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A STUDY ON QUASI-DUO RINGS

CHOL ON KIM, HONG KEE KIM AND SUNG HEE JANG

ABSTRACT. In this paper we study the connections between right quasi-duo rings and 2-primal rings, including several counterexamples for answers to some questions that occur naturally in the process. Actually we concern following three questions and modified ones: (1) Are right quasi-duo rings 2-primal?, (2) Are formal power series rings over weakly right duo rings also weakly right duo? and (3) Are 2-primal rings right quasi-duo? Moreover we consider some conditions under which the answers of them may be affirmative, obtaining several results which are related to the questions.

1. Introduction

Throughout this paper, all rings are associative with identity. Observing the properties of quasi-duo rings was initiated by Yu in [14], related to the Bass' conjecture in [1]. Given a ring R the polynomial ring over R and the formal power series ring over R are denoted by $R[x]$ and $R[[x]]$, respectively. In [9], if $R[x]$ is right quasi-duo then R is right quasi-duo but the converse is not true in general: similarly R is 2-primal if and only if $R[x]$ is 2-primal by [2, Proposition 2.6] and [7, Proposition 4]. R is right quasi-duo if and only if $R[[x]]$ is right quasi-duo [9, Proposition 6]: similarly by [7, Proposition 12] if $R[[x]]$ is 2-primal then R is 2-primal, but the converse is not true in general by

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[10, Example 1.1]. Considering the preceding results, we observe the similarities or differences between right quasi-duo rings and 2-primal rings.

A ring R is called *right* (*left*) *duo* if every right (left) ideal of R is two-sided. A ring R is called *right* (*left*) *quasi-duo* if every maximal right (left) ideal of R is two-sided. Commutative rings are clearly right and left duo; right (left) duo rings are right (left) quasi-duo obviously. The n by n upper triangular matrix rings over right quasi-duo rings are also right quasi-duo by [14, Proposition 2.1]. But the n by n full matrix rings over right quasi-duo rings are not right quasi-duo.

As another generalization of commutative rings, there are 2-primal rings. The term 2-*primal* was come upon originally by Birkenmeier-Heatherly-Lee [2] in the context of left near rings. Shin [13] proved that a ring R is 2-primal if and only if every minimal prime ideal of R is completely prime, which was one of the earliest results known to us about 2-primal rings (although not so called at the time.) A ring R is called 2-*primal* if $P(R) = N(R)$, where $P(R)$ is the prime radical of R and $N(R)$ is the set of all nilpotent elements in R. It is straightforward to check that a ring R is 2-primal if and only if $R/P(R)$ is a reduced ring (i.e., a ring without nonzero nilpotent elements). Commutative rings and reduced rings are 2-primal obviously, and the n by n upper triangular matrix rings over 2-primal rings are also 2-primal by [2, Proposition 2.5]. But the n by n full matrix rings over 2-primal rings are not 2-primal.

2. Counterexamples and Related Results

An ideal I of a ring R is called *completely prime* if R/I is a domain.

PROPOSITION 1. *Right* (*or left*) *duo rings are 2-primal.*

Proof. Let R be a right duo ring and $r \in R$. Then we have $RrR \subseteq rR$. Suppose that P is a prime ideal of R and $ab \in P$ for $a, b \in R$. Then

$$RaRRbR \subseteq aRbR \subseteq abR \subseteq P$$

and so $RaR \subseteq P$ or $RbR \subseteq P$. It follows that P is completely prime, and hence R is 2-primal by [13]. The proof of the left case is similar.□

By Proposition 1 and the fact that right duo rings are right quasi-duo, we may raise the following question:

QUESTION (1). Are right quasi-duo rings 2-primal?

But the answer is negative by the following Example 2.

EXAMPLE 2. There exists a right quasi-duo ring but not 2-primal.

Proof. We take the ring R in [10, Example 1.1]. Let F be a field and let V be a infinite dimensional left vector space over F with $\{v_1, v_2, \dots\}$ a basis. For the endomorphism ring $A = End_F(V)$, define

$$A_1 = \{f \in A \mid \text{rank}(f) < \infty \text{ and } f(v_i) = a_1 v_1 + \cdots + a_i v_i \text{ for } i = 1, 2, \dots \text{ with } a_j \in F\}$$

and let R be the F-subalgebra of A generated by A_1 and 1_A. Let M be a maximal right ideal of R. Then M is of the form

$$M = \{r \in R \mid (i, i) - \text{entry of } r \text{ is } zero\}$$

for some $i \in \{1, 2, \dots\}$. But M is also a 2-sided ideal of R and so R is right quasi-duo. By [9, Proposition 6] the formal power series ring $R[[x]]$ over R is also right quasi-duo. However $R[[x]]$ is not 2-primal by the argument in [10, Example 1.1]. □

The ring $R[[x]]$ in Example 2 is not semiprimitive. We have an affirmative answer to Question (1) when given a ring is semiprimitive.

PROPOSITION 3. *Semiprimitive right (or left) quasi-duo rings are reduced (hence 2-primal).*

Proof. A semiprimitive right (or left) quasi-duo ring is isomorphic to a subdirect product of division rings by [9, Corollary 3(1)]; hence it is reduced. □

REMARK. The converse of Proposition 3 (i.e., for a semiprimitive ring R, is R right quasi-duo if R is reduced?) is not true in general by the following Example 9.

There is another condition under which the answer to Question (1) is affirmative. A subset of a ring is said to be *nil* if each element of it is nilpotent. The *index of nilpotency* of a nilpotent element x in a ring R is the least positive integer n such that $x^n = 0$. The *index of nilpotency* of a two-sided ideal I of R is the supremum of the indices of nilpotency of all nilpotent elements in I. If such a supremum is finite, then I is said to be *of bounded index of nilpotency.* By [6, Theorem 6], if R is a right quasi-duo ring of bounded index of nilpotency and the Jacobson radical of R is nil then R is 2-primal. Notice that for the ring $R[[x]]$ in Example 2, the preceding conditions are not satisfied.

A ring R is called *weakly right* (*left*) *duo* if for each a in R there exists a positive integer $n = n(a)$, depending on a, such that a^nR (Ra^n) is a two-sided ideal of R. Weakly right duo rings are abelian right quasi-duo rings by [14, Proposition 2.2].

LEMMA 4. *Let R be a ring. Then we have the following statements:*

(1) *$R[[x]]$ is right quasi-duo if and only if R is right quasi-duo.*
(2) *If $R[[x]]$ is right duo then R is right duo.*
(3) *If $R[[x]]$ is weakly right duo then R is weakly right duo.*

Proof. (1). By [9, Proposition 6]. (2). Let $a \in R$. Since $R[[x]]$ is right duo, $aR[[x]]$ is a 2-sided ideal of $R[[x]]$ and so $R[[x]]a \subseteq aR[[x]]$ implies $Ra \subseteq aR$. (3). Similar to the proof of (2). □

REMARK. We may obtain the same results for the left cases by replacing "right" by "left" in the preceding lemma and its proof.

The converse of (2) in Lemma 4 is not true in general by [5, Example 4]. So we raise the following as the converse of (3).

QUESTION (2). Are formal power series rings over weakly right duo rings also weakly right duo?

But the answer is negative by the following Example 5. Recall that a *right Ore domain* is a domain R in which every two nonzero elements have a nonzero common right multiple, i.e., for each nonzero $x, y \in R$ there exist $r, s \in R$ such that $xr = ys \neq 0$. Right Noetherian domains are right Ore by [4, Corollary 5.16].

EXAMPLE 5. There exists a weakly right duo ring such that the formal power series ring over it is not weakly right duo.

Proof. We hire the method in [5, Example 4]. Let F be a field of characteristic zero, $A = F[y]$ be the polynomial ring over F. Define $\sigma : A \to A$ with $\sigma(y) = 1+y$, then σ is an automorphism of A. Let $B = A[z;\sigma]$ be the skew polynomial ring over A, subject to $az = z\sigma(a)$ for all $a \in A$. Then since A is right Noetherian and σ is an automorphism, B is a right Noetherian domain by [11, Theorem 1.2.9]. Hence B is also right Ore by the preceding argument and thus B is a right order in a division ring by [4, Theorem 5.17], say D is the division ring. Then D is a noncommutative division ring.

For each $i = 1, 2, \ldots$, let $D_i = D$ and $S = \prod D_i$ be the direct product of D_i's. Now define

$$R = \{(d_i) \in S \mid \text{ there exists } n \text{ such that } d_i = d_n \text{ for all } i \geq n\}.$$

Then R is a strongly regular ring and so it is right duo (hence weakly right duo). Let $e_j = (d_i) \in R$ such that $d_j = 1$ and $d_i = 0$ for $i \neq j$, and consider $f = ze_1 + z^2e_2x + z^3e_3x^2 + \cdots \in R[[x]]$. Then for any positive integer k, $f^k = z^ke_1 + z^{2k}e_2x^k + z^{3k}e_3x^{2k} + \cdots$. Assume that $R[[x]]$ is weakly right duo. Then for some positive integer k, $f^kR[[x]]$ is a 2-sided ideal of $R[[x]]$; hence for $g = (y, y, y, \ldots) \in R$ there exists $h \in R[[x]]$ such that $gf^k = f^kh$. Then

$$ye_nz^{nk}x^{(n-1)k} = e_ngf^k = e_nf^kh = z^{nk}e_nx^{(n-1)k}(e_nh)$$

and so $e_nh = e_nz^{-nk}yz^{nk} = e_n(nk + y)$. This implies that $h = (nk + y) \in S \backslash R$ because the characteristic of F is zero, a contradiction. Therefore $R[[x]]$ is not weakly right duo. $\square$

From [5, Theorem 4] we obtain conditions under which the converses in Lemma 4 may be true as in the following.

THEOREM 6. *Suppose that R is a right self-injective von Neumann regular ring. Then the following statements are equivalent:*

(1) *R is right (left) duo.*
(2) *R is weakly right (left) duo.*
(3) *R is right (left) quasi-duo.*
(4) *R is a reduced ring.*
(5) *R is a 2-primal ring.*
(6) *$R[[x]]$ is right (left) duo.*
(7) *$R[[x]]$ is weakly right (left) duo.*
(8) *$R[[x]]$ is right (left) quasi-duo.*
(9) *$R[[x]]$ is a reduced ring.*
(10) *$R[[x]]$ is a 2-primal ring.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5), (6) $\Rightarrow$ (7) $\Rightarrow$ (8) and (9) $\Rightarrow$ (10) are straightforward. Since R is semiprimitive, we get (3) $\Rightarrow$ (4) by Proposition 3. (5) $\Rightarrow$ (1): Since R is a semiprime 2-primal ring, R is reduced and so R is right (left) duo by [3, Theorem 3.2]. (8) $\Rightarrow$ (9): By Lemma 4 R is right quasi-duo because $R[[x]]$ is right quasi-duo. So R is reduced by the proof of (3) $\Rightarrow$ (4) and hence $R[[x]]$ is also reduced. (5) $\Rightarrow$ (10): Since R is semiprime and 2-primal, $R[[x]]$ is also 2-primal by [10, Proposition 1.2]. (10) $\Rightarrow$ (5): By [7, Proposition 12]. (10) $\Rightarrow$ (6): R is 2-primal by the proof of (10) $\Rightarrow$ (5) and so R is right (left) duo by the proof of (5) $\Rightarrow$ (1); hence R is strongly regular by [3, Theorem 3.2 and 3.5]. Now [5, Theorem 4] implies that $R[[x]]$ is right (left) duo because R is right self-injective by hypothesis. □

REMARK. The equivalences of (1), (2), (3), (4) and (5) in Theorem 6 hold only when R is von Neumann regular.

Now we consider a characterization of right quasi-duo rings, obtained from [9, Proposition 1] and the proof of [14, Proposition 2.1].

LEMMA 7. *For a ring R, the following statements are equivalent:*

(1) *R is right quasi-duo.*
(2) *Every right primitive factor ring of R is a division ring.*
(3) *$R/J(R)$ is right quasi-duo ($J(R)$ is the Jacobson radical of R).*

By Lemma 7, every right primitive factor ring of a right quasi-duo ring is Artinian. Whence we have the following useful result if given a ring is both 2-primal and right quasi-duo.

PROPOSITION 8. *Suppose that R is a 2-primal right quasi-duo ring. Then the following statements are equivalent:*

(1) *R is strongly π-regular.*
(2) *R is π-regular.*
(3) *R is weakly π-regular.*
(4) *R is right weakly π-regular.*
(5) *$R/J(R)$ is right weakly π-regular and $J(R)$ is nil where $J(R)$ is the Jacobson radical of R.*
(6) *Every prime ideal of R is maximal.*

Proof. By Lemma 7 and [6, Theorem 3]. □

By this result, Question (1) and the following Question (3) may be meaningful. Recall that a ring R is called a *PI-ring* if R satisfies a polynomial identity with coefficients in the ring of integers. PI-rings are another generalization of commutative rings. As the converse of Question (1) one may ask the following.

QUESTION (3). Are 2-primal rings right quasi-duo?

But the answer is also negative by the following example, although it is a semiprimitive PI-ring.

EXAMPLE 9. There exists a ring S such that

(1) S is a domain (hence reduced and so 2-primal),
(2) S is a semiprimitive PI-ring and
(3) S is not right quasi-duo.

Proof. We take the ring $R[x]$ in [9, Example 9]. Let R be the Hamilton quaternion over the field of real numbers and $S = R[x]$ be the polynomial ring over R. Then clearly $R[x]$ is a domain and so it is semiprimitive by [12, Reproof of Amitsur's Theorem (2.5.23) after Lemma 2.3.41]. $R[x]$ is also a PI-ring because R is a PI-ring by [4, Prologue]. But $R[x]$ is not right quasi-duo by [9, Lemma 8]. □

As we see in the proof of Theorem 6, R is 2-primal if and only if R is right (left) duo if and only if R is weakly right (left) duo if and only if R is right (left) quasi-duo if and only if R is reduced, when R is a von Neumann regular ring.

Note that a semiprimitive right (or left) quasi-duo ring is a subdirect product of division rings, and if the polynomial ring over a ring R is right quasi-duo then $R/J(R)$ is commutative [9, Theorem 12] where $J(R)$ is the Jacobson radical of R; hence one may suspect that semiprimitive right quasi-duo rings are PI-rings. However the answer is negative because there is a division ring which is not a PI-ring. Recall that a ring R is right quasi-duo if the polynomial ring $R[x]$ over R is right quasi-duo. The preceding suspicion is affirmative when $R[x]$ is right quasi-duo by [9, Corollary 14]. However this argument does not hold in general for the formal power series rings. Let D be a noncommutative division ring, then D is semiprimitive right quasi-duo obviously and $D[[x]]$ is right quasi-duo by [9, Proposition 6]; but $D[[x]]$ is noncommutative.

Lastly we obtain similar results to [14, Proposition 2.1].

PROPOSITION 10. *Let R, S be rings and ${}_RM_S$ be a (R,S)-bimodule. Let $E = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$. Then E is a right quasi-duo ring if and only if R and S are right quasi-duo rings.*

Proof. We use Lemma 7. $(\Rightarrow)$: Take

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then clearly e_1 and e_2 are nonzero idempotents of E. By [9, Theorem 5], $e_1Ee_1 \cong R$ and $e_2Ee_2 \cong S$ are right quasi-duo because E is right quasi-duo. $(\Leftarrow)$: First note that every right primitive factor ring of E is one of the following forms:

$$\begin{pmatrix} R/P & 0 \\ 0 & 0 \end{pmatrix} \cong R/P \text{ and } \begin{pmatrix} 0 & 0 \\ 0 & S/Q \end{pmatrix} \cong S/Q,$$

where P, Q are right primitive ideals of R, S respectively. Since R and S are right quasi-duo, every right primitive factor ring of E is a division ring and hence E is right quasi-duo. □

For a ring R, let $T(R,R) = \{(a,x) \mid a,x \in R\}$ with the addition componentwise and the multiplication defined by $(a_1,x_1)(a_2,x_2) = (a_1a_2, a_1x_2 + x_1a_2)$. Then $T(R,R)$ is a ring which is called the *trivial extension* of R by R. $T(R,R)$ is isomorphic to the ring of matrices $\begin{pmatrix} a & x \\ 0 & a \end{pmatrix}$ with $a,x \in R$.

PROPOSITION 11. *For a ring R, $T(R,R)$ is a right quasi-duo ring if and only if R is a right quasi-duo ring.*

Proof. Similar to the proof of Proposition 10. □

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CHOL ON KIM, DEPARTMENT OF MATHEMATICS, BUSAN NATIONAL UNIVERSITY, BUSAN 609-735, KOREA

HONG KEE KIM, DEPARTMENT OF MATHEMATICS AND RESEARCH INSTITUTE OF NATURAL SCIENCE, GYEONGSANG NATIONAL UNIVERSITY, JINJU 660-701, KOREA

SUNG HEE JANG, DEPARTMENT OF MATHEMATICS EDUCATION, KYUNGPOOK NATIONAL UNIVERSITY, TAEGU 702-701, KOREA