arbitrary linear operators $L$ and $M$ by which we perturb $\sigma$. We first find a necessary and sufficient condition for $\langle \cdot, \cdot \rangle_{\lambda,\mu}$ to be quasi-definite (see Theorem 2.2), which extends results in [6, 8, 14], in which $L = M = Id$ or $\mu = 0$. We then express $\{R_n(x)\}_{n=0}^{\infty}$ in terms of orthogonal polynomials $\{P_n(x)\}_{n=0}^{\infty}$ relative to $\sigma$ and discuss their algebraic properties such as long term recurrence relation, quasi-orthogonality, and semi-classical character, including several non-standard examples. Finally, when $\mu = 0$ and $\{P_n(x)\}_{n=0}^{\infty}$ are eigenfunctions of a certain linear operator on $\Pi$, we show that $\{R_n(x)\}_{n=0}^{\infty}$ must also be eigenfunctions of a possibly infinite order operator equation.

This last result (see Theorem 4.1) explains many previous works ([1, 3, 7, 9, 10]) in a unified manner. Bavinck [2] considered a similar problem in a different viewpoint: perturb not the moment functional but the PS $\{P_n(x)\}_{n=0}^{\infty}$ (which need not be orthogonal) by another PS $\{Q_n(x)\}_{n=0}^{\infty}$. 

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2. Necessary and Sufficient Conditions

We always assume that $\sigma$ is quasi-definite and $\{P_n(x)\}_{n=0}^{\infty}$ is the MOPS relative to $\sigma$. Let $\langle \cdot , \cdot \rangle_{\lambda, \mu}$ be the bilinear form as in (1.1). We set for $p$ and $q = 0, 1, \cdots$

\begin{align}
K^{(p,q)}_n(x, y) &= \sum_{j=0}^{n} \frac{L^p[P_j](x) L^q[P_j](y)}{\langle \sigma, P^2_j \rangle} \\
G^{(p,q)}_n(x, y) &= \sum_{j=0}^{n} \frac{L^p[P_j](x) M^q[P_j](y)}{\langle \sigma, P^2_j \rangle} \\
J^{(p,q)}_n(x, y) &= \sum_{j=0}^{n} \frac{M^p[P_j](x) M^q[P_j](y)}{\langle \sigma, P^2_j \rangle},
\end{align}

where $L^{p+1} = L[L^p]$ and $M^{p+1} = M[M^p]$. Then

$$K^{(0,0)}_n(x, y) = G^{(q,0)}_n(y, x)$$ and $$J^{(0,q)}_n(x, y) = G^{(0,q)}_n(x, y).$$

**Proposition 2.1.** The kernels $K^{(p,q)}_n(x, y)$, $G^{(p,q)}_n(x, y)$ and $J^{(p,q)}_n(x, y)$ as in (2.1) ~ (2.3) have reproducing properties, i.e.,

\begin{align}
\langle \sigma, K^{(0,q)}_n(x, y) \psi(x) \rangle &= \langle \sigma, G^{(q,0)}_n(y, x) \psi(x) \rangle = L^q[\psi](y) \\
\langle \sigma, J^{(0,q)}_n(x, y) \psi(x) \rangle &= \langle \sigma, G^{(0,q)}_n(x, y) \psi(x) \rangle = M^q[\psi](y)
\end{align}

for any polynomial $\psi(x)$ of degree $\leq n$.

**Proof.** We set $\psi(x) = \sum_{k=0}^{n} c_k P_k(x)$. Then

$$\langle \sigma, K^{(0,q)}_n(x, y) \psi(x) \rangle = \langle \sigma, G^{(q,0)}_n(y, x) \psi(x) \rangle = \sum_{j=0}^{n} \frac{L^q[P_j](y)}{\langle \sigma, P^2_j \rangle} \langle \sigma, P_j \psi \rangle$$

$$= \sum_{j=0}^{n} \frac{L^q[P_j](y)}{\langle \sigma, P^2_j \rangle} \langle \sigma, P_j \sum_{k=0}^{n} c_k P_k \rangle = \sum_{j=0}^{n} c_j L^q[P_j](y) = L^q[\psi](y)$$

by the orthogonality of $\{P_n(x)\}_{n=0}^{\infty}$ relative to $\sigma$. Hence we have (2.4). (2.5) can be proved similarly. \(\square\)

Now, we have:

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**Theorem 2.2.** The bilinear form $\langle \cdot, \cdot \rangle_{\lambda, \mu}$ in (1.1) is quasi-definite if and only if

$$d_n := \begin{vmatrix} 1 + \lambda K^{(1,1)}_{n-1}(a, a) & \mu G^{(1,1)}_{n-1}(a, b) \\ \lambda G^{(1,1)}_{n-1}(a, b) & 1 + \mu J^{(1,1)}_{n-1}(b, b) \end{vmatrix} \neq 0, \quad n \geq 0.$$ 

In this case, $\{R_n(x)\}_{n=0}^{\infty}$ is given by

$$R_n(x) = P_n(x) - \frac{1}{d_{n+1}} \begin{vmatrix} L[P_n](a) & \mu G^{(1,1)}_{n-1}(a, b) \\ M[P_n](b) & 1 + \mu J^{(1,1)}_{n-1}(b, b) \end{vmatrix} G^{(1,0)}_{n-1}(a, x)$$ 

$$- \frac{\mu}{d_{n+1}} \begin{vmatrix} 1 + \lambda K^{(1,1)}_{n-1}(a, a) & L[P_n](a) \\ \lambda G^{(1,1)}_{n-1}(a, b) & M[P_n](b) \end{vmatrix} G^{(0,1)}_{n-1}(x, b), \quad n \geq 0,$$

where $d_{-1} = 1$.

Moreover, we have

$$\langle R_n, R_n \rangle_{\lambda, \mu} = \langle R_n, P_n \rangle_{\lambda, \mu} = \frac{d_n}{d_{n+1}} \langle \sigma, P^2_n \rangle.$$

**Proof.** Assume that the bilinear form $\langle \cdot, \cdot \rangle_{\lambda, \mu}$ in (1.1) is quasi-definite and let $\{R_n(x)\}_{n=0}^{\infty}$ be the MOPS relative to $\langle \cdot, \cdot \rangle_{\lambda, \mu}$. Then we may write $R_n(x)$ as

$$R_n(x) = P_n(x) + \sum_{j=0}^{n-1} C^n_j P_j(x), \quad n \geq 0.$$

By the orthogonality of $\{R_n(x)\}_{n=0}^{\infty}$ and $\{P_n(x)\}_{n=0}^{\infty}$, we have for $0 \leq j \leq n-1$

$$C^n_j = \frac{\langle \sigma, P_j R_n \rangle}{\langle \sigma, P^2_j \rangle} = \frac{-\lambda L[R_n](a)L[P_j](a) - \mu M[R_n](b)M[P_j](b)}{\langle \sigma, P^2_j \rangle}$$

and so

$$R_n(x) = P_n(x) - \lambda L[R_n](a)G^{(1,0)}_{n-1}(a, x) - \mu M[R_n](b)G^{(0,1)}_{n-1}(x, b).$$

Acting $L$ and $M$ to (2.8) and evaluating at $x = a$ and $x = b$ respectively, we obtain

$$\begin{pmatrix} 1 + \lambda K^{(1,1)}_{n-1}(a, a) & \mu G^{(1,1)}_{n-1}(a, b) \\ \lambda G^{(1,1)}_{n-1}(a, b) & 1 + \mu J^{(1,1)}_{n-1}(b, b) \end{pmatrix} \begin{pmatrix} L[R_n](a) \\ M[R_n](b) \end{pmatrix} = \begin{pmatrix} L[P_n](a) \\ M[P_n](b) \end{pmatrix},$$

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\[ n \geq 0, \]
where \( G^{(1,1)}_{-1}(x, y) = J^{(1,1)}_{-1}(x, y) = 0. \)
We now show that \( d_n \neq 0 \) for \( n \geq 0 \) by induction on \( n \). For \( n = 0 \),
\[
d_0 = 1 + \mu \frac{(M[1](b))^2}{\langle \sigma, 1 \rangle} + \lambda \frac{(L[1](a))^2}{\langle \sigma, 1 \rangle} = \frac{\langle 1, 1 \rangle_{\lambda, \mu}}{\langle \sigma, 1 \rangle} \neq 0.
\]
Assume that \( d_n \neq 0 \) for \( 0 \leq n \leq m \) for some integer \( m \geq 0 \). Then, the system (2.9) is uniquely solvable for \( L[R_n](a) \) and \( M[R_n](b) \) as
\[
(2.10) \quad \begin{pmatrix}
L[R_n](a) \\
M[R_n](b)
\end{pmatrix}
= \frac{1}{d_{n+1}} \begin{pmatrix}
1 + \mu \gamma^{(1,1)}_{n+1}(b, b) & -\mu G^{(1,1)}_{n+1}(a, b) \\
-\mu G^{(1,1)}_{n+1}(a, b) & 1 + \mu K^{(1,1)}_{n+1}(a, b)
\end{pmatrix}
\begin{pmatrix}
L[P_n](a) \\
M[P_n](b)
\end{pmatrix}
\]
for \( 0 \leq n \leq m + 1 \). Substituting (2.10) into (2.8), we obtain (2.6). Hence
\[
(2.11) \quad \langle R_n, P_k \rangle_{\lambda, \mu} = \langle \sigma, R_n P_k \rangle + \lambda L[R_n](a)L[P_k](a) + \mu M[R_n](b)M[P_k](b)
= \langle \sigma, P_n P_k \rangle + \lambda L[R_n](a)L[P_k](a)\delta_{kn} + \mu M[R_n](b)M[P_k](b)\delta_{kn}
= \frac{d_n}{d_{n+1}} \langle \sigma, P^2 \rangle \delta_{kn}, \quad 0 \leq k \leq n \leq m + 1
\]
since
\[
\langle \sigma, R^{(0,1)}_{n-1}(x, a)P_k(x) \rangle = L[P_k](a)(1 - \delta_{kn}),
\]
\[
\langle \sigma, J^{(0,1)}_{n-1}(x, b)P_k(x) \rangle = L[P_k](b)(1 - \delta_{kn}),
\]
and
\[
d_n = d_{n+1} + \mu \frac{\{1 + \lambda K^{(1,1)}_{n+1}(a, a)\}(M[P_n](b))^2}{\langle \sigma, P^2 \rangle}
+ \lambda \frac{\{1 + \mu \gamma^{(1,1)}_{n+1}(b, b)\}(L[P_n](a))^2}{\langle \sigma, P^2 \rangle}.
\]
In particular,
\[
d_{m+1} = \frac{\langle R_{m+1}, R_{m+1} \rangle_{\lambda, \mu}}{\langle \sigma, P^2_{m+1} \rangle} d_m \neq 0.
\]
Conversely, assume \( d_n \neq 0, n \geq 0 \) and define \( R_n(x) \) by (2.6). Then \( \{R_n(x)\}_{n=0}^{\infty} \) is a monic PS and (2.8) holds. Then the equation (2.11) implies that \( \{R_n(x)\}_{n=0}^{\infty} \) is the MOPS relative to \( \langle \cdot, \cdot \rangle_{\lambda, \mu} \) so that \( \langle \cdot, \cdot \rangle_{\lambda, \mu} \) is quasi-definite.
\[\square\]
COROLLARY 2.3. If \( \sigma \) is positive-definite, then \( \langle \cdot, \cdot \rangle_{\lambda, \mu} \) is also positive-definite if and only if \( d_n > 0, \ n \geq 0 \).

Some special cases of Theorem 2.2 were handled in [6, 8, 14, 15] when \( L = Id \) and \( \mu = 0 \) or \( L = D^r, \ M = D^s, \) where \( D = \frac{d}{dx} \) and \( r \) and \( s \) are non-negative integers.

COROLLARY 2.4. When \( \mu = 0 \), \( \langle \cdot, \cdot \rangle_\lambda := \langle \cdot, \cdot \rangle_{\lambda, \mu} \) is quasi-definite if and only if \( d_n := 1 + \lambda K_n^{(1,1)}(a, a) \neq 0, \ n \geq 0 \). In this case, we have

\[
R_n(x) = P_n(x) - \frac{\lambda L[P_n](a)}{1 + \lambda K_n^{(1,1)}(a, a)} G_n^{(1,0)}(a, x)
\]

and

\[
\langle R_n, R_n \rangle_\lambda = \frac{1 + \lambda K_n^{(1,1)}(a, a)}{1 + \lambda K_n^{(1,1)}(a, a)} \langle \sigma, P_n^2 \rangle, \ n \geq 0.
\]

From now on, we always assume that \( \langle \cdot, \cdot \rangle_{\lambda, \mu} \) is quasi-definite and let \( \{R_n(x)\}_{n=0}^\infty \) be the MOPS relative to \( \langle \cdot, \cdot \rangle_{\lambda, \mu} \).

If there exists a polynomial \( \pi(x) \) of degree \( t(\geq 1) \) such that

(2.12) \[
L[\pi \phi](a) = M[\pi \phi](b) = 0, \ \phi \in \Pi,
\]

then

(2.13) \[
\langle \pi \phi, \psi \rangle_{\lambda, \mu} = \langle \sigma, \pi \phi \psi \rangle = \langle \phi, \pi \psi \rangle_{\lambda, \mu}, \ \phi \text{ and } \psi \in \Pi.
\]

For example, we have:
If \( L = D^r \), where \( D = \frac{d}{dx} \) and \( r \geq 0, \) then

\[
L[(x - a)^{r+1}f(x)](a) = 0, \ f \in \Pi.
\]

If \( L = \Delta^r \), where \( \Delta f(x) = f(x + 1) - f(x) \) is the forward difference operator and \( r \geq 0, \) then

\[
L \left[ \prod_{k=0}^{r} (x - a - k) f(x) \right](a) = 0, \ f \in \Pi.
\]

THEOREM 2.5. Assume that there exists a monic polynomial \( \pi(x) \) of degree \( t(\geq 1) \) satisfying (2.12). Then
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(i) *(Long term recurrence relation)*

\[ \pi(x) R_n(x) = \sum_{j=n-t}^{n+t} s_{nj} R_j(x) \]

where \( s_{nj} = \frac{(\pi R_n R_j)_{\mu_{\lambda}}}<(\pi R_n R_j)_{\lambda_{\mu}}^{d_{j-1}} d_j, \quad n - t \leq j \leq n + t \) \((s_{n,n+t} = 1, s_{n,n-t} \neq 0, n \geq t)\).

(ii) *(Quasi-orthogonality relative to \( \sigma \))*

\[ \pi(x) R_n(x) = \sum_{j=n-t}^{n+t} r_{nj} P_j(x), \]

where \( r_{nj} = \frac{<(\sigma R_n R_j)_{\sigma_{\lambda}}}<(\sigma R_j)_{\sigma_{\lambda}}^{d_{j-1}} d_j, \quad n - t \leq j \leq n + t \) \((r_{n,n+t} = 1, r_{n,n-t} \neq 0, n \geq t)\).

*Proof.* Since \( \text{deg}(\pi R_n) \leq n + t \), we may write it as

\[ \pi(x) R_n(x) = \sum_{j=0}^{n+t} s_{nj} R_j(x), \quad n \geq 0. \]

Multiplying by \( R_k(x) \) and applying \( \langle \cdot, \cdot \rangle_{\lambda_{\mu}} \) on both sides, we have

\[ s_{nk} \langle R_k, R_k \rangle_{\lambda_{\mu}} = \langle \sigma, R_n R_k \rangle = \langle R_n, \pi R_k \rangle_{\lambda_{\mu}} \]

so that \( s_{nk} = 0, 0 \leq k < n - t, s_{n,n+1} = 1 \), and

\[ s_{nk} = \frac{\langle R_n, \pi R_k \rangle_{\lambda_{\mu}}}{\langle R_k, R_k \rangle_{\lambda_{\mu}}} = \frac{\langle \sigma, R_n R_k \rangle_{\lambda_{\mu}} d_{k-1}}{\langle \sigma, P_k^2 \rangle_{\lambda_{\mu}}} d_k, \quad n - t \leq k \leq n + t \]

by (2.13) and (2.14). In particular,

\[ s_{n,n-t} = \frac{\langle R_n, \pi R_{n-t} \rangle_{\lambda_{\mu}}}{\langle R_{n-t}, R_{n-t} \rangle_{\lambda_{\mu}}} = \frac{\langle R_n, R_n \rangle_{\lambda_{\mu}}}{\langle R_{n-t}, R_{n-t} \rangle_{\lambda_{\mu}}} \neq 0. \]

Hence, we have (i). For (ii), the proof is essentially the same as the one for (i).

*Note that*

\[ \frac{s_{n-i,n-j}}{\langle R_{n-i}, R_{n-i} \rangle_{\lambda_{\mu}}} = \frac{s_{n-j,n-i}}{\langle R_{n-j}, R_{n-j} \rangle_{\lambda_{\mu}}} \]

and

\[ \frac{r_{n-i,n-j}}{\langle \sigma, P_i^2 \rangle_{\lambda_{\mu}}} = \frac{r_{n-j,n-i}}{\langle \sigma, P_i^2 \rangle_{\lambda_{\mu}}}, \quad 0 \leq i \leq n - t, i - t \leq j \leq i + t. \]

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It is well known (cf. [4]) that if $\sigma$ is positive-definite, then $P_n(x)$, $n \geq 1$, has $n$ real simple zeros.

**COROLLARY 2.6.** Assume $\sigma$ is positive-definite and let $[\xi, \eta]$ be the true interval of orthogonality of $\sigma$, that is, the smallest interval containing all zeros of $P_n(x)$, $n \geq 1$, in $(\xi, \eta)$. If there exists a polynomial $\pi(x)$ satisfying (2.12), then $\pi(x)R_n(x)$ has at least $n - t$ nodal zeros, that is, zeros of odd multiplicity, in $(\xi, \eta)$ so that $R_n(x)$ has at least $n - t - k$ nodal zeros in $(\xi, \eta)$, where $k$ is the number of zeros of $\pi(x)$ which have odd multiplicity in $(\xi, \eta)$.

**Proof.** Let $x_1 < x_2 < \cdots < x_\ell$ be the nodal zeros of $\pi(x)R_n(x)$ in $(\xi, \eta)$ and $\phi(x) = \prod_{i=1}^{\ell} (x - x_i)$. Then either $\phi(x)\pi(x)R_n(x) \geq 0$ or $\phi(x)\pi(x)R_n(x) \leq 0$ on $[\xi, \eta]$ so that
$$\langle \sigma, \phi(x)\pi(x)R_n(x) \rangle \neq 0.$$

Now, by Theorem 2.5 (ii),
$$\langle \sigma, \phi(x)\pi(x)R_n(x) \rangle = \sum_{j=n-t}^{n+t} r_{nj} \langle \sigma, \phi(x)P_j(x) \rangle$$
so that $\deg(\phi) = \ell \geq n - t$. □

We now ask: When is the symmetric bilinear form $\langle \cdot, \cdot \rangle_{\lambda, \mu}$ induced by a moment functional? That is, when is there a moment functional $\tau$ such that
$$\langle f, g \rangle_{\lambda, \mu} = \langle \tau, fg \rangle, \ f \text{ and } g \in \Pi?$$

**LEMMA 2.7.** If both $L[\cdot]$ and $M[\cdot]$ are linear algebra homomorphisms, then
$$\langle f, g \rangle_{\lambda, \mu} = \langle \sigma, fg \rangle + \lambda \langle \delta(x - a), L[fg] \rangle + \mu \langle \delta(x - b), M[fg] \rangle = \langle \tau, fg \rangle$$
where $\tau$ is a moment functional defined by
$$\langle \tau, f \rangle = \langle \sigma, f \rangle + \lambda \langle \delta(x - a), L[f] \rangle + \mu \langle \delta(x - b), M[f] \rangle, \ f \in \Pi.$$

If $L[\cdot] : \Pi \rightarrow \Pi$ is any linear algebra homomorphism, then for any
$$f(x) = \sum_{k=0}^{n} a_k x^k \text{ in } \Pi$$
$$L[f] = \sum_{k=0}^{n} a_k L[x^k] = a_0 L[1] + \sum_{k=1}^{n} a_k L[x]^k$$

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so that \( L[\cdot] \) is completely determined by \( L[1] \) and \( L[x] \). Moreover, since
\( L[x] = L[x]L[1] \), either \( L[x] = 0 \) or \( L[1] = 1 \).

If \( L[1] = 0 \), then \( L[f] = a_0 L[1] = f(0)L[1] \) so that \( L[xf(x)] = 0, \ f \in \Pi \).
If \( L[1] = 1 \), then \( L[f] = a_0 + \sum_{k=1}^{n} a_k L[x]^k = \sum_{k=0}^{n} a_k L[x]^k = L(f[x]) \) so that \( L([x-a)f(x)] = 0, f \in \Pi \).

Hence, if both \( L[\cdot] \) and \( M[\cdot] \) are linear algebra homomorphisms so that
\( \langle f, g \rangle_{\lambda, \mu} = \langle \tau, fg \rangle \), then there exists a polynomial \( \pi(x) \) of degree \( t \), \( 1 \leq t \leq 2 \), such that (2.12) holds. To be precise, we have

\[
\pi(x) = \begin{cases} 
  x, & \text{if } L[x] = M[x] = 0 \\
  x(x-b), & \text{if } L[x] = 0 \text{ and } M[1] = 1 \\
  (x-a)x, & \text{if } L[1] = 1 \text{ and } M[x] = 0 \\
\end{cases}
\]

(2.16)

THEOREM 2.8. Assume that both \( L[\cdot] \) and \( M[\cdot] \) are linear algebra homomorphisms so that \( \langle \cdot, \cdot \rangle_{\lambda, \mu} \) is induced by a moment functional \( \tau \) in (2.15). Let

\[
P_{n+1}(x) = (x-b_n)P_n(x) - c_n P_{n-1}(x), \quad n \geq 0,
\]

where \( b_n \) and \( c_n \) are real numbers and \( c_n \neq 0 \) for \( n \geq 1 \), be the three term recurrence relation for MOPS \( \{P_n(x)\}_{n=0}^{\infty} \) relative to \( \sigma \). Then the MOPS \( \{R_n(x)\}_{n=0}^{\infty} \) relative to \( \tau \) satisfy a three term recurrence relation

\[
R_{n+1}(x) = (x-\beta_n)R_n(x) - \gamma_n R_{n-1}(x), \quad n \geq 0,
\]

where

\[
\beta_n = b_{n+t} + r_{n,n+t-1} - r_{n+1,n+t}, \quad n \geq 0; \quad \gamma_n = \frac{d_n d_{n+1}}{d_{n+t}}, \quad n \geq 1.
\]

Proof. We have (2.14) : \( \pi(x)R_n(x) = \sum_{j=n-t}^{n+t} r_{nj} P_j(x), \quad n \geq 0 \), where \( r_{nj} = 0 \) for \( j < 0 \) and \( \pi(x) \) is the one in (2.16). Multiplying (2.17) by \( \pi(x) \) and applying (2.14), we obtain

\[
\sum_{j=n+1-t}^{n+1+t} r_{n+1,j} P_j(x) = (x-\beta_n) \sum_{j=n-t}^{n+t} r_{nj} P_j(x) - \gamma_n \sum_{j=n-1-t}^{n-1+t} r_{n-1,j} P_j(x).
\]

(2.18)

Multiplying (2.18) by \( P_{n+t}(x) \) and applying \( \sigma \), we have

\[
\tau_{n+1,n+t}(\sigma, P_{n+t}^2) = \tau_{n,n+t}(\sigma, s P_{n+t}^2) - \beta_n \tau_{n,n+t}(\sigma, P_{n+t}^2) + \tau_{n,n+t-1}(\sigma, s P_{n+t-1}P_{n+t}),
\]

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which gives

\[ r_{n+1,n+t} = \frac{\langle \sigma, xP_{n+1}^2 \rangle}{\langle \sigma, P_{n+1}^2 \rangle} - \beta_n + r_{n,n+t-1} = b_{n+t} - \beta_n + r_{n,n+t-1}, \quad n \geq 0 \]

since \( b_n = \frac{\langle \sigma, xP_n^2 \rangle}{\langle \sigma, P_n^2 \rangle} \) and \( \langle \sigma, xP_nP_{n+1} \rangle = \langle \sigma, P_{n+1}^2 \rangle \).

On the other hand, we have by (2.7)

\[ \gamma_n = \frac{\langle \tau, R_{n-1}^2 \rangle}{\langle \tau, R_{n-1}^2 \rangle} = \frac{d_n}{d_{n-1}} \frac{\langle \sigma, P_n^2 \rangle}{\langle \sigma, P_{n-1}^2 \rangle} - \frac{1}{d_{n-1}} \frac{\langle \sigma, P_{n-1}^2 \rangle}{\langle \sigma, P_{n-1}^2 \rangle} = \frac{d_n^{-2}d_n}{d_{n-1}^2} c_n, \quad n \geq 1. \]

\[ \square \]

Theorem 2.8 was proved in [14, Theorem 4.5] when \( L[\cdot] = M[\cdot] = Id \).

**Theorem 2.9.** Assume that both \( L[\cdot] \) and \( M[\cdot] \) are linear algebra homomorphisms so that \( \langle \cdot, \cdot \rangle_{\lambda,\mu} \) is induced by a moment functional \( \tau \) in (2.15). If \( \sigma \) is semiclassical satisfying \( (\alpha\sigma)' = \beta\sigma \), where \( \alpha(x) \) and \( \beta(x) \) are polynomials with \( \text{deg}(\beta) \geq 1 \), then \( \tau \) is also semiclassical and satisfies

\[ (\pi \alpha \tau)' = (\pi' \alpha + \pi \beta)\sigma, \]

where \( \pi(x) \) is the one in (2.16).

**Proof.** We have for any \( f \in \Pi \)

\[ \langle (\pi \alpha \tau)', f \rangle = -\langle \pi \alpha \tau, f' \rangle \]
\[ = -\langle \sigma, \alpha f' \rangle - \lambda \delta(x - a), L[\pi \alpha f'] \rangle - \mu \langle \delta(x - b), M[\pi \alpha f'] \rangle \]
\[ = \langle (\pi \alpha \sigma)', f \rangle = \langle (\pi' \alpha + \pi \beta)\sigma, f \rangle \]

so that \( (\pi \alpha \tau)' = (\pi' \alpha + \pi \beta)\sigma \). \[ \square \]

In case \( L[\cdot] = M[\cdot] = Id \), the class number of \( \tau \) is computed in [14, Section 5].

### 3. Examples

Almost all the previously known examples are concerned with the symmetric bilinear form \( \langle \cdot, \cdot \rangle_{\lambda,\mu} \), where linear operators \( L[\cdot] \) and \( M[\cdot] \) are of the same kind, e.g., \( L[\cdot] = M[\cdot] = Id \) or \( L[\cdot] = D^r, M[\cdot] = D^s \) or \( L[\cdot] = M[\cdot] = \Delta \). Here, we give some interesting non-standard examples.
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**Example 3.1.** Consider a symmetric bilinear form defined by
\[
(f, g)_\lambda = \langle \sigma, fg \rangle + \lambda L[f](a)L[g](a),
\]
where \(L[f](x) = f(x^2)\). Then \(L[\pi \phi](a) = 0, \phi \in \Pi\), where \(\pi(x) = x - a^2\).
Let \(\{P_n(x)\}_{n=0}^{\infty}\) and \(\{R_n(x)\}_{n=0}^{\infty}\) be the MOPS's relative to \(\sigma\) and \(\langle \cdot, \cdot \rangle_\lambda\), respectively. Then by Corollary 2.4, we obtain
\[
d_n = 1 + \lambda \sum_{j=0}^{n} \frac{(L[P_j](a))^2}{\langle \sigma, P_j^2 \rangle} = 1 + \lambda \sum_{j=0}^{n} \frac{(P_j(a))^2}{\langle \sigma, P_j^2 \rangle}.
\]
Thus if \(\lambda \neq -(\sum_{j=0}^{n} \frac{(P_j(a))^2}{\langle \sigma, P_j^2 \rangle})^{-1}\), then \(\langle \cdot, \cdot \rangle_\lambda\) is quasi-definite, and
\[
R_n(x) = P_n(x) - \lambda \frac{P_n(a^2)}{d_{n-1}} \sum_{j=0}^{n-1} \frac{P_j(a^2)}{\langle \sigma, P_j^2(x) \rangle} P_j(x).
\]
If moreover, \(\sigma\) is positive-definite, then \(R_n(x)\) has at least \(n - 2\) nodal zeros.

Now, consider another bilinear form defined by
\[
(f, g)_{\lambda, \mu} = \langle \sigma, fg \rangle + \lambda L[f](a)L[g](a) + \mu L[f](b)L[g](b),
\]
where \(L[\cdot]\) is the same as above. Assume that \(\sigma\) is positive-definite. If \(a^2 = b^2\), then \(\langle \pi f, g \rangle_{\lambda, \mu} = \langle f, \pi g \rangle_{\lambda, \mu}\), \(f\) and \(g \in \Pi\), for \(\pi(x) = x - a^2\) and \(R_n(x)\) has at least \(n - 2\) nodal zeros. If \(a^2 \neq b^2\), then \(\langle f, g \rangle_{\lambda, \mu} = \langle f, \pi g \rangle_{\lambda, \mu}\), \(f\) and \(g \in \Pi\), for \(\pi(x) = (x - a^2)(x - b^2)\) and \(R_n(x)\) has at least \(n - 4\) nodal zeros.

**Example 3.2.** Let \(\{P_n(x)\}_{n=0}^{\infty}\) be a Bochner-Krall OPS relative to \(\sigma\) satisfying
\[
L_N[P_n](x) = \sum_{i=0}^{N} \ell_i(x)P_n^{(i)}(x) = \lambda_n P_n(x), \quad n \geq 0,
\]
where \(\ell_i(x) = \sum_{j=0}^{i} \ell_{ij}x^j\) is a polynomial of degree \(\leq i\), \(\ell_N(x) \neq 0\), and
\[
\lambda_n = \ell_{11}n + \ell_{22}n(n - 1) + \cdots + \ell_{NN}n(n - 1)\cdots(n - N + 1)
\]
is the eigenvalue parameter. Note here that \(N\) must be an even integer (cf. [12, 13]). We now consider
\[
\langle f, g \rangle_\lambda = \langle \sigma, fg \rangle + \lambda L_N[f](a)L_N[g](a), \quad f \text{ and } g \in \Pi.
\]
Then

$$\langle \pi f, g \rangle_\lambda = \langle \sigma, \pi f g \rangle = \langle f, \pi g \rangle_\lambda, \text{ } f \text{ and } g \in \Pi$$

for \( \pi(x) = (x - a)^{N+1} \) and

$$d_n = 1 + \lambda \sum_{j=0}^{n} \frac{(L_N[x_j]^2(a))^2}{\langle \sigma, P_j^2 \rangle} = 1 + \lambda \sum_{j=0}^{n} \frac{\lambda_j P_j^2(a)}{\langle \sigma, P_j^2 \rangle}, \quad n \geq 0.$$ 

Hence, if \( \lambda \neq -\left(\sum_{j=0}^{n} \frac{\lambda_j P_j^2(a)}{\langle \sigma, P_j^2 \rangle}\right)^{-1}, \quad n \geq 0, \) then \( \langle \cdot, \cdot \rangle_\lambda \) is quasi-definite and the corresponding MOPS \( \{R_n(x)\}_{n=0}^{\infty} \) is given by

$$R_n(x) = P_n(x) - \frac{\lambda \lambda_n P_n(a)}{d_{n-1}} \sum_{j=0}^{n-1} \frac{\lambda_j P_j(a)}{\langle \sigma, P_j^2 \rangle} P_j(x), \quad n \geq 0.$$ 

If moreover, \( \sigma \) is positive-definite, then \( R_n(x) \) has at least \( n - 2N - 2 \) real nodal zeros. In particular, let's take \( a = 0 \) and

$$\langle \sigma, f \rangle = \int_0^\infty x^\alpha e^{-x} f(x) \, dx \quad (\alpha > -1)$$

so that \( \{P_n(x)\}_{n=0}^{\infty} = \{L_n^{(\alpha)}(x)\}_{n=0}^{\infty} \) is the Laguerre polynomials:

$$L_n^{(\alpha)}(x) = (-a)^n n! \sum_{j=0}^{n} \binom{n+\alpha}{n-j} \frac{(-x)^j}{j!}, \quad n \geq 0$$

satisfying

$$xL_n^{(\alpha)}(x)'' + (\alpha + 1 - x)L_n^{(\alpha)}(x)' = -nL_n^{(\alpha)}(x), \quad n \geq 0.$$ 

In this case, the symmetric bilinear form (3.1) becomes

$$\langle f, g \rangle_\lambda = \langle \sigma, fg \rangle + \lambda(\alpha + 1)^2 f'(0)g'(0), \quad f \text{ and } g \in \Pi.$$ 

Since \( L_n^{(\alpha)}(0) = (-1)^n n! \binom{n+\alpha}{n} \) and \( \langle \sigma, (L_n^{(\alpha)}(x))^2 \rangle = (n!)^2, \quad d_n = 1 + \lambda \sum_{j=0}^{n} j^2 (j+\alpha), \quad n \geq 0. \) Hence, if \( \lambda \neq -\left(\sum_{j=0}^{n} j^2 (j+\alpha)^2\right)^{-1}, \quad n \geq 0, \) then the MOPS \( \{R_n(x)\}_{n=0}^{\infty} \) relative to \( \langle \cdot, \cdot \rangle_\lambda \) in (3.2) is given by

$$R_n(x) = L_n^{(\alpha)}(x) + (-1)^n \frac{n \lambda n! \binom{n+\alpha}{n}}{d_{n-1}} \sum_{j=0}^{n-1} \frac{(-1)^{j+1} j^2 (j+\alpha)}{j!} L_j^{(\alpha)}(x), \quad n \geq 0.$$ 

Moreover, \( R_n(x) \) has at least \( n - 2 \) nodal zeros in \((0, \infty)\) since

$$\langle x^2 f, g \rangle_\lambda = \langle \sigma, x^2 fg \rangle = \langle f, x^2 g \rangle_\lambda, \quad f \text{ and } g \in \Pi.$$
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**Example 3.3.** Let \( \sigma \) be a positive-definite moment functional defined by

\[
\langle \sigma, f(x) \rangle = \sum_{x=0}^{\infty} \frac{e^{-\mu} \mu^x}{x!} f(x), \ f(x) \in \Pi (\mu > 0).
\]

Then the corresponding MOPS is the Charlier polynomials \( \{C_n^{(\mu)}(x)\}_{n=0}^{\infty} \) ([3, 16]):

\[
C_n^{(\mu)}(x) = \sum_{j=0}^{n} \frac{n!}{(n-j)!} (-\mu)^{n-j} \binom{x}{j}, \ n \geq 0
\]

satisfying

\[
\langle \sigma, (C_n^{(\mu)}(x))^2 \rangle = \mu^n n!, \ n \geq 0
\]

and

\[
x \Delta \nabla C_n^{(\mu)}(x) + (\mu - x) \Delta C_n^{(\mu)}(x) = -n C_n^{(\mu)}(x), \ n \geq 0,
\]

where \( \Delta f(x) = f(x+1) - f(x) \) and \( \nabla f(x) = f(x) - f(x-1) \) are forward and backward difference operators.

We first consider a symmetric bilinear form defined by

\[
\langle f, g \rangle_\lambda = \langle \sigma, fg \rangle + \lambda \Delta^r f(0) \Delta^r g(0),
\]

where \( r \geq 1 \) is an integer. From the facts that \( C_n^{(\mu)}(0) = (-\mu)^n \) and

\[
\Delta C_n^{(\mu)}(x) = n C_{n-1}^{(\mu)}(x), \ n \geq 0,
\]

we have

\[
d_n = \begin{cases} 
1, & \text{if } 0 \leq n < r \\
1 + \lambda \sum_{j=r}^{n} \frac{j! \mu^{j-2r}}{(j-r)!^2}, & \text{if } n \geq r.
\end{cases}
\]

Hence, if \( \lambda \neq -\left( \sum_{j=r}^{n} \frac{j! \mu^{j-2r}}{(j-r)!^2} \right)^{-1} \), \( n \geq r \), then \( \langle \cdot, \cdot \rangle_\lambda \) is quasi-definite and the corresponding MOPS \( \{R_n^{(\mu,r)}(x)\}_{n=0}^{\infty} \) is given by

\[
R_n^{(\mu,r)}(x) = \begin{cases} 
C_n^{(\mu)}(x), & \text{if } 0 \leq n < r \\
C_n^{(\mu)}(x) - \frac{\lambda^r \binom{n}{r} \mu^{n-r}}{d_{n-1}} \sum_{j=r}^{n-1} \frac{(-1)^{j-r}}{\mu^r (j-r)!} C_j^{(\mu)}(x), & \text{if } n \geq r.
\end{cases}
\]

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Moreover, since \( \langle \pi f, g \rangle_\lambda = \langle \sigma, \pi f g \rangle = \langle f, \pi g \rangle_\lambda \), where
\[
\pi(x) = x(x-1) \cdots (x-r)
\]
\( R^{(\mu,r)}_n(x) \) has at least \( n - 2r - 1 \) nodal zeros in (0, \( \infty \)). OPS's \( \{R^{(\mu,0)}_n\}_{n=0}^\infty \) and \( \{R^{(\mu,1)}_n\}_{n=0}^\infty \) for \( \lambda > 0 \) (note that in these cases, \( \langle \cdot, \cdot \rangle_\lambda \) in (3.4) is always positive-definite) were already considered by Bavinck and Koekoek [3] and Bavinck [1], respectively. They express \( R^{(\mu,r)}_n(x) \) for \( r = 0, 1 \) in terms of \( C^{(\mu)}_n(x) \), \( C^{(\mu)}_{n-1}(x-1) \), and \( C^{(\mu)}_{n-2}(x-2) \) and find infinite order difference equations having them as eigenfunctions.

Now, consider another symmetric bilinear form defined by
\[
\langle f, g \rangle_\lambda = \langle \sigma, fg \rangle + \lambda L[f](0)L[g](0), \quad f \text{ and } g \in \Pi
\]
where \( \sigma \) is the Charlier moment functional as in (3.3) and \( L[\cdot] \) is a hypergeometric type difference operator given by
\[
L[y](x) = \mu \Delta^2 y(x) - (x + 1 - \mu) \Delta y(x).
\]
Then using (3.5) and the three term recurrence relation for \( \{C^{(\mu)}_n(x)\}_{n=0}^\infty \):
\[
C^{(\mu)}_{n+1}(x) = [x - (\mu + n)]C^{(\mu)}_n(x) - \mu n C^{(\mu)}_{n-1}(x), \quad n \geq 0
\]
we have
\[
L[C^{(\mu)}_n](x) = -n \Delta C^{(\mu)}_n(x) - n C^{(\mu)}_n(x), \quad n \geq 0
\]
so that
\[
d_n = \begin{cases} 
1, & \text{if } n = 0 \\
1 + \lambda \sum_{j=1}^{n} \frac{j \mu^{j-2}(j-\mu)}{(j-1)!}, & \text{if } n \geq 1.
\end{cases}
\]
Hence, if \( \lambda \neq \left( \sum_{j=1}^{n} \frac{j \mu^{j-2}(j-\mu)}{(j-1)!} \right)^{-1} \), \( n \geq 1 \), then \( \langle \cdot, \cdot \rangle_\lambda \) in (3.6) is quasideterminate and the corresponding MOPS \( \{R_n(x)\}_{n=0}^\infty \) is given by
\[
R_n(x) = \begin{cases} 
C^{(\mu)}_0(x) = 1, & \text{if } n = 0 \\
C^{(\mu)}_n(x) + \frac{\lambda n(n+\mu)(-\mu)^{n-1}}{d_n} \sum_{j=1}^{n-1} \frac{(-1)^j(j-\mu)}{\mu(j-1)!} C^{(\mu)}_j(x), & \text{if } n \geq 1.
\end{cases}
\]
Moreover, since \( \langle \pi f, g \rangle_\lambda = \langle \sigma, \pi f g \rangle = \langle f, \pi g \rangle_\lambda \), where
\[
\pi(x) = x(x-1)(x-2),
\]
\( R_n(x) \) has at least \( n - 5 \) nodal zeros in (0, \( \infty \)).
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4. Operator Equations of Infinite Order

In this section, we consider a symmetric bilinear form

\[ (f, g)_\lambda = (\sigma, fg) + \lambda L^*[f](a)L^*[g](a), \]

where \( L[\cdot] \) is a linear operator with \( \deg(L[f]) \leq \deg(f) - 1 \), \( \lambda \) and \( a \) are real numbers and \( r \) is a positive integer. We assume that the MOPS \( \{P_n(x)\}_{n=0}^\infty \) are eigenfunctions of another linear operator \( M[\cdot] \), that is,

\[ M[P_n](x) = \lambda_n P_n(x), \quad n \geq 0. \]

By setting for \( p \) and \( q = 0, 1, 2, \ldots \)

\[ K^{(r,q)}_n(x, y) = \sum_{j=0}^n \frac{L^p[P_j](x) L^q[P_j](y)}{(\sigma, P_j^2)}, \]

we have from Corollary 2.4 that \( \langle \cdot, \cdot \rangle_\lambda \) is quasi-definite if and only if

\[ 1 + \lambda K^{(r,r)}_n(a, a) \neq 0, \quad n \geq 0. \]

We always assume that the condition (4.2) holds so that \( \langle \cdot, \cdot \rangle_\lambda \) is quasi-definite and let \( \{R_n(x)\}_{n=0}^\infty \) be the MOPS relative to \( \langle \cdot, \cdot \rangle_\lambda \). Then

\[ R_n(x) = P_n(x) - \frac{\lambda L^*[P_n](a)}{1 + \lambda K^{(r,r)}_{n-1}(a, a)} K^{(r,r)}_{n-1}(a, x), \quad n \geq 0. \]

In the following, all the summations are understood to be equal to 0 if the upper limit of the sum is less than the lower limit of the sum.

**Theorem 4.1.** The MOPS \( \{R_n(x)\}_{n=0}^\infty \) relative to \( \langle \cdot, \cdot \rangle_\lambda \) in (4.1) satisfies the following operator equations

\[ \lambda \left\{ \sum_{i=1}^\infty \alpha_i(x)L^i[y](x) + \alpha_0(x, n)y(x) \right\} + M[y](x) - \lambda_n y(x) = 0, \]

where

\[ \alpha_i(x) = \frac{-1}{L^i[P_i](x)} \left\{ \alpha_0(x, i) P_i(x) + \sum_{j=1}^{i-1} \alpha_j(x)L^j[P_i](x) \\
+ L^*[P_i](a) \sum_{j=i-r}^{i-1} \frac{(\lambda_i - \lambda_j)L^*[P_j](a)P_j(x)}{(\sigma, P_j^2)} \right\}, \quad i \geq 1 \]
(4.6) \[
\alpha_0(x, n) = \begin{cases} 
0, & n = 0 \\
\text{arbitrary constant}, & 1 \leq n \leq r \\
\alpha_0(x, n - 1) - K_{n-1}^{(r,r)}(a, a)(\lambda_n - \lambda_{n-1}) \\
= \alpha_0(x, r) - \sum_{i=r}^{n-1} K_{i}^{(r,r)}(a, a)(\lambda_{i+1} - \lambda_i), & n \geq r + 1.
\end{cases}
\]

Proof. Note that deg(α_i) ≤ i and α_i(x) is independent of n, n ≥ 1. Substituting \{1 + λK_{n-1}^{(r,r)}(a, a)\}R_n(x) for y in (4.4) gives

\[
\{1 + λK_{n-1}^{(r,r)}(a, a)\} \times \lambda \sum_{i=1}^{\infty} \left\{ α_i(x)L^i[R_n](x) + M[R_n](x) - μ_nR_n(x) \right\}
\]

\[
= \lambda \left\{ α_0(x, n)P_n(x) + \sum_{i=1}^{\infty} α_i(x)L^i[P_n](x) + L^r[P_n](a) \right. \\
\times \sum_{i=r}^{n-1} \frac{(\lambda_n - \lambda_i)L^r[P_i](a)P_i(x)}{\langle σ, P_i^2 \rangle} \left. \right\} + λ^2 \left\{ α_0(x, n)K_{n-1}^{(r,r)}(a, a)P_n(x) \\
+ K_{n-1}^{(r,r)}(a, a) \sum_{i=1}^{\infty} α_i(x)L^i[P_n](x) - α_0(x, n)L^r[P_n](a)K_{n-1}^{(r,0)}(a, x) \\
- L^r[P_n](a) \sum_{i=1}^{\infty} α_i(x)L^i[K_{n-1}^{(r,0)}(a, x)] \right\} = 0.
\]

Since λ can be any real number satisfying 1 + λK_n^{(r,r)}(a, a) ≠ 0, n ≥ 0, (4.7) is equivalent to

(4.8) \[
α_0(x, n)P_n(x) + \sum_{i=1}^{\infty} α_i(x)L^i[P_n](x) \\
+ L^r[P_n](a) \sum_{i=r}^{n-1} \frac{(\lambda_n - \lambda_i)L^r[P_i](a)P_i(x)}{\langle σ, P_i^2 \rangle} = 0
\]
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and

\[(4.9) \quad K^{(r,r)}_{n-1}(a, a) \left\{ \alpha_0(x, n) P_n(x) + \sum_{i=1}^{\infty} \alpha_i(x) L^i[P_n](x) \right\} \]

\[-L^r[P_n](a) \left\{ \alpha_0(x, n) K^{(r,0)}_{n-1}(a, x) + \sum_{i=1}^{\infty} \alpha_i(x) K^{(r,i)}_{n-1}(a, x) \right\} = 0 \]

for all \(x \in \mathbb{R}\) and \(n \geq 0\). Thus to prove this theorem, it is sufficient to show that \(\{\alpha_i(x)\}_{i=0}^{\infty}\) defined by (4.5) and (4.6) satisfy (4.8) and (4.9).

Multiplying (4.8) by \(K^{(r,r)}_{n-1}(a, a)\) and then subtracting (4.9) gives

\[L^r[P_n](a) \left\{ \alpha_0(x, n) K^{(r,0)}_{n-1}(a, x) + \sum_{i=1}^{\infty} \alpha_i(x) K^{(r,i)}_{n-1}(a, x) \right\} + K^{(r,r)}_{n-1}(a, a) \sum_{i=r}^{n-1} \frac{(\lambda_n - \lambda_i) L^l[P_i](a) P_i(x)}{\langle \sigma, P^2_i \rangle} = 0.\]

Hence it is sufficient to show that \(\{\alpha_i(x)\}_{i=0}^{\infty}\) satisfy (4.8) and

\[\alpha_0(x, n) K^{(r,0)}_{n-1}(a, x) + \sum_{i=1}^{\infty} \alpha_i(x) K^{(r,i)}_{n-1}(a, x) \]

\[+ K^{(r,r)}_{n-1}(a, a) \sum_{i=r}^{n-1} \frac{(\lambda_n - \lambda_i) L^l[P_i](a) P_i(x)}{\langle \sigma, P^2_i \rangle} = 0.\]

In fact, (4.8) is equivalent to (4.5) and (4.10) holds trivially for \(0 \leq n \leq r\).

Assume that (4.10) holds up to \(n = m\). Note that

\[(4.10) \quad K^{(p,q)}_m(x, y) = K^{(p,q)}_{m-1}(x, y) + \frac{L^p[P_m](x)L^q[P_m](y)}{\langle \sigma, P^2_m \rangle}.\]
For \( n = m + 1 \), the left-hand side of (4.10) becomes by (4.6) and (4.10)

\[
\begin{align*}
\alpha_0(x, m + 1)K_m^{(r,0)}(a, x) &+ \sum_{i=1}^{\infty} \alpha_i(x)K_m^{(r,i)}(a, x) \\
+ K_m^{(r,r)}(a, a) \sum_{i=r}^{m} \frac{\lambda_{m+1} - \lambda_i}{\langle \sigma, P_i^2 \rangle} L^*[P_i](a)P_i(x) &+ \sum_{i=r}^{m} \frac{\lambda_m - \lambda_i}{\langle \sigma, P_i^2 \rangle} L^*[P_i](a)P_i(x) \\
= & \left\{ \alpha_0(x, m) - K_m^{(r,r)}(a, a)\right\}_{m+1} - \lambda_m \} K_m^{(r,0)}(a, x) \\
&+ \sum_{i=1}^{\infty} \alpha_i(x)K_m^{(r,i)}(a, x) + K_m^{(r,r)}(a, a) \\
&\left\{ \sum_{i=r}^{m} \frac{\lambda_{m+1} - \lambda_m}{\langle \sigma, P_i^2 \rangle} L^*[P_i](a)P_i(x) + \sum_{i=r}^{m-1} \frac{\lambda_m - \lambda_i}{\langle \sigma, P_i^2 \rangle} L^*[P_i](a)P_i(x) \right\} \\
&+ \sum_{i=r}^{m} \frac{\lambda_m - \lambda_i}{\langle \sigma, P_i^2 \rangle} L^*[P_i](a)P_i(x) \\
&= \alpha_0(x, m)K_{m-1}^{(r,0)}(a, x) + \sum_{i=1}^{\infty} \alpha_i(x)K_{m-1}^{(r,i)}(a, x) \\
&+ K_{m-1}^{(r,r)}(a, a) \sum_{i=r}^{m-1} \frac{\lambda_m - \lambda_i}{\langle \sigma, P_i^2 \rangle} L^*[P_i](a)P_i(x) + \sum_{i=r}^{m-1} \frac{\lambda_m - \lambda_i}{\langle \sigma, P_i^2 \rangle} L^*[P_i](a)P_i(x) \\
&+ \sum_{i=r}^{m} \frac{\lambda_m - \lambda_i}{\langle \sigma, P_i^2 \rangle} L^*[P_i](a)P_i(x) \\
&\left\{ \alpha_0(x, m)P_m(x) + \sum_{i=1}^{\infty} \alpha_i(x)L^*[P_i](a)P_i(x) \\
&+ \sum_{i=r}^{m-1} \frac{\lambda_m - \lambda_i}{\langle \sigma, P_i^2 \rangle} L^*[P_i](a)P_i(x) \right\} \\
\end{align*}
\]

which is equal to 0 by the induction hypothesis for \( n = m \) and (4.10).
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**Example 4.1.** Consider a bilinear form \( \langle \cdot, \cdot \rangle_\lambda \) as in (3.4). Then,

\[
K^{(r,r)}_n(0,0) = \sum_{j=r}^{n} \frac{j! \mu^{j-2r}}{(j-r)!^2}.
\]

Since \( \{C_n^{(\mu)}(x)\}_{n=0}^\infty \) satisfy a hypergeometric type difference equation

\[
x\Delta \nabla y(x) + (\mu - x)\Delta y(x) = -ny(x),
\]

\( \{P_n^{(\mu,r)}(x)\}_{n=0}^\infty \) satisfy

\[
\lambda \left\{ \sum_{i=1}^{\infty} \alpha_i(x) \Delta^i y(x) + \alpha_0(x, n)y(x) \right\}
\]

\[
+ x\Delta \nabla y(x) + (\mu - x)\Delta y(x) + ny(x) = 0,
\]

where

\[
\alpha_i(x) = \frac{-1}{i!} \left\{ \alpha_0(x, i) C_i^{(\mu)}(x) + \sum_{j=1}^{i-1} \alpha_j(x) \Delta^j C_i^{(\mu)}(x) \right.
\]

\[
+ \frac{i!(-\mu)^{i-r}}{(i-r)!} \sum_{j=r}^{i-1} \frac{(j-i) C_j^{(\mu)}(x)}{\mu^j (j-r)!} \}
\]

\[
= \frac{-1}{i!} \left\{ \alpha_0(x, i) C_i^{(\mu)}(x) + \sum_{j=1}^{i-1} \frac{i!}{(i-j)!} \alpha_j(x) C_{i-j}^{(\mu)}(x) \right.
\]

\[
+ \frac{i!(-\mu)^{i-r}}{(i-r)!} \sum_{j=r}^{i-1} \frac{(j-i) C_j^{(\mu)}(x)}{\mu^j (j-r)!} \}, \quad i \geq 1
\]

and

\[
\alpha_0(x, n) = \begin{cases} 
0, & n = 0 \\
\text{arbitrary}, & 1 \leq n \leq r \\
\alpha_0(x, r) + \sum_{i=r}^{n-1} \sum_{j=r}^{i-1} \frac{j! \mu^{j-2r}}{(j-r)! (j-r)!}, & n \geq r + 1.
\end{cases}
\]
As a special case, if we choose $r = 0$, then
\[
\alpha_i(x) = \frac{-1}{i!} \left\{ \alpha_0(x, i) C_i^{(\mu)}(x) + \sum_{j=1}^{i-1} \frac{i!}{(i-j)!} \alpha_i(x) C_{i-j}^{(\mu)}(x) \right. \\
+ \frac{i!(-\mu)^i}{i!} \sum_{j=0}^{i-1} \frac{(j-i) C_j^{(\mu)}(x)}{\mu^j j!} \left. \right\}.
\]

Bavinck and Koekoek [3] have found a difference equation of infinite order for \( \{R_n^{(\mu, \beta)}(x)\}_{n=0}^{\infty} \) (see also [1] for the case $r = 1$).

**EXAMPLE 4.2.** Consider a bilinear form
\[
\langle f, g \rangle_\lambda = \langle \sigma, fg \rangle + \lambda f^{(r)}(0)g^{(r)}(0),
\]
where $\sigma$ is a positive-definite moment functional defined by
\[
\langle \sigma, f \rangle = \int_{0}^{\infty} f(x)x^{\alpha}e^{-x} \, dx, \quad (\alpha > -1)
\]
so that \( \{P_n(x)\}_{n=0}^{\infty} = \{L_n^{(\alpha)}(x)\}_{n=0}^{\infty} \) is the monic Laguerre polynomials (see Example 3.2). Then,
\[
K_n^{(r,r)}(0,0) = \sum_{j=0}^{n} \frac{(L_j^{(\alpha)})^{(r)}(0)(L_j^{(\alpha)})^{(r)}(0)}{\langle \sigma, (L_j^{(\alpha)}(x))^2 \rangle} = \sum_{j=0}^{n-r} \binom{j + r + \alpha}{r + \alpha}^2.
\]

Since \( \{L_n^{(\alpha)}(x)\}_{n=0}^{\infty} \) satisfies a second order differential equation
\[
xy''(x) + (\alpha + 1 - x)y'(x) = -ny(x), \quad n \geq 0,
\]
the corresponding MOPS \( \{R_n(x)\}_{n=0}^{\infty} \) relative to \( \langle \cdot, \cdot \rangle_\lambda \) satisfy
\[
\lambda \left\{ \sum_{i=1}^{\infty} \alpha_i(x) y^{(i)}(x) + \alpha_0(x, n)y(x) \right\} \\
+ xy''(x) + (\alpha + 1 - x)y'(x) + ny(x) = 0,
\]

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where

\[ \alpha_i(x) = \frac{-1}{i!} \left\{ \alpha_0(x, i)L_i^{(\alpha)}(x) + \sum_{j=1}^{i-1} \alpha_j(x)(L_i^{(\alpha)})^{(j)}(x) \right\} \]

\[ + (L_i^{(\alpha)})^{(r)}(0) \sum_{j=r}^{i-1} \frac{(j-i)(L_j^{(\alpha)})^{(r)}(0)L_i^{(\alpha)}(x)}{(j!)^2} \}

\[ = \frac{-1}{i!} \left\{ \alpha_0(x, i)L_i^{(\alpha)}(x) + \sum_{j=1}^{i-1} \frac{i!}{(i-j)!} \alpha_j(x)L_{i-j}^{(\alpha)}(x) \right\} \]

\[ + i!(-1)^{i-r} \left( \begin{array}{c} i + \alpha \\ r + \alpha \end{array} \right) \sum_{j=r}^{i-1} \frac{(j-i)(-1)^{j-r}(j+\alpha)L_j^{(\alpha)}(x)}{j!} \} \]

and

\[ \alpha_0(x, n) = \begin{cases} 0, & n = 0 \\ \text{arbitrary}, & 1 \leq n \leq r \\ \alpha_0(x, r) + \sum_{i=r}^{n-1} \sum_{j=0}^{i-r} \left( \begin{array}{c} j + r + \alpha \\ r + \alpha \end{array} \right)^2, & n \geq r + 1. \end{cases} \]

As a special case, if we choose \( r = 0 \), then

\[ \alpha_i(x) = \frac{-1}{i!} \left\{ \alpha_0(x, i)L_i^{(\alpha)}(x) + \sum_{j=1}^{i-1} \frac{i!}{(i-j)!} \alpha_j(x)L_i^{(\alpha)}(x) \right\} \]

\[ + i!(-1)^i \left( \begin{array}{c} i + \alpha \\ \alpha \end{array} \right) \sum_{j=0}^{i-1} \frac{(j-i)(-1)^{j+\alpha}L_j^{(\alpha)}(x)}{j!}. \]

J. Koekoek and R. Koekoek [9] have found a differential equation of infinite order when \( r = 0 \).

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References


[11] T. H. Koornwinder, Orthogonal Polynomials with weight function \((1 - x)^\alpha(1 + x)^\beta + M\delta(x - 1) + N\delta(x + 1)

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