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CONDENSING MAPPINGS AND APPLICATIONS TO EXISTENCE THEOREMS FOR COMMON SOLUTION OF DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper some common fixed point theorems for pairs of condensing mappings in a Banach space are proved and applications are given to a pair of nonlinear first order ordinary differential equations in Banach spaces for proving the existence of common solution under suitable conditions.

1. Introduction

Ambrosetti [2] first used the measure of noncompactness for proving the existence theorems for differential equations in Banach spaces. But at present there is an extensive literature on the existence theorems for differential equations in abstract spaces which involves the use of measure of noncompactness. The main idea using the measure of noncompactness in proving the solution of differential equations in question is to convert it first into an equivalent operator equation and secondly exploit the fixed point theorem of Darbo type or the comparison function of Kamke type and the detail treatment of this aspect is given in Banas and Goebel [5], Deimling [9] and Martin [14]. But to the knowledge of the present author, the problem of the existence of common solution of differential equations in abstract spaces is not discussed and it is the aim of this paper to establish some results in this direction. It seems that the present discussion is new to the literature and with that many interesting results would be possible in the theory of nonlinear differential equations in abstract spaces. In the main, in section 2, we

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prove some common fixed point theorems of Darbo [8] type for a pair of condensing mappings in Banach spaces. Section 3 deals with the existence theorems for the common solution of two nonlinear differential equations in Banach spaces.

2. Fixed Point Theorems

Let E denote a real Banach space with a norm $\|\cdot\|_E$. A non-empty closed subset K of E is said to be an *order cone* in E if

(i) $K + K \subseteq K$,
(ii) $\lambda K \subseteq K$, for $\lambda \geq 0$ and
(iii) $K \cap -K = \{\theta\}$, where θ denotes the zero element of E.

Then the relation $x \leq y$ if and only if $y - x \in K$ defines the partial ordering in E. We do not require any property of the cones in the present discussion, however, the details of order cones and their properties may be found in Guo and Lakshmikantham [11].

The measure of noncompactness of a bounded set A in E is a nonnegative real number $\alpha(A)$ with the following properties:

(M1) $\alpha(A) = 0$ if and only if A is precompact, and

(M2) $\alpha(coA) = \alpha(\overline{co}A) = \alpha(A)$, where coA and $\overline{co}A$ are the convex and the closed convex hull of A in E, respectively.

We also suppose that the measure of noncompactness α satisfies the following three properties:

(M3) $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$,

(M4) $\alpha(\{A \cup B\}) = \max\{\alpha(A), \alpha(B)\}$, and

(M5) $\alpha(\lambda A) = |\lambda|\alpha(A), \quad \lambda \in R$.

There does exist the measure of noncompactness satisfying the properties (M3)-(M5). In fact, the Kuratowskii [12] and Hausdorff [6] measures of noncompactness satisfy the above properties. The details of different type of measures of noncompactness and their properties are given in Banas and Goebel [5]. Below we give some definitions of the contraction mappings on the Banach space E with respect to the measure of noncompactness in E.

DEFINITION 2.1. A mapping $T : E \to E$ is said to be *k-set-contraction* if for any bounded set A in E, $T(A)$ is bounded and $\alpha(T(A)) \leq$

$k\alpha(A)$ holds for some $k > 0$. In particular, if $k < 1$, then T is called a *strict-set-contraction* on E.

DEFINITION 2.2. A mapping $T : E \to E$ is called *ϕ-set-contraction* if for a bounded set A in E, $T(A)$ is bounded and $\alpha(T(A)) \leq \phi(\alpha(A))$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and nondecreasing function. In particular if $\phi(r) < r$, $r > 0$, then T is called a *nonlinear-set-contraction* on E

DEFINITION 2.3. A mapping $T : E \to E$ is called *condensing* if for any bounded set A in E, $T(A)$ is bounded and $\alpha(T(A)) < \alpha((A))$ for $\alpha(A) > 0$.

REMARK 2.1. It is clear that strict-set-contraction $\Rightarrow$ nonlinear-set-contraction $\Rightarrow$ condensing mapping.

A noteworthy fixed point theorem using the measures of non-compactness in a Banach spaces is due to Sadovskii [15] which is a generalization of the fixed point theorem of Darbo [8] and includes the well-known fixed point theorems of Schauder [16], Banach [4] and Krasnoselskii [13], etc. as the special cases. This fixed point theorem is as follows.

THEOREM A. *Let X be a non-empty, closed, convex and bounded subset of the Banach space E and let $T : X \to X$ be a continuous and condensing mapping. Then T has a fixed point.*

In this section we first prove some common fixed point theorems for a pair of condensing mappings on a Banach space E and then derive some interesting corollaries. For we need the following definitions in the sequel.

DEFINITION 2.4. A mapping $T : E \to E$ is said to be *isotone increasing* if $x, y \in E$ with $x \leq y$ then $Tx \leq Ty$, where E is an ordered Banach space with some order relation $\leq$ in it.

DEFINITION 2.5. Two mappings $S, T : E \to E$ are said to be *weakly isotone* increasing if $Sx \leq TSx$ and $Tx \leq STx$ hold for all $x \in E$. Similarly the mappings $S, T : E \to E$ are said to be weakly isotone decreasing if $Tx \geq STx$ and $Sx \geq TSx$ holds for all $x \in E$. We say two mappings S, T are *weakly isotone* if they are either weakly isotone increasing or weakly isotone decreasing on E.

EXAMPLE 2.1. Let $\mathbb{R}$ denote the real line with the usual norm $|\cdot|$ and the order relation $\leq$ and let $X = [0,1] \subset \mathbb{R}$. Consider two mappings $f, g : [0,1] \to [0,1]$ defined by $f(x) = \frac{x}{2}$ and $g(x) = \frac{x^2}{3}$ for $x \in [0,1]$. Then the pair of mappings f and g is weakly isotone decreasing on $[0,1]$. To see this,

$$f(x) = \frac{x}{2} \geq \frac{x^2}{12} = g\left(\frac{x}{2}\right) = g(f(x))$$

and

$$g(x) = \frac{x^2}{3} \geq \frac{x^2}{6} = f\left(\frac{x^2}{3}\right) = f(g(x))$$

for all $x \in [0,1]$. Also note that the mappings f and g are isotone increasing on $[0,1]$.

In the sequel, throughout this section, let E denote an ordered Banach space with the order relation $\leq$ induced by the order cone K in E and let X denote a non-empty, closed, convex and bounded subset of E.

THEOREM 2.1. *Let $S, T : X \to X$ be two continuous and condensing mappings. Further if S and T are weakly isotone, then they have a common fixed point, i.e., there is a point x^* in X such that $Sx^* = x^* = Tx^*$.*

Proof. Let $x \in X$ be arbitrary and consider the sequence $\{x_n\}$ in X defined by

$$x_0 = x, \ x_{2n+1} = Sx_{2n}, \ x_{2n+2} = Tx_{2n+1}, \quad n = 0, 1, 2, \cdots \tag{2.1}$$

Suppose that the mappings S and T are weakly isotone increasing on X. Then form (2.1), it follows that

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \tag{2.2}$$

Let

$$\begin{aligned} A &= \{x_1, x_2, \cdots, x_n, \cdots\} \\ &= \{x_1\} \cup \{x_3, x_5, \cdots, x_{2n+1}, \cdots\} \cup \{x_2, x_4, \cdots, x_{2n}, \cdots\} \\ &= \{x_1\} \cup S(A_1) \cup T(A_2), \end{aligned}$$

where $A_1 = \{x_2, x_4, \cdots, x_{2n}, \cdots\} \subset A$ and $A_2 = \{x_1, x_3, \cdots, x_{2n+1}, \cdots\} \subset A$. Clearly $A \subset X$ and hence A is bounded. we shall prove that A is precompact. Suppose not. Then by nature of α, we get

$$\begin{aligned} \alpha(A) &= \alpha(\{x_1\} \cup S(A_1) \cup T(A_2)) \\ &= \max\{\alpha(S(A_1)), \alpha(T(A_2))\} \\ &< \alpha(A). \end{aligned}$$

This is a contradiction, and hence A is precompact and $\overline{A}$ is compact. In view of (2.2), the sequence $\{x_n\}$ is monotone increasing in $\overline{A}$. Therefore, there is a unique limit point x^* in $\overline{A}$ such that $\lim_{n\to\infty} x_n = x^*$. Again every subsequence of the sequence $\{x_n\}$ converges to the same limit point $x^* \in X$. Thus we have

$$\lim_{n\to\infty} x_{2n+1} = x^* \quad \text{and} \quad \lim_{n\to\infty} x_{2n+2} = x^*.$$

By continuity of S and T, we obtain

$$x^* = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} Sx_{2n} = S\big(\lim_{n\to\infty} x_{2n}\big) = Sx^*$$

and

$$x^* = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} x_{2n+1} = T\big(\lim_{n\to\infty} x_{2n+1}\big) = Tx^*.$$

Similarly if S and T are weakly isotone decreasing on X, then it can be proved that the sequence $\{x_n\}$ is monotone decreasing and converges to the unique limit point $x_* \in X$, which is again a common fixed point of S and T. This completes the proof. □

As a consequence of Theorem 2.1, we obtain the following interesting corollaries:

Corollary 2.1. *Let $S, T : X \to X$ be two completely continuous mappings. Further if S and T are weakly isotone mappings, then they have a common fixed point.*

Corollary 2.2. *Let $S, T : X \to X$ be two continuous and nonlinear-set-contraction mappings. Further if S and T are weakly isotone mappings, then they have a common fixed point.*

Theorem 2.2. *Let $S, T : X \to X$ be two continuous and condensing mappings. Further suppose that*

(i) *S and T are isotone increasing,*
(ii) *S and T are commutative, i.e., $S(T(x)) = T(S(x))$ for all $x \in X$, and*
(iii) *$x \leq Sx$ and $x \leq Tx$ for some $x \in X$.*

Then S and T have a common fixed point.

Proof. Define a sequence $\{x_n\}$ in X by (2.1). Then in view of the hypotheses (i) - (ii), it follows that

$$x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots .$$

The remainder of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.3. *Let $S, T : X \to X$ be two completely continuous mappings. Further if the hypotheses* (i)-(iii) *of* Theorem 2.2 *hold, then S and T have a common fixed point.*

Corollary 2.4. *Let $S, T : X \to X$ be two continuous and nonlinear-set-contraction mappings. Further if the hypotheses* (i)-(iii) *of* Theorem 2.2 *hold, then S and T have a common fixed point.*

Next we obtain some interesting results about the existence of the unique common fixed point for a pair of mappings on Banach spaces by the application of Theorem 2.1. These results do not require the compactness type conditions, but the mappings under consideration are required to satisfy certain contraction type conditions. Also these results have some nice applications for proving the existence as well as uniqueness of the common solution of certain nonlinear differential and integral problems. We need the following definition in the sequel.

DEFINITION 2.6. A mapping $T : E \to E$ is called a *nonlinear contraction* if there exists a continuous and nondecreasing real function $\phi_T : [0, \infty) \to [0, \infty)$ such that

$$\|Tx - Ty\|_E \leq \phi_T(\|x - y\|_E) \tag{2.3}$$

for all $x, y \in E$, where $\phi_T(r) < r$ for $r > 0$. In particular, if $\phi_T(r) = kr$, $0 \leq k < 1$, then T is called a contraction mapping with contraction constant k.

REMARK 2.2. It is easy to prove that every nonlinear contraction mapping is a nonlinear set-contraction with respect to the Kuratowskii measure of noncompactness and hence condensing, but the converse is not necessarily true.

THEOREM 2.3. *Let $S, T : X \to X$ be two mappings satisfying*

(i) *S is nonlinear contraction,*
(ii) *T is continuous and condensing, and*
(iii) *S and T are weakly isotone.*

Then S and T have unique common fixed point which is the unique fixed point of S.

Proof. Since S is a nonlinear contraction mappings, it is continuous on X. If S has a fixed point, it is unique in view of the condition (2.3). Also by Remark 2.2, S is a condensing mapping with respect to the Kuratowskii measure of noncompactness α. Thus all the conditions of Theorem 2.1 are fulfilled and hence an application of it yields that S and T have a common fixed point. Since S cannot have two fixed points, S and T have a unique common fixed point. This completes the proof. □

COROLLARY 2.5. *Let $S, T : X \to X$ be two mappings satisfying*

(i) *S is nonlinear contraction,*
(ii) *T is continuous and nonlinear set-contraction and*
(iii) *S and T are weakly isotone.*

Then S and T have a unique common fixed point which is also a unique fixed point of S.

Corollary 2.6. *Let $S, T : X \to X$ be two mappings satisfying*

(i) *S is nonlinear contraction, and*
(ii) *S and T are weakly isotone.*

Then S and T have a unique common fixed point.

Remark 2.3. We note that Theorems 2.3 and 2.4 give the numerical iterative method for finding the unique common fixed point of two mappings S and T. In this case the sequence $\{S^n x\}$, $x \in X$, of iterates of S, converges to the unique common fixed point of S and T by the following fixed point theorem of Boyd and Wong [7].

Theorem B. *Let X be a non-empty, closed, convex and bounded subset of the Banach X and let $T : X \to X$ be a nonlinear contraction. Then T has a unique fixed point x^* and the sequence of successive iterations $\{S^n x\}$, $x \in X$, converges to x^*.*

Remark 2.4. We note that in Dhage et al. [10] some common fixed point theorems for a pair of mappings in an ordered Banach space using the properties of the cones are proved which are further applied to a pair of discontinuous nonlinear differential equations for proving the existence of their common solution. The common fixed point theorems presented here in this paper do not invoke any property of the cones in a Banach space, and so these results are different from that of Dhage et al. [10]. Also the uniqueness of common fixed point of a pair of mappings established here in this paper is not discussed in Dhage et al. [10].

To illustrate the abstract theory developed in this section, some applications will be given to nonlinear differential equations for proving the existence of common solution under certain compactness and Lipschitzicity conditions on the nonlinearity involved in the equations.

3. Differential Equations in Banach Spaces

Let $\mathbb{R}$ denote the real line and $\mathbb{R}_+$ the set of nonnegative real numbers. Let $J = [t_0, t_0 + a] \subset \mathbb{R}$ for some t_0, $a \in \mathbb{R}$, $a > 0$, be a closed and bounded interval. Let E denote the real Banach space with a norm $\|\cdot\|_E$ and an order relation $\leq$ induced by the order cone K in E. By

α we denote the Kuratowskii measure of noncompactness in E. Now consider the two nonlinear differential equations with the same initial condition (for convenience)

$$\tag{3.1} \begin{cases} x' = f(t,x), & t \in J, \\ x(t_0) = x_0 \in E \end{cases}$$

and

$$\tag{3.2} \begin{cases} x' = g(t,x), & t \in J, \\ x(t_0) = x_0 \in E, \end{cases}$$

where $f, g : J \times E \to E$ are continuous functions. Let $C(J,E)$ denote the space of all continuous E-valued functions on J. Define a norm $\|x\|_X$ by

$$\tag{3.3} \|x\|_X = \sup_{t \in J} \|x(t)\|_E.$$

We define an order relation $\leq$ in $C(J,E)$ by the order cone $\overline{K}$ in $C(J,E)$ defined by

$$\tag{3.4} \overline{K} = \{x \in C(J,E) \mid x(t) \in K \text{ for all } t \in J\}.$$

Clearly the space $C(J,E)$ with the norm $\|\cdot\|_X$ and order relation $\leq$ becomes a ordered Banach space. Let $B \subset C(J,E)$ be a set, then

$$B(t) = \{f(t) \mid f \in B\} \subset E \text{ and } B(J) = \cup_{t \in J} B(t).$$

To prove the main existence theorem, we need the following lemma, the proof of which is well-known, see for example, Ambrosetti [2] and Banas and Goebel [5].

LEMMA 3.1. *For any bounded, equicontinuous set B in $C(J,E)$,*

(a) $\alpha\left(\int_{t_0}^t B(s)\,ds\right) \leq \int_{t_0}^t \alpha(B(s))\,ds$, $t \in J$, *and*
(b) $\alpha(B) = \max_{t \in J} \alpha(B(t))$.

We consider the following assumptions:

(H1) *The function f and g are bounded on $J \times E$ with bound M.*

(H2) *f and g are uniformly continuous on $J \times E$.*

(H3) *For $t \in J$, $\alpha(f(t, B)) \leq \Psi_f(\alpha(B))$ and $\alpha(g(t, B)) \leq \Psi_g(\alpha(B))$ for any bounded set $B \subset E$, where Ψ_f and Ψ_g are non-negative continuous and nondecreasing real functions on $\mathbb{R}_+$.*

(H4) *There exist continuous and nondecreasing functions $\Phi_f, \Phi_g : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|f(t, x) - f(t, y)\|_E \leq \Phi_f(\|x - y\|_E)$$

and

$$\|g(t, x) - g(t, y)\|_E \leq \Phi_g(\|x - y\|)_E)$$

for all $(t, x), (t, y) \in J \times E$.

(G1) *The functions $f(t, x)$ and $g(t, x)$ are nondecreasing in $x \in E$ for all $t \in J$.*

(G2) *$f(t, x) \leq g(t, f(t, x))$ and $g(t, x) \leq f(t, g(t, x))$ for all $(t, x) \in J \times E$.*

(G3) *$f(t, x(t)) \leq x_0 + \int_{t_0}^t f(\tau, x(\tau))\, d\tau$ and $g(t, x(t)) \leq x_0 + \int_{t_0}^t g(\tau, x(\tau))\, d\tau$ for all $(t, x) \in J \times C(J, E)$ and for a fixed element $x_0 \in E$ given in (3.1) and (3.2).*

THEOREM 3.1. *Assume (H1)-(H3) and (G1)-(G3) hold. Further if $a\Psi_f(r) < r$ and $a\Psi_g(r) < r$, for $r > 0$, then the differential equations (3.1) and (3.2) have a common solution on J.*

Proof. Define a subset X of the Banach space $C(J, E)$ by

$$X = \{x \in C(J, E) : x(t_0) = x_0, |x(t) - x(s)| \leq M|t - s|\}. \tag{3.5}$$

Clearly X is closed, convex, bounded and equi-continuous set in $C(J, E)$. Now the differential equations (3.1) and (3.2) are equivalent to the integral equations

$$x(t) = x_0 + \int_{t_0}^t f(s, x(s))ds,\ t \in J \tag{3.6}$$

and

$$x(t) = x_0 + \int_{t_0}^{t} g(s, x(s))ds, \; t \in J, \tag{3.7}$$

respectively.

Define the mappings S and T on X by

$$Sx(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds, \; t \in J, \tag{3.8}$$

$$Tx(t) = x_0 + \int_{t_0}^{t} g(s, x(s))ds, \; t \in J. \tag{3.9}$$

Then the problem of the common solution of the differential equations is just reduced to finding the common fixed points of the operators S and T on X. Obviously the mappings S and T are continuous and map X into itself. We show that S and T are condensing mappings on X with respect to the measure of noncompactness $\overline{\alpha}$ in $C(J, E)$ defined by $\overline{\alpha}(B) = \max_{t \in J} \alpha(B(t))$, where B is a bounded set in $C(J, E)$. Now for any bounded set B of X, by Lemma 3.1, we have

$$\begin{aligned}
\alpha(S(B(t))) &\leq \int_{t_0}^{t} \alpha(f(s, B(s)))ds + \alpha(\{x_0\}) \\
&\qquad (\text{since } \alpha(A + B) \leq \alpha(A) + \alpha(B)) \\
&= \int_{t_0}^{t} \alpha(f(s, B(s)))ds \\
&= \int_{t_0}^{t} \Psi_f(\alpha(B(s)))ds \\
&= a\Psi_f(\overline{\alpha}(B)).
\end{aligned} \tag{3.10}$$

Taking maximum over t in (3.10), we get

$$\overline{\alpha}(S(B)) \leq a\Psi_f(\overline{\alpha}(B)) < \overline{\alpha}(B) \;\text{ if }\; \overline{\alpha}(B) > 0.$$

This shows that S is condensing on X. Similarly it is shown that T is also a condensing mapping on X. Further for any $x \in X$,

$$\begin{aligned} Sx(t) &= x_0 + \int_{t_0}^{t} f(s, x(s))ds \\ &= x_0 + \int_{t_0}^{t} g(s, f(s, x(s)))ds \\ &\leq x_0 + \int_{t_0}^{t} g(s, x_0 + \int_{t_0}^{s} f(\tau, x(\tau))d\tau)ds \\ &= x_0 + \int_{t_0}^{t} g(s, Sx(s))ds \\ &= TSx(t) \end{aligned}$$

for all $t \in J$, i.e., $Sx \leq TSx$ for all $x \in X$. Similarly $Tx \leq STx$ for all $x \in X$. This shows that the mappings S and T are weakly isotone increasing on X. Thus all the conditions of Theorem 2.1 are satisfied and hence an application of it yields that the mappings S and T have a common fixed point in X. This completes the proof. □

Next we prove the uniqueness theorem for common solution of the differential equations (3.1) and (3.2) under weaker condition of the uniform continuity of the functions f and g. However, in this case the functions f and g are required to satisfy certain contraction type conditions.

THEOREM 3.2. *Assume* (H1), (H4) *and* (G1)-(G2) *hold. Further if $a\Phi_f(r) < r$ and $a\Phi_g(r) < r$ for $r > 0$, then the differential equation* (3.1) *and* (3.2) *have a unique common solution x^* on J, and the sequences $\{x_n\}$ and $\{y_n\}$ defined by*

$$x_0 = x_0, \; x_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, x_n(s))ds, \tag{3.11}$$

$$y_0 = x_0, \; y_{n+1}(t) = x_0 + \int_{t_0}^{t} g(s, y_n(s))ds \tag{3.12}$$

converge to x^*.

Proof. Define a subset X of the Banach space $C(J,E)$ by

$$X = \{x \in C(J,E) : \|x\| \leq Ma + x_0\}. \tag{3.13}$$

Clearly X is a closed, convex and bounded subset of the Banach space $C(J,E)$. Define two mappings S and T on X by (3.8) and (3.9) respectively. Obviously S and T map X into itself. We show that S and T are nonlinear contraction mappings on X. Let $x, y \in X$, then by (H4), we have

$$\begin{aligned}
\|Sx(t) - Sy(t)\|_E &\leq \int_{t_0}^{t} \|f(s,x(s)) - f(s,y(s))\|_E ds \\
&\leq \int_{t_0}^{t} \Phi_f(\|x(s) - y(s)\|_E) ds \\
&\leq \int_{t_0}^{t} \Phi_f(\|x - y\|_X) ds.
\end{aligned}$$

Therefore,

$$\|Sx - Sy\|_X \leq \beta(\|x - y\|_X), \tag{3.14}$$

where $\beta(r) = a\Phi_f(r) < r$, $r > 0$.

This shows that S is a nonlinear contraction on X. Similarly, it can be shown the mapping T is also a nonlinear contraction on X. Further proceeding as in the proof of Theorem 3.1, it can be proved that the mappings S and T are weakly isotone increasing on X. Now an application of Theorem 2.4 yields that S and T have a unique common fixed point in X. This further implies that the differential equations (3.1) and (3.2) have a unique common solution x^* in X, and by Remark 2.3, the sequences $\{x_n\}$ and $\{y_n\}$ of successive iterations of S and T converge to x^*. This completes the proof. □

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