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ORTHOGONAL POLYNOMIALS RELATIVE TO LINEAR PERTURBATIONS OF QUASI-DEFINITE MOMENT FUNCTIONALS

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ABSTRACT. Consider a symmetric bilinear form defined on $\Pi \times \Pi$ by

$$\langle f,g\rangle_{\lambda,\mu} = \langle \sigma, fg\rangle + \lambda L[f](a)L[g](a) + \mu M[f](b)M[g](b),$$

where σ is a quasi-definite moment functional, L and M are linear operators on Π, the space of all real polynomials and a, b, λ, and μ are real constants. We find a necessary and sufficient condition for the above bilinear form to be quasi-definite and study various properties of corresponding orthogonal polynomials. This unifies many previous works which treated cases when both L and M are differential or difference operators. Finally, infinite order operator equations having such orthogonal polynomials as eigenfunctions are given when $\mu = 0$.

1. Introduction

All polynomials in this work are assumed to be real polynomials in one variable and we let Π be the space of all real polynomials. We let $\deg(\pi)$ be the degree of a polynomial $\pi(x)$ with the convention $\deg(0) = -1$. By a polynomial system (PS), we mean a sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ with $\deg(P_n) = n$, $n \geq 0$.

Let L and M be any two linear operators defined on Π. We now consider a symmetric bilinear form on $\Pi \times \Pi$ given by

(1.1)
$$\langle f,g\rangle_{\lambda,\mu} := \langle \sigma, fg\rangle + \lambda L[f](a)L[g](a) + \mu M[f](b)M[g](b), \ (f,g \in \Pi)$$

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where σ is a moment functional (i.e., a linear functional on Π) and λ, μ, a, and b are real numbers. We say that the bilinear form $\langle\cdot,\cdot\rangle_{\lambda,\mu}$ is quasi-definite (respectively, positive-definite) if

$$\Delta_n := \begin{vmatrix} \langle 1,1\rangle_{\lambda,\mu} & \langle 1,x\rangle_{\lambda,\mu} & \cdots & \langle 1,x^n\rangle_{\lambda,\mu} \\ \langle x,1\rangle_{\lambda,\mu} & \langle x,x\rangle_{\lambda,\mu} & \cdots & \langle x,x^n\rangle_{\lambda,\mu} \\ \vdots & \vdots & \ddots & \vdots \\ \langle x^n,1\rangle_{\lambda,\mu} & \langle x^n,x\rangle_{\lambda,\mu} & \cdots & \langle x^n,x^n\rangle_{\lambda,\mu} \end{vmatrix} \neq 0,$$

$$(\text{respectively}, \quad \Delta_n > 0) \qquad n \geq 0.$$

When $\langle\cdot,\cdot\rangle_{\lambda,\mu}$ is quasi-definite (respectively, positive-definite), there is a unique monic PS $\{R_n(x)\}_{n=0}^{\infty}$ such that

$$\langle R_m, R_n\rangle_{\lambda,\mu} = K_n\delta_{mn}, \quad m \text{ and } n \geq 0,$$

where K_n is a non-zero (respectively, a positive) constant and vice versa. In this case, we call $\{R_n(x)\}_{n=0}^{\infty}$ a monic orthogonal polynomial system (MOPS) relative to $\langle\cdot,\cdot\rangle_{\lambda,\mu}$ (or a MOPS relative to σ when $\lambda = \mu = 0$). When $L = M = Id$, the identity operator,

$$\langle f,g\rangle_{\lambda,\mu} = \langle \sigma + \lambda\delta(x-a) + \mu\delta(x-b), fg\rangle, \ f \text{ and } g \in \Pi$$

is just a point mass perturbation of σ, which is already handled by many authors ([5, 6, 11, 14, 15]). When L and M are differential or difference operators, $\{R_n(x)\}_{n=0}^{\infty}$ is called a Sobolev-type orthogonal polynomials (see [1, 7, 8] and references therein). Generalizing these examples, we consider arbitrary linear operators L and M by which we perturb σ. We first find a necessary and sufficient condition for $\langle\cdot,\cdot\rangle_{\lambda,\mu}$ to be quasi-definite (see Theorem 2.2), which extends results in [6, 8, 14], in which $L = M = Id$ or $\mu = 0$. We then express $\{R_n(x)\}_{n=0}^{\infty}$ in terms of orthogonal polynomials $\{P_n(x)\}_{n=0}^{\infty}$ relative to σ and discuss their algebraic properties such as long term recurrence relation, quasi-orthogonality, and semi-classical character, including several non-standard examples. Finally, when $\mu = 0$ and $\{P_n(x)\}_{n=0}^{\infty}$ are eigenfunctions of a certain linear operator on Π, we show that $\{R_n(x)\}_{n=0}^{\infty}$ must also be eigenfunctions of a possibly infinite order operator equation.

This last result (see Theorem 4.1) explains many previous works ([1, 3, 7, 9, 10]) in a unified manner. Bavinck [2] considered a similar problem in a different viewpoint: perturb not the moment functional but the PS $\{P_n(x)\}_{n=0}^{\infty}$ (which need not be orthogonal) by another PS $\{Q_n(x)\}_{n=0}^{\infty}$.

2. Necessary and Sufficient Conditions

We always assume that σ is quasi-definite and $\{P_n(x)\}_{n=0}^{\infty}$ is the MOPS relative to σ. Let $\langle \cdot, \cdot \rangle_{\lambda,\mu}$ be the bilinear form as in (1.1). We set for p and $q = 0, 1, \cdots$

$$K_n^{(p,q)}(x,y) = \sum_{j=0}^{n} \frac{L^p[P_j](x) L^q[P_j](y)}{\langle \sigma, P_j^2 \rangle} \tag{2.1}$$

$$G_n^{(p,q)}(x,y) = \sum_{j=0}^{n} \frac{L^p[P_j](x) M^q[P_j](y)}{\langle \sigma, P_j^2 \rangle} \tag{2.2}$$

$$J_n^{(p,q)}(x,y) = \sum_{j=0}^{n} \frac{M^p[P_j](x) M^q[P_j](y)}{\langle \sigma, P_j^2 \rangle}, \tag{2.3}$$

where $L^{p+1} = L[L^p]$ and $M^{p+1} = M[M^p]$. Then

$$K_n^{(0,q)}(x,y) = G_n^{(q,0)}(y,x) \text{ and } J_n^{(0,q)}(x,y) = G_n^{(0,q)}(x,y).$$

PROPOSITION 2.1. *The kernels $K_n^{(p,q)}(x,y)$, $G_n^{(p,q)}(x,y)$ and $J_n^{(p,q)}(x,y)$ as in (2.1) $\sim$ (2.3) have reproducing properties, i.e.,*

$$\langle \sigma, K_n^{(0,q)}(x,y)\psi(x) \rangle = \langle \sigma, G_n^{(q,0)}(y,x)\psi(x) \rangle = L^q[\psi](y) \tag{2.4}$$

$$\langle \sigma, J_n^{(0,q)}(x,y)\psi(x) \rangle = \langle \sigma, G_n^{(0,q)}(x,y)\psi(x) \rangle = M^q[\psi](y) \tag{2.5}$$

for any polynomial $\psi(x)$ of degree $\leq n$.

Proof. We set $\psi(x) = \sum_{k=0}^{n} c_k P_k(x)$. Then

$$\langle \sigma, K_n^{(0,q)}(x,y)\psi(x) \rangle = \langle \sigma, G_n^{(q,0)}(y,x)\psi(x) \rangle = \sum_{j=0}^{n} \frac{L^q[P_j](y)}{\langle \sigma, P_j^2 \rangle} \langle \sigma, P_j \psi \rangle$$

$$= \sum_{j=0}^{n} \frac{L^q[P_j](y)}{\langle \sigma, P_j^2 \rangle} \langle \sigma, P_j \sum_{k=0}^{n} c_k P_k \rangle = \sum_{j=0}^{n} c_j L^q[P_j](y) = L^q[\psi](y)$$

by the orthogonality of $\{P_n(x)\}_{n=0}^{\infty}$ relative to σ. Hence we have (2.4). (2.5) can be proved similarly. □

Now, we have:

THEOREM 2.2. *The bilinear form* $\langle \cdot, \cdot \rangle_{\lambda,\mu}$ *in* (1.1) *is quasi-definite if and only if*

$$d_n := \begin{vmatrix} 1+\lambda K_n^{(1,1)}(a,a) & \mu G_n^{(1,1)}(a,b) \\ \lambda G_n^{(1,1)}(a,b) & 1+\mu J_n^{(1,1)}(b,b) \end{vmatrix} \neq 0, \quad n \geq 0.$$

In this case, MOPS $\{R_n(x)\}_{n=0}^\infty$ *is given by*

(2.6)

$$\begin{aligned} R_n(x) = P_n(x) &- \tfrac{\lambda}{d_{n-1}} \begin{vmatrix} L[P_n](a) & \mu G_{n-1}^{(1,1)}(a,b) \\ M[P_n](b) & 1+\mu J_{n-1}^{(1,1)}(b,b) \end{vmatrix} G_{n-1}^{(1,0)}(a,x) \\ &- \tfrac{\mu}{d_{n-1}} \begin{vmatrix} 1+\lambda K_{n-1}^{(1,1)}(a,a) & L[P_n](a) \\ \lambda G_{n-1}^{(1,1)}(a,b) & M[P_n](b) \end{vmatrix} G_{n-1}^{(0,1)}(x,b), \quad n \geq 0, \end{aligned}$$

where $d_{-1} = 1$.
Moreover, we have

$$\langle R_n, R_n \rangle_{\lambda,\mu} = \langle R_n, P_n \rangle_{\lambda,\mu} = \frac{d_n}{d_{n-1}} \langle \sigma, P_n^2 \rangle. \tag{2.7}$$

Proof. Assume that the bilinear form $\langle \cdot, \cdot \rangle_{\lambda,\mu}$ in (1.1) is quasi-definite and let $\{R_n(x)\}_{n=0}^\infty$ be the MOPS relative to $\langle \cdot, \cdot \rangle_{\lambda,\mu}$. Then we may write $R_n(x)$ as

$$R_n(x) = P_n(x) + \sum_{j=0}^{n-1} C_j^n P_j(x), \quad n \geq 0.$$

By the orthogonality of $\{R_n(x)\}_{n=0}^\infty$ and $\{P_n(x)\}_{n=0}^\infty$, we have for $0 \leq j \leq n-1$

$$C_j^n = \frac{\langle \sigma, P_j R_n \rangle}{\langle \sigma, P_j^2 \rangle} = \frac{-\lambda L[R_n](a) L[P_j](a) - \mu M[R_n](b) M[P_j](b)}{\langle \sigma, P_j^2 \rangle}$$

and so

$$R_n(x) = P_n(x) - \lambda L[R_n](a) G_{n-1}^{(1,0)}(a,x) - \mu M[R_n](b) G_{n-1}^{(0,1)}(x,b). \tag{2.8}$$

Acting L and M to (2.8) and evaluating at $x = a$ and $x = b$ respectively, we obtain

(2.9)

$$\begin{pmatrix} 1+\lambda K_{n-1}^{(1,1)}(a,a) & \mu G_{n-1}^{(1,1)}(a,b) \\ \lambda G_{n-1}^{(1,1)}(a,b) & 1+\mu J_{n-1}^{(1,1)}(b,b) \end{pmatrix} \begin{pmatrix} L[R_n](a) \\ M[R_n](b) \end{pmatrix} = \begin{pmatrix} L[P_n](a) \\ M[P_n](b) \end{pmatrix},$$

$$n \geq 0,$$

where $G_{-1}^{(1,1)}(x,y) = J_{-1}^{(1,1)}(x,y) = 0$.

We now show that $d_n \neq 0$ for $n \geq 0$ by induction on n. For $n = 0$,

$$d_0 = 1 + \mu\frac{(M[1](b))^2}{\langle\sigma, 1\rangle} + \lambda\frac{(L[1](a))^2}{\langle\sigma, 1\rangle} = \frac{\langle 1, 1\rangle_{\lambda,\mu}}{\langle\sigma, 1\rangle} \neq 0.$$

Assume that $d_n \neq 0$ for $0 \leq n \leq m$ for some integer $m \geq 0$. Then, the system (2.9) is uniquely solvable for $L[R_n](a)$ and $M[R_n](b)$ as

$$(2.10)\quad \begin{pmatrix} L[R_n](a) \\ M[R_n](b) \end{pmatrix} = \frac{1}{d_{n-1}}\begin{pmatrix} 1 + \mu J_{n-1}^{(1,1)}(b,b) & -\mu G_{n-1}^{(1,1)}(a,b) \\ -\lambda G_{n-1}^{(1,1)}(a,b) & 1 + \lambda K_{n-1}^{(1,1)}(a,a) \end{pmatrix}\begin{pmatrix} L[P_n](a) \\ M[P_n](b) \end{pmatrix}$$

for $0 \leq n \leq m+1$. Substituting (2.10) into (2.8), we obtain (2.6). Hence

(2.11)

$$\begin{aligned}\langle R_n, P_k\rangle_{\lambda,\mu} &= \langle\sigma, R_nP_k\rangle + \lambda L[R_n](a)L[P_k](a) + \mu M[R_n](b)M[P_k](b) \\ &= \langle\sigma, P_nP_k\rangle + \lambda L[R_n](a)L[P_k](a)\delta_{kn} + \mu M[R_n](b)M[P_k](b)\delta_{kn} \\ &= \tfrac{d_n}{d_{n-1}}\langle\sigma, P_n^2\rangle\delta_{kn},\ 0 \leq k \leq n \leq m+1\end{aligned}$$

since

$$\langle\sigma, K_{n-1}^{(0,1)}(x,a)P_k(x)\rangle = L[P_k](a)(1-\delta_{kn}),$$
$$\langle\sigma, J_{n-1}^{(0,1)}(x,b)P_k(x)\rangle = L[P_k](b)(1-\delta_{kn}),$$

and

$$\begin{aligned}d_n = d_{n-1} \quad &+ \quad \mu\frac{\{1 + \lambda K_{n-1}^{(1,1)}(a,a)\}(M[P_n](b))^2}{\langle\sigma, P_n^2\rangle} \\ &+ \quad \lambda\frac{\{1 + \mu J_{n-1}^{(1,1)}(b,b)\}(L[P_n](a))^2}{\langle\sigma, P_n^2\rangle}.\end{aligned}$$

In particular,

$$d_{m+1} = \frac{\langle R_{m+1}, R_{m+1}\rangle_{\lambda,\mu}}{\langle\sigma, P_{m+1}^2\rangle}d_m \neq 0.$$

Conversely, assume $d_n \neq 0$, $n \geq 0$ and define $R_n(x)$ by (2.6). Then $\{R_n(x)\}_{n=0}^{\infty}$ is a monic PS and (2.8) holds. Then the equation (2.11) implies that $\{R_n(x)\}_{n=0}^{\infty}$ is the MOPS relative to $\langle\cdot,\cdot\rangle_{\lambda,\mu}$ so that $\langle\cdot,\cdot\rangle_{\lambda,\mu}$ is quasi-definite. □

COROLLARY 2.3. *If σ is positive-definite, then $\langle\cdot,\cdot\rangle_{\lambda,\mu}$ is also positive-definite if and only if $d_n > 0$, $n \geq 0$.*

Some special cases of Theorem 2.2 were handled in [6, 8, 14, 15] when $L = Id$ and $\mu = 0$ or $L = D^r$, $M = D^s$, where $D = \frac{d}{dx}$ and r and s are non-negative integers.

COROLLARY 2.4. *When $\mu = 0$, $\langle\cdot,\cdot\rangle_\lambda := \langle\cdot,\cdot\rangle_{\lambda,\mu}$ is quasi-definite if and only if $d_n := 1 + \lambda K_n^{(1,1)}(a,a) \neq 0$, $n \geq 0$. In this case, we have*

$$R_n(x) = P_n(x) - \frac{\lambda L[P_n](a)}{1 + \lambda K_{n-1}^{(1,1)}(a,a)} G_{n-1}^{(1,0)}(a,x)$$

and

$$\langle R_n, R_n\rangle_\lambda = \frac{1 + \lambda K_n^{(1,1)}(a,a)}{1 + \lambda K_{n-1}^{(1,1)}(a,a)} \langle\sigma, P_n^2\rangle,\ n \geq 0.$$

From now on, we always assume that $\langle\cdot,\cdot\rangle_{\lambda,\mu}$ is quasi-definite and let $\{R_n(x)\}_{n=0}^\infty$ be the MOPS relative to $\langle\cdot,\cdot\rangle_{\lambda,\mu}$.
If there exists a polynomial $\pi(x)$ of degree $t(\geq 1)$ such that

$$L[\pi\phi](a) = M[\pi\phi](b) = 0,\ \phi \in \Pi, \tag{2.12}$$

then

$$\langle\pi\phi, \psi\rangle_{\lambda,\mu} = \langle\sigma, \pi\phi\psi\rangle = \langle\phi, \pi\psi\rangle_{\lambda,\mu},\ \phi \text{ and } \psi \in \Pi. \tag{2.13}$$

For example, we have:
If $L = D^r$, where $D = \frac{d}{dx}$ and $r \geq 0$, then

$$L[(x-a)^{r+1} f(x)](a) = 0,\ f \in \Pi.$$

If $L = \Delta^r$, where $\Delta f(x) = f(x+1) - f(x)$ is the forward difference operator and $r \geq 0$, then

$$L\left[\prod_{k=0}^{r}(x-a-k)f(x)\right](a) = 0,\ f \in \Pi.$$

THEOREM 2.5. *Assume that there exists a monic polynomial $\pi(x)$ of degree $t(\geq 1)$ satisfying (2.12). Then*

(i) (*Long term recurrence relation*)

$$\pi(x)R_n(x) = \sum_{j=n-t}^{n+t} s_{nj}R_j(x)$$

where $s_{nj} = \frac{\langle R_n,\pi R_j\rangle_{\lambda,\mu}}{\langle R_j,R_j\rangle_{\lambda,\mu}} = \frac{\langle \sigma,R_n\pi R_j\rangle}{\langle\sigma,R_j^2\rangle}\frac{d_{j-1}}{d_j}$, $n-t \le j \le n+t$ ($s_{n,n+t}=1$, $s_{n,n-t} \neq 0$, $n \ge t$);

(ii) (*Quasi-orthogonality relative to* σ)

$$\pi(x)R_n(x) = \sum_{j=n-t}^{n+t} r_{nj}P_j(x), \tag{2.14}$$

where $r_{nj} = \frac{\langle\sigma,R_n\pi R_j\rangle}{\langle\sigma,R_j^2\rangle}$, $n-t \le j \le n+t$ ($r_{n,n+t}=1$, $r_{n,n-t}\neq 0$, $n \ge t$).

Proof. Since $\deg(\pi R_n) \le n+t$, we may write it as

$$\pi(x)R_n(x) = \sum_{j=0}^{n+t} s_{nj}R_j(x), \qquad n \ge 0.$$

Multiplying by $R_k(x)$ and applying $\langle\cdot,\cdot\rangle_{\lambda,\mu}$ on both sides, we have

$$s_{nk}\langle R_k,R_k\rangle_{\lambda,\mu} = \langle\sigma,R_n\pi R_k\rangle = \langle R_n,\pi R_k\rangle_{\lambda,\mu}$$

so that $s_{nk}=0$, $0 \le k < n-t$, $s_{n,n+1}=1$, and

$$s_{nk} = \frac{\langle R_n,\pi R_k\rangle_{\lambda,\mu}}{\langle R_k,R_k\rangle\lambda,\mu} = \frac{\langle\sigma,R_n\pi R_k\rangle_{\lambda,\mu}}{\langle\sigma,P_k^2\rangle}\frac{d_{k-1}}{d_k},\ n-t \le k \le n+t$$

by (2.13) and (2.14). In Particular,

$$s_{n,n-t} = \frac{\langle R_n,\pi R_{n-t}\rangle_{\lambda,\mu}}{\langle R_{n-t},R_{n-t}\rangle_{\lambda,\mu}} = \frac{\langle R_n,R_n\rangle_{\lambda,\mu}}{\langle R_{n-t},R_{n-t}\rangle_{\lambda,\mu}} \neq 0.$$

Hence, we have (i). For (ii), the proof is essentially the same as the one for (i). □

Note that

$$\frac{s_{n-i,n-j}}{\langle R_{n-i},R_{n-i}\rangle_{\lambda,\mu}} = \frac{s_{n-j,n-i}}{\langle R_{n-j},R_{n-j}\rangle_{\lambda,\mu}}$$

and

$$\frac{r_{n-i,n-j}}{\langle\sigma,P_i^2\rangle} = \frac{r_{n-j,n-i}}{\langle\sigma,P_i^2\rangle},\ 0 \le i \le n-t,\ i-t \le j \le i+t.$$

It is well known (cf. [4]) that if σ is positive-definite, then $P_n(x)$, $n \geq 1$, has n real simple zeros.

COROLLARY 2.6. *Assume σ is positive-definite and let $[\xi, \eta]$ be the true interval of orthogonality of σ, that is, the smallest interval containing all zeros of $P_n(x)$, $n \geq 1$, in (ξ, η). If there exists a polynomial $\pi(x)$ satisfying (2.12), then $\pi(x)R_n(x)$ has at least $n - t$ nodal zeros, that is, zeros of odd multiplicity, in (ξ, η) so that $R_n(x)$ has at least $n - t - k$ nodal zeros in (ξ, η), where k is the number of zeros of $\pi(x)$ which have odd multiplicity in (ξ, η).*

Proof. Let $x_1 < x_2 < \cdots < x_\ell$ be the nodal zeros of $\pi(x)R_n(x)$ in (ξ, η) and $\phi(x) = \prod_{i=1}^{\ell}(x - x_i)$. Then either $\phi(x)\pi(x)R_n(x) \geq 0$ or $\phi(x)\pi(x)R_n(x) \leq 0$ on $[\xi, \eta]$ so that

$$\langle \sigma, \phi(x)\pi(x)R_n(x) \rangle \neq 0.$$

Now, by Theorem 2.5 (ii),

$$\langle \sigma, \phi(x)\pi(x)R_n(x) \rangle = \sum_{j=n-t}^{n+t} r_{nj} \langle \sigma, \phi(x)P_j(x) \rangle$$

so that $\deg(\phi) = \ell \geq n - t$. □

We now ask: When is the symmetric bilinear form $\langle \cdot, \cdot \rangle_{\lambda,\mu}$ induced by a moment functional? That is, when is there a moment functional τ such that

$$\langle f, g \rangle_{\lambda,\mu} = \langle \tau, fg \rangle, \ f \text{ and } g \in \Pi \ ?$$

LEMMA 2.7. *If both $L[\cdot]$ and $M[\cdot]$ are linear algebra homomorphisms, then*

$$\langle f, g \rangle_{\lambda,\mu} = \langle \sigma, fg \rangle + \lambda \langle \delta(x - a), L[fg] \rangle + \mu \langle \delta(x - b), M[fg] \rangle = \langle \tau, fg \rangle$$

where τ is a moment functional defined by

$$\langle \tau, f \rangle = \langle \sigma, f \rangle + \lambda \langle \delta(x - a), L[f] \rangle + \mu \langle \delta(x - b), M[f] \rangle, \ f \in \Pi. \tag{2.15}$$

If $L[\cdot] : \Pi \to \Pi$ is any linear algebra homomorphism, then for any $f(x) = \sum_{k=0}^{n} a_k x^k$ in Π

$$L[f] = \sum_{k=0}^{n} a_k L[x^k] = a_0 L[1] + \sum_{k=1}^{n} a_k L[x]^k$$

so that $L[\cdot]$ is completely determined by $L[1]$ and $L[x]$. Moreover, since $L[x] = L[x]L[1]$, either $L[x] = 0$ or $L[1] = 1$.
If $L[x] = 0$, then $L[f] = a_0 L[1] = f(0)L[1]$ so that $L[xf(x)] = 0$, $f \in \Pi$.
If $L[1] = 1$, then $L[f] = a_0 + \sum_{k=1}^{n} a_k L[x]^k = \sum_{k=0}^{n} a_k L[x]^k = f(L[x])$ so that $L[(x-a)f(x)] = 0$, $f \in \Pi$.
Hence, if both $L[\cdot]$ and $M[\cdot]$ are linear algebra homomorphisms so that $\langle f, g\rangle_{\lambda,\mu} = \langle \tau, fg\rangle$, then there exists a polynomial $\pi(x)$ of degree t, $1 \leq t \leq 2$, such that (2.12) holds. To be precise, we have

$$\pi(x) = \begin{cases} x, & \text{if } L[x] = M[x] = 0 \\ x(x-b), & \text{if } L[x] = 0 \text{ and } M[1] = 1 \\ (x-a)x, & \text{if } L[1] = 1 \text{ and } M[x] = 0 \\ (x-a)(x-b), & \text{if } L[1] = M[1] = 1. \end{cases} \tag{2.16}$$

THEOREM 2.8. *Assume that both $L[\cdot]$ and $M[\cdot]$ are linear algebra homomorphisms so that $\langle \cdot, \cdot\rangle_{\lambda,\mu}$ is induced by a moment functional τ in* (2.15). *Let*

$$P_{n+1}(x) = (x - b_n)P_n(x) - c_n P_{n-1}(x), \ n \geq 0,$$

where b_n and c_n are real numbers and $c_n \neq 0$ for $n \geq 1$, be the three term recurrence relation for MOPS $\{P_n(x)\}_{n=0}^{\infty}$ relative to σ. Then the MOPS $\{R_n(x)\}_{n=0}^{\infty}$ relative to τ satisfy a three term recurrence relation

$$R_{n+1}(x) = (x - \beta_n)R_n(x) - \gamma_n R_{n-1}(x), \ n \geq 0, \tag{2.17}$$

where

$$\beta_n = b_{n+t} + r_{n,n+t-1} - r_{n+1,n+t}, \ n \geq 0;$$
$$\gamma_n = \frac{d_{n-2}d_n}{d_{n-1}^2}, \ n \geq 1.$$

Proof. We have (2.14) : $\pi(x)R_n(x) = \sum_{j=n-t}^{n+t} r_{nj}P_j(x)$, $n \geq 0$, where $r_{nj} = 0$ for $j < 0$ and $\pi(x)$ is the one in (2.16). Multiplying (2.17) by $\pi(x)$ and applying (2.14), we obtain

$$\sum_{j=n+1-t}^{n+1+t} r_{n+1,j}P_j(x) = (x - \beta_n)\sum_{j=n-t}^{n+t} r_{nj}P_j(x) - \gamma_n \sum_{j=n-1-t}^{n-1+t} r_{n-1,j}P_j(x). \tag{2.18}$$

Multiplying (2.18) by $P_{n+t}(x)$ and applying σ, we have

$$r_{n+1,n+t}\langle \sigma, P_{n+t}^2\rangle = r_{n,n+t}\langle \sigma, xP_{n+t}^2\rangle - \beta_n r_{n,n+t}\langle \sigma, P_{n+t}^2\rangle + r_{n,n+t-1}\langle \sigma, xP_{n+t-1}P_{n+t}\rangle,$$

which gives

$$r_{n+1,,n+t} = \frac{\langle \sigma, xP_{n+t}^2 \rangle}{\langle \sigma, P_{n+t}^2 \rangle} - \beta_n + r_{n,n+t-1} = b_{n+t} - \beta_n + r_{n,n+t-1}, \ n \geq 0$$

since $b_n = \frac{\langle \sigma, xP_n^2 \rangle}{\langle \sigma, P_n^2 \rangle}$ and $\langle \sigma, xP_nP_{n+1} \rangle = \langle \sigma, P_{n+1}^2 \rangle$.
On the other hand, we have by (2.7)

$$\gamma_n = \frac{\langle \tau, R_n^2 \rangle}{\langle \tau, R_{n-1}^2 \rangle} = \frac{d_n}{d_{n-1}} \langle \sigma, P_n^2 \rangle \frac{d_{n-2}}{d_{n-1}} \frac{1}{\langle \sigma, P_{n-1}^2 \rangle} = \frac{d_{n-2}d_n}{d_{n-1}^2} c_n, \ n \geq 1.$$

□

Theorem 2.8 was proved in [14, Theorem 4.5] when $L[\cdot] = M[\cdot] = Id$.

THEOREM 2.9. *Assume that both $L[\cdot]$ and $M[\cdot]$ are linear algebra homomorphisms so that $\langle \cdot, \cdot \rangle_{\lambda,\mu}$ is induced by a moment functional τ in (2.15). If σ is semiclassical satisfying $(\alpha\sigma)' = \beta\sigma$, where $\alpha(x)$ and $\beta(x)$ are polynomials with $\deg(\beta) \geq 1$, then τ is also semiclassical and satisfies*

$$(\pi\alpha\tau)' = (\pi'\alpha + \pi\beta)\sigma,$$

where $\pi(x)$ is the one in (2.16).

Proof. We have for any $f \in \Pi$

$$\begin{aligned} &\langle (\pi\alpha\tau)', f \rangle = -\langle \pi\alpha\tau, f' \rangle \\ &= -\langle \sigma, \alpha f' \rangle - \lambda \langle \delta(x-a), L[\pi\alpha f'] \rangle - \mu \langle \delta(x-b), M[\pi\alpha f'] \rangle \\ &= \langle (\pi\alpha\sigma)', f \rangle = \langle (\pi'\alpha + \pi\beta)\sigma, f \rangle \end{aligned}$$

so that $(\pi\alpha\tau)' = (\pi'\alpha + \pi\beta)\sigma$. □

In case $L[\cdot] = M[\cdot] = Id$, the class number of τ is computed in [14, Section 5].

3. Examples

Almost all the previously known examples are concerned with the symmetric bilinear form $\langle \cdot, \cdot \rangle_{\lambda,\mu}$, where linear operators $L[\cdot]$ and $M[\cdot]$ are of the same kind, e.g., $L[\cdot] = M[\cdot] = Id$ or $L[\cdot] = D^r$, $M[\cdot] = D^s$ or $L[\cdot] = M[\cdot] = \Delta$. Here, we give some interesting non-standard examples.

EXAMPLE 3.1. Consider a symmetric bilinear form defined by

$$\langle f, g\rangle_\lambda = \langle \sigma, fg\rangle + \lambda L[f](a)L[g](a),$$

where $L[f](x) = f(x^2)$. Then $L[\pi\phi](a) = 0$, $\phi \in \Pi$, where $\pi(x) = x - a^2$. Let $\{P_n(x)\}_{n=0}^\infty$ and $\{R_n(x)\}_{n=0}^\infty$ be the MOPS's relative to σ and $\langle\cdot,\cdot\rangle_\lambda$, respectively. Then by Corollary 2.4, we obtain

$$d_n = 1 + \lambda \sum_{j=0}^{n} \frac{(L[P_j](a))^2}{\langle \sigma, P_j^2\rangle} = 1 + \lambda \sum_{j=0}^{n} \frac{(P_j(a))^2}{\langle \sigma, P_j^2\rangle}.$$

Thus if $\lambda \neq -(\sum_{j=0}^{n} \frac{(P_j(a))^2}{\langle\sigma, P_j^2\rangle})^{-1}$, then $\langle\cdot,\cdot\rangle_\lambda$ is quasi-definite, and

$$R_n(x) = P_n(x) - \lambda \frac{P_n(a^2)}{d_{n-1}} \sum_{j=0}^{n-1} \frac{P_j(a^2)P_j(x)}{\langle \sigma, P_j^2(x)\rangle}.$$

If moreover, σ is positive-definite, then $R_n(x)$ has at least $n-2$ nodal zeros.

Now, consider another bilinear form defined by

$$\langle f, g\rangle_{\lambda,\mu} = \langle \sigma, fg\rangle + \lambda L[f](a)L[g](a) + \mu L[f](b)L[g](b),$$

where $L[\cdot]$ is the same as above. Assume that σ is positive-definite. If $a^2 = b^2$, then $\langle \pi f, g\rangle_{\lambda,\mu} = \langle f, \pi g\rangle_{\lambda,\mu}$, f and $g \in \Pi$, for $\pi(x) = x - a^2$ and $R_n(x)$ has at least $n-2$ nodal zeros. If $a^2 \neq b^2$, then $\langle \pi f, g\rangle_{\lambda,\mu} = \langle f, \pi g\rangle_{\lambda,\mu}$, f and $g \in \Pi$, for $\pi(x) = (x-a^2)(x-b^2)$ and $R_n(x)$ has at least $n-4$ nodal zeros.

EXAMPLE 3.2. Let $\{P_n(x)\}_{n=0}^\infty$ be a Bochner-Krall OPS relative to σ satisfying

$$L_N[P_n](x) = \sum_{i=0}^{N} \ell_i(x) P_n^{(i)}(x) = \lambda_n P_n(x), \ n \geq 0,$$

where $\ell_i(x) = \sum_{j=0}^{i} \ell_{ij} x^j$ is a polynomial of degree $\leq i$, $\ell_N(x) \neq 0$, and

$$\lambda_n = \ell_{11} n + \ell_{22} n(n-1) + \cdots + \ell_{NN} n(n-1)\cdots(n-N+1)$$

is the eigenvalue parameter. Note here that N must be an even integer (cf. [12, 13]). We now consider

$$\langle f, g\rangle_\lambda = \langle \sigma, fg\rangle + \lambda L_N[f](a) L_N[g](a), \ f \text{ and } g \in \Pi. \tag{3.1}$$

Then

$$\langle \pi f, g\rangle_\lambda = \langle \sigma, \pi f g\rangle = \langle f, \pi g\rangle_\lambda,\ f \text{ and } g \in \Pi$$

for $\pi(x) = (x-a)^{N+1}$ and

$$d_n = 1 + \lambda \sum_{j=0}^{n} \frac{(L_N[P_j](a))^2}{\langle \sigma, P_j^2\rangle} = 1 + \lambda \sum_{j=0}^{n} \frac{\lambda_j^2 P_j^2(a)}{\langle \sigma, P_j^2\rangle},\ n \geq 0.$$

Hence, if $\lambda \neq -(\sum_{j=0}^{n} \frac{\lambda_j^2 P_j^2(a)}{\langle \sigma, P_j^2\rangle})^{-1}$, $n \geq 0$, then $\langle \cdot, \cdot\rangle_\lambda$ is quasi-definite and the corresponding MOPS $\{R_n(x)\}_{n=0}^{\infty}$ is given by

$$R_n(x) = P_n(x) - \frac{\lambda\lambda_n P_n(a)}{d_{n-1}} \sum_{j=0}^{n-1} \frac{\lambda_j P_j(a)}{\langle \sigma, P_j^2\rangle} P_j(x),\ n \geq 0.$$

If moreover, σ is positive-definite, then $R_n(x)$ has at least $n-2N-2$ real nodal zeros. In particular, let's take $a = 0$ and

$$\langle \sigma, f\rangle = \int_0^\infty x^\alpha e^{-x} f(x)\, dx\ (\alpha > -1)$$

so that $\{P_n(x)\}_{n=0}^{\infty} = \{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ is the Laguerre polynomials:

$$L_n^{(\alpha)}(x) = (-a)^n n! \sum_{j=0}^{n} \binom{n+\alpha}{n-j} \frac{(-x)^j}{j!},\ n \geq 0$$

satisfying

$$x L_n^{(\alpha)}(x)'' + (\alpha + 1 - x) L_n^{(\alpha)}(x)' = -n L_n^{(\alpha)}(x),\ n \geq 0.$$

In this case, the symmetric bilinear form (3.1) becomes

(3.2) $$\langle f, g\rangle_\lambda = \langle \sigma, fg\rangle + \lambda(\alpha+1)^2 f'(0) g'(0),\ f \text{ and } g \in \Pi.$$

Since $L_n^{(\alpha)}(0) = (-1)^n n! \binom{n+\alpha}{n}$ and $\langle \sigma, (L_n^{(\alpha)}(x))^2\rangle = (n!)^2$, $d_n = 1 + \lambda \sum_{j=0}^{n} j^2 \binom{j+\alpha}{j}$, $n \geq 0$. Hence, if $\lambda \neq -(\sum_{j=0}^{n} j^2 \binom{j+\alpha}{j}^2)^{-1}$, $n \geq 0$, then the MOPS $\{R_n(x)\}_{n=0}^{\infty}$ relative to $\langle \cdot, \cdot\rangle_\lambda$ in (3.2) is given by

$$R_n(x) = L_n^{(\alpha)}(x) + (-1)^n \lambda \frac{n n! \binom{n+\alpha}{n}}{d_{n-1}} \sum_{j=0}^{n-1} \frac{(-1)^{j+1} j \binom{j+\alpha}{j}}{j!} L_j^{(\alpha)}(x),\ n \geq 0.$$

Moreover, $R_n(x)$ has at least $n-2$ nodal zeros in $(0, \infty)$ since

$$\langle x^2 f, g\rangle_\lambda = \langle \sigma, x^2 f g\rangle = \langle f, x^2 g\rangle_\lambda,\ f \text{ and } g \in \Pi.$$

EXAMPLE 3.3. Let σ be a positive-definite moment functional defined by

$$\langle \sigma, f(x) \rangle = \sum_{x=0}^{\infty} \frac{e^{-\mu}\mu^x}{x!} f(x), \ f(x) \in \Pi \ (\mu > 0). \tag{3.3}$$

Then the corresponding MOPS is the Charlier polynomials $\{C_n^{(\mu)}(x)\}_{n=0}^{\infty}$ ([3, 16]):

$$C_n^{(\mu)}(x) = \sum_{j=0}^{n} \frac{n!}{(n-j)!}(-\mu)^{n-j}\binom{x}{j}, \ n \geq 0$$

satisfying

$$\langle \sigma, (C_n^{(\mu)}(x))^2 \rangle = \mu^n n!, \ n \geq 0$$

and

$$x\Delta\nabla C_n^{(\mu)}(x) + (\mu - x)\Delta C_n^{(\mu)}(x) = -nC_n^{(\mu)}(x), \ n \geq 0,$$

where $\Delta f(x) = f(x+1) - f(x)$ and $\nabla f(x) = f(x) - f(x-1)$ are forward and backward difference operators.

We first consider a symmetric bilinear form defined by

$$\langle f, g \rangle_\lambda = \langle \sigma, fg \rangle + \lambda \Delta^r f(0) \Delta^r g(0), \tag{3.4}$$

where $r \geq 1$ is an integer. From the facts that $C_n^{(\mu)}(0) = (-\mu)^n$ and

$$\Delta C_n^{(\mu)}(x) = nC_{n-1}^{(\mu)}(x), \quad n \geq 0, \tag{3.5}$$

we have

$$d_n = \begin{cases} 1, & \text{if } 0 \leq n < r \\ 1 + \lambda \displaystyle\sum_{j=r}^{n} \frac{j!\mu^{j-2r}}{((j-r)!)^2}, & \text{if } n \geq r. \end{cases}$$

Hence, if $\lambda \neq -(\sum_{j=r}^{n} \frac{j!\mu^{j-2r}}{((j-r)!)^2})^{-1}$, $n \geq r$, then $\langle \cdot, \cdot \rangle_\lambda$ is quasi-definite and the corresponding MOPS $\{R_n^{(\mu,r)}(x)\}_{n=0}^{\infty}$ is given by

$$R_n^{(\mu,r)}(x) = \begin{cases} C_n^{(\mu)}(x), & \text{if } 0 \leq n < r \\ C_n^{(\mu)}(x) - \dfrac{\lambda r! \binom{n}{r} \mu^{n-r}}{d_{n-1}} \displaystyle\sum_{j=r}^{n-1} \frac{(-1)^{j-r}}{\mu^r (j-r)!} C_j^{(\mu)}(x), & \text{if } n \geq r. \end{cases}$$

Moreover, since $\langle \pi f, g\rangle_\lambda = \langle \sigma, \pi f g\rangle = \langle f, \pi g\rangle_\lambda$, where

$$\pi(x) = x(x-1)\cdots(x-r)$$

$R_n^{(\mu,r)}(x)$ has at least $n-2r-1$ nodal zeros in $(0,\infty)$. OPS's $\{R_n^{(\mu,0)}\}_{n=0}^\infty$ and $\{R_n^{(\mu,1)}\}_{n=0}^\infty$ for $\lambda > 0$ (note that in these cases, $\langle\cdot,\cdot\rangle_\lambda$ in (3.4) is always positive-definite) were already considered by Bavinck and Koekoek [3] and Bavinck [1], respectively. They express $R_n^{(\mu,r)}(x)$ for $r = 0, 1$ in terms of $C_n^{(\mu)}(x)$, $C_{n-1}^{(\mu)}(x-1)$, and $C_{n-2}^{(\mu)}(x-2)$ and find infinite order difference equations having them as eigenfunctions.

Now, consider another symmetric bilinear form defined by

$$\langle f, g\rangle_\lambda = \langle \sigma, fg\rangle + \lambda L[f](0)L[g](0),\ f \text{ and } g \in \Pi \tag{3.6}$$

where σ is the Charlier moment functional as in (3.3) and $L[\cdot]$ is a hypergeometric type difference operator given by

$$L[y](x) = \mu\Delta^2 y(x) - (x+1-\mu)\Delta y(x).$$

Then using (3.5) and the three term recurrence relation for $\{C_n^{(\mu)}(x)\}_{n=0}^\infty$:

$$C_{n+1}^{(\mu)}(x) = [x-(\mu+n)]C_n^{(\mu)}(x) - \mu n C_{n-1}^{(\mu)}(x),\ n \geq 0$$

we have

$$L[C_n^{(\mu)}](x) = -n\Delta C_n^{(\mu)}(x) - nC_n^{(\mu)}(x),\ n \geq 0$$

so that

$$d_n = \begin{cases} 1, & \text{if } n = 0 \\ 1 + \lambda \displaystyle\sum_{j=1}^{n} \frac{j\mu^{j-2}(j-\mu)}{(j-1)!}, & \text{if } n \geq 1. \end{cases}$$

Hence, if $\lambda \neq (\sum_{j=1}^n \frac{j\mu^{j-2}(j-\mu)}{(j-1)!})^{-1}$, $n \geq 1$, then $\langle\cdot,\cdot\rangle_\lambda$ in (3.6) is quasi-definite and the corresponding MOPS $\{R_n(x)\}_{n=0}^\infty$ is given by

$$R_n(x) = \begin{cases} C_0^{(\mu)}(x) = 1, & \text{if } n = 0 \\ C_n^{(\mu)}(x) + \dfrac{\lambda n(n+\mu)(-\mu)^{n-1}}{d_{n-1}} \displaystyle\sum_{j=1}^{n-1} \frac{(-1)^j(j-\mu)}{\mu(j-1)!} C_j^{(\mu)}(x), & \text{if } n \geq 1. \end{cases}$$

Moreover, since $\langle \pi f, g\rangle_\lambda = \langle \sigma, \pi f g\rangle = \langle f, \pi g\rangle_\lambda$, where

$$\pi(x) = x(x-1)(x-2),$$

$R_n(x)$ has at least $n-5$ nodal zeros in $(0,\infty)$.

4. Operator Equations of Infinite Order

In this section, we consider a symmetric bilinear form

$$\langle f,g\rangle_\lambda = \langle \sigma, fg\rangle + \lambda L^r[f](a)L^r[g](a), \tag{4.1}$$

where $L[\cdot]$ is a linear operator with $\deg(L[f]) \le \deg(f)-1$, λ and a are real numbers and r is a positive integer. We assume that the MOPS $\{P_n(x)\}_{n=0}^\infty$ are eigenfunctions of another linear operator $M[\cdot]$, that is,

$$M[P_n](x) = \lambda_n P_n(x),\ n \ge 0.$$

By setting for p and $q = 0, 1, 2, \cdots$

$$K_n^{(p,q)}(x,y) = \sum_{j=0}^{n} \frac{L^p[P_j](x)\, L^q[P_j](y)}{\langle \sigma, P_j^2\rangle},$$

we have from Corollary 2.4 that $\langle\cdot,\cdot\rangle_\lambda$ is quasi-definite if and only if

$$1+\lambda K_n^{(r,r)}(a,a) \ne 0, \quad n \ge 0. \tag{4.2}$$

We always assume that the condition (4.2) holds so that $\langle\cdot,\cdot\rangle_\lambda$ is quasi-definite and let $\{R_n(x)\}_{n=0}^\infty$ be the MOPS relative to $\langle\cdot,\cdot\rangle_\lambda$. Then

$$R_n(x) = P_n(x) - \frac{\lambda L^r[P_n](a)}{1+\lambda K_{n-1}^{(r,r)}(a,a)} K_{n-1}^{(r,r)}(a,x),\ n \ge 0. \tag{4.3}$$

In the following, all the summations are understood to be equal to 0 if the upper limit of the sum is less than the lower limit of the sum.

THEOREM 4.1. *The MOPS $\{R_n(x)\}_{n=0}^\infty$ relative to $\langle\cdot,\cdot\rangle_\lambda$ in (4.1) satisfies the following operator equations*

$$\lambda\left\{\sum_{i=1}^{\infty} \alpha_i(x)L^i[y](x) + \alpha_0(x,n)y(x)\right\} + M[y](x) - \lambda_n y(x) = 0, \tag{4.4}$$

where

$$\alpha_i(x) = \frac{-1}{L^i[P_i](x)}\left\{\alpha_0(x,i)P_i(x) + \sum_{j=1}^{i-1}\alpha_j(x)L^j[P_i](x) + L^r[P_i](a)\sum_{j=r}^{i-1}\frac{(\lambda_i-\lambda_j)L^r[P_j](a)P_j(x)}{\langle\sigma,P_j^2\rangle}\right\}, \quad i \ge 1 \tag{4.5}$$

and

(4.6)

$$\alpha_0(x,n) = \begin{cases} 0, & n=0 \\ \text{arbitrary constant}, & 1 \le n \le r \\ \alpha_0(x,n-1) - K_{n-1}^{(r,r)}(a,a)(\lambda_n - \lambda_{n-1}) & \\ \quad = \alpha_0(x,r) - \sum\limits_{i=r}^{n-1} K_i^{(r,r)}(a,a)(\lambda_{i+1} - \lambda_i), & n \ge r+1. \end{cases}$$

Proof. Note that $\deg(\alpha_i) \le i$ and $\alpha_i(x)$ is independent of n, $n \ge 1$. Substituting $\{1 + \lambda K_{n-1}^{(r,r)}(a,a)\} R_n(x)$ for y in (4.4) gives

$$\begin{aligned} &\{1 + \lambda K_{n-1}^{(r,r)}(a,a)\} \times \lambda \sum_{i=1}^{\infty} \{\alpha_i(x) L^i[R_n](x) + M[R_n](x) - \mu_n R_n(x)\} \\ &= \lambda \Bigg\{ \alpha_0(x,n) P_n(x) + \sum_{i=1}^{\infty} \alpha_i(x) L^i[P_n](x) + L^r[P_n](a) \\ &\times \sum_{i=r}^{n-1} \frac{(\lambda_n - \lambda_i) L^r[P_i](a) P_i(x)}{\langle \sigma, P_i^2 \rangle} \Bigg\} + \lambda^2 \Bigg\{ \alpha_0(x,n) K_{n-1}^{(r,r)}(a,a) P_n(x) \\ &+ K_{n-1}^{(r,r)}(a,a) \sum_{i=1}^{\infty} \alpha_i(x) L^i[P_n](x) - \alpha_0(x,n) L^r[P_n](a) K_{n-1}^{(r,0)}(a,x) \\ &- L^r[P_n](a) \sum_{i=1}^{\infty} \alpha_i(x) L^i[K_{n-1}^{(r,0)}(a,x)] \Bigg\} = 0. \end{aligned} \tag{4.7}$$

Since λ can be any real number satisfying $1 + \lambda K_n^{(r,r)}(a,a) \neq 0$, $n \ge 0$, (4.7) is equivalent to

$$\begin{aligned} &\alpha_0(x,n) P_n(x) + \sum_{i=1}^{\infty} \alpha_i(x) L^i[P_n](x) \\ &+ L^r[P_n](a) \sum_{i=r}^{n-1} \frac{(\lambda_n - \lambda_i) L^r[P_i](a) P_i(x)}{\langle \sigma, P_i^2 \rangle} = 0 \end{aligned} \tag{4.8}$$

and

$$\text{(4.9)}\qquad K_{n-1}^{(r,r)}(a,a)\left\{\alpha_0(x,n)P_n(x)+\sum_{i=1}^{\infty}\alpha_i(x)L^i[P_n](x)\right\}$$
$$-L^r[P_n](a)\left\{\alpha_0(x,n)K_{n-1}^{(r,0)}(a,x)+\sum_{i=1}^{\infty}\alpha_i(x)K_{n-1}^{(r,i)}(a,x)\right\}=0$$

for all $x \in \mathbb{R}$ and $n \geq 0$. Thus to prove this theorem, it is sufficient to show that $\{\alpha_i(x)\}_{i=0}^{\infty}$ defined by (4.5) and (4.6) satisfy (4.8) and (4.9). Multiplying (4.8) by $K_{n-1}^{(r,r)}(a,a)$ and then subtracting (4.9) gives

$$L^r[P_n](a)\left\{\alpha_0(x,n)K_{n-1}^{(r,0)}(a,x)+\sum_{i=1}^{\infty}\alpha_i(x)K_{n-1}^{(r,i)}(a,x)\right.$$
$$\left.+K_{n-1}^{(r,r)}(a,a)\sum_{i=r}^{n-1}\frac{(\lambda_n-\lambda_i)L^r[P_i](a)P_i(x)}{\langle\sigma,P_i^2\rangle}\right\}=0.$$

Hence it is sufficient to show that $\{\alpha_i(x)\}_{i=0}^{\infty}$ satisfy (4.8) and

$$\alpha_0(x,n)K_{n-1}^{(r,0)}(a,x)+\sum_{i=1}^{\infty}\alpha_i(x)K_{n-1}^{(r,i)}(a,x)$$
$$+K_{n-1}^{(r,r)}(a,a)\sum_{i=r}^{n-1}\frac{(\lambda_n-\lambda_i)L^rP_i(a)P_i(x)}{\langle\sigma,P_i^2\rangle}=0.$$

In fact, (4.8) is equivalent to (4.5) and (4.10) holds trivially for $0 \leq n \leq r$. Assume that (4.10) holds up to $n = m$. Note that

$$\text{(4.10)}\qquad K_m^{(p,q)}(x,y)=K_{m-1}^{(p,q)}(x,y)+\frac{L^p[P_m](x)L^q[P_m](y)}{\langle\sigma,P_m^2\rangle}.$$

For $n = m+1$, the left-hand side of (4.10) becomes by (4.6) and (4.10)

$$\begin{aligned}
&\alpha_0(x,m+1)K_m^{(r,0)}(a,x) + \sum_{i=1}^{\infty} \alpha_i(x) K_m^{(r,i)}(a,x) \\
&+K_m^{(r,r)}(a,a)\sum_{i=r}^{m} \frac{(\lambda_{m+1}-\lambda_i)L^r[P_i](a)P_i(x)}{\langle \sigma, P_i^2\rangle} \\
=\ & \left\{\alpha_0(x,m) - K_m^{(r,r)}(a,a)(\lambda_{m+1}-\lambda_m)\right\} K_m^{(r,0)}(a,x) \\
&+\sum_{i=1}^{\infty} \alpha_i(x) K_m^{(r,i)}(a,x) + K_m^{(r,r)}(a,a) \\
&\cdot\left\{\sum_{i=r}^{m} \frac{(\lambda_{m+1}-\lambda_m)L^r[P_i](a)P_i(x)}{\langle \sigma, P_i^2\rangle} + \sum_{i=r}^{m-1} \frac{(\lambda_m-\lambda_i)L^r[P_i](a)P_i(x)}{\langle \sigma, P_i^2\rangle}\right\} \\
=\ & \alpha_0(x,m)K_m^{(r,0)}(a,x) + \sum_{i=1}^{\infty} \alpha_i(x) K_m^{(r,i)}(a,x) \\
&+K_m^{(r,r)}(a,a)\sum_{i=r}^{m-1} \frac{(\lambda_m-\lambda_i)L^r[P_i](a)P_i(x)}{\langle \sigma, P_i^2\rangle} \\
=\ & \alpha_0(x,m)K_{m-1}^{(r,0)}(a,x) + \sum_{i=1}^{\infty} \alpha_i(x) K_{m-1}^{(r,i)}(a,x) \\
&+K_{m-1}^{(r,r)}(a,a)\sum_{i=r}^{m-1} \frac{(\lambda_m-\lambda_i)L^r[P_i](a)P_i(x)}{\langle \sigma, P_i^2\rangle} + \frac{L^r[P_m](a)}{\langle \sigma, P_m^2\rangle} \\
&\cdot\left\{\alpha_0(x,m)P_m(x) + \sum_{i=1}^{\infty}\alpha_i(x)L^i[P_m](x)\right. \\
&\qquad \left. + L^r[P_m](a)\sum_{i=r}^{m-1} \frac{(\lambda_m-\lambda_i)L^r[P_i](a)P_i(x)}{\langle \sigma, P_i^2\rangle}\right\}
\end{aligned}$$

which is equal to 0 by the induction hypothesis for $n = m$ and (4.10).

EXAMPLE 4.1. Consider a bilinear form $\langle\cdot,\cdot\rangle_\lambda$ as in (3.4). Then,

$$K_n^{(r,r)}(0.0)=\sum_{j=r}^{n}\frac{j!\,\mu^{j-2r}}{((j-r)!)^2}.$$

Since $\{C_n^{(\mu)}(x)\}_{n=0}^{\infty}$ satisfy a hypergeometric type difference equation

$$x\Delta\nabla y(x)+(\mu-x)\Delta y(x)=-ny(x),$$

$\{R_n^{(\mu,r)}(x)\}_{n=0}^{\infty}$ satisfy

$$\begin{aligned}&\lambda\left\{\sum_{i=1}^{\infty}\alpha_i(x)\Delta^i y(x)+\alpha_0(x,n)y(x)\right\}\\&+x\Delta\nabla y(x)+(\mu-x)\Delta y(x)+ny(x)=0,\end{aligned}$$

where

$$\begin{aligned}\alpha_i(x)&=\frac{-1}{i!}\left\{\alpha_0(x,i)C_i^{(\mu)}(x)+\sum_{j=1}^{i-1}\alpha_j(x)\Delta^j C_i^{(\mu)}(x)\right.\\&\quad\left.+\frac{i!(-\mu)^{i-r}}{(i-r)!}\sum_{j=r}^{i-1}\frac{(j-i)C_j^{(\mu)}(x)}{\mu^j(j-r)!}\right\}\\&=\frac{-1}{i!}\left\{\alpha_0(x,i)C_i^{(\mu)}(x)+\sum_{j=1}^{i-1}\frac{i!}{(i-j)!}\alpha_i(x)C_{i-j}^{(\mu)}(x)\right.\\&\quad\left.+\frac{i!(-\mu)^{i-r}}{(i-r)!}\sum_{j=r}^{i-1}\frac{(j-i)C_j^{(\mu)}(x)}{\mu^j(j-r)!}\right\},\qquad i\geq 1\end{aligned}$$

and

$$\alpha_0(x,n)=\begin{cases}0, & n=0\\ \text{arbitrary}, & 1\leq n\leq r\\ \alpha_0(x,r)+\displaystyle\sum_{i=r}^{n-1}\sum_{j=r}^{i-1}\frac{j!\,\mu^{j-2r}}{(j-r)!\,(j-r)!}, & n\geq r+1.\end{cases}$$

As a special case, if we choose $r = 0$, then

$$\alpha_i(x) = \frac{-1}{i!}\Big\{\alpha_0(x,i)C_i^{(\mu)}(x) + \sum_{j=1}^{i-1} \frac{i!}{(i-j)!}\alpha_i(x)C_{i-j}^{(\mu)}(x) + \frac{i!(-\mu)^i}{i!}\sum_{j=0}^{i-1}\frac{(j-i)C_j^{(\mu)}(x)}{\mu^j j!}\Big\}.$$

Bavinck and Koekoek [3] have found a difference equation of infinite order for $\{R_n^{(\mu,0)}(x)\}_{n=0}^{\infty}$ (see also [1] for the case $r = 1$).

EXAMPLE 4.2. Consider a bilinear form

$$\langle f, g\rangle_\lambda = \langle \sigma, fg\rangle + \lambda f^{(r)}(0)g^{(r)}(0),$$

where σ is a positive-definite moment functional defined by

$$\langle \sigma, f\rangle = \int_0^\infty f(x)x^\alpha e^{-x}\,dx,\ (\alpha > -1)$$

so that $\{P_n(x)\}_{n=0}^{\infty} = \{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ is the monic Laguerre polynomials (see Example 3.2). Then,

$$K_n^{(r,r)}(0,0) = \sum_{j=0}^{n} \frac{(L_j^{(\alpha)})^{(r)}(0)(L_j^{(\alpha)})^{(r)}(0)}{\langle \sigma, (L_j^{(\alpha)}(x))^2} = \sum_{j=0}^{n-r}\binom{j+r+\alpha}{r+\alpha}^2.$$

Since $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ satisfies a second order differential equation

$$xy''(x) + (\alpha+1-x)y'(x) = -ny(x),\ n \geq 0,$$

the corresponding MOPS $\{R_n(x)\}_{n=0}^{\infty}$ relative to $\langle\cdot,\cdot\rangle_\lambda$ satisfy

$$\lambda\Big\{\sum_{i=1}^{\infty}\alpha_i(x)y^{(i)}(x) + \alpha_0(x,n)y(x)\Big\} + xy''(x) + (\alpha+1-x)y'(x) + ny(x) = 0,$$

where

$$\begin{aligned}\alpha_i(x) &= \frac{-1}{i!}\Bigg\{\alpha_0(x,i)L_i^{(\alpha)}(x)+\sum_{j=1}^{i-1}\alpha_j(x)(L_i^{(\alpha)})^{(j)}(x)\\ &\quad +(L_i^{(\alpha)})^{(r)}(0)\sum_{j=r}^{i-1}\frac{(j-i)(L_j^{(\alpha)})^{(r)}(0)L_j^{(\alpha)}(x)}{(j!)^2}\Bigg\}\\ &= \frac{-1}{i!}\Bigg\{\alpha_0(x,i)L_i^{(\alpha)}(x)+\sum_{j=1}^{i-1}\frac{i!}{(i-j)!}\alpha_j(x)L_{i-j}^{(\alpha)}(x)\\ &\quad +i!(-1)^{i-r}\binom{i+\alpha}{r+\alpha}\sum_{j=r}^{i-1}\frac{(j-i)(-1)^{j-r}\binom{j+\alpha}{r+\alpha}L_j^{(\alpha)}(x)}{j!}\Bigg\}\end{aligned}$$

and

$$\alpha_0(x,n)=\begin{cases}0, & n=0\\ \text{arbitrary}, & 1\le n\le r\\ \alpha_0(x,r)+\displaystyle\sum_{i=r}^{n-1}\sum_{j=0}^{i-r}\binom{j+r+\alpha}{r+\alpha}^2, & n\ge r+1.\end{cases}$$

As a special case, if we choose $r=0$, then

$$\begin{aligned}\alpha_i(x) &= \frac{-1}{i!}\Bigg\{\alpha_0(x,i)L_i^{(\alpha)}(x)+\sum_{j=1}^{i-1}\frac{i!}{(i-j)!}\alpha_j(x)L_{i-j}^{(\alpha)}(x)\\ &\quad +i!(-1)^{i}\binom{i+\alpha}{\alpha}\sum_{j=0}^{i-1}\frac{(j-i)(-1)^{j}\binom{j+\alpha}{\alpha}L_j^{(\alpha)}(x)}{j!}\Bigg\}.\end{aligned}$$

J. Koekoek and R. Koekoek [9] have found a differential equation of infinite order when $r=0$.

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