Bull. Korean Math. Soc. **36** (1999), No. 3, pp. 589–597

BEST APPROXIMATIONS IN $L_P(S, X)$

MUN BAE LEE, SUNG HO PARK* AND HYANG JOO RHEE

ABSTRACT. Let G be a closed subspace of a Banach space X and let (S, Ω, μ) be a σ-finite measure space. It was known that $L_1(S, G)$ is proximinal in $L_1(S, X)$ if and only if $L_p(S, G)$ is proximinal in $L_p(S, X)$ for $1 < p < \infty$. In this article we show that this result remains true when "proximinal" is replaced by "Chebyshev". In addition, it is shown that if G is a proximinal subspace of X such that either G or the kernel of the metric projection P_G is separable then, for $0 < p \leq \infty$, $L_p(S, G)$ is proximinal in $L_p(S, X)$.

1. Introduction

Throughout this paper, X is a Banach space, G is a closed subspace of X and (S, Ω, μ) is a σ-finite measure space.

Let K be a nonempty subset of X and let $x \in X$. An element k_0 in K satisfying

$$\|x - k_0\| = d(x, K) := \inf_{k \in K} \|x - k\|$$

is called a best approximation of x in K. For any $x \in X$, the set of all best approximations of x in K is denoted by

$$P_K(x) = \{k \in K : \|x - k\| = d(x, K)\}.$$

The set K is called proximinal (resp., Chebyshev) if for every $x \in X$, $P_K(x)$ is nonempty (resp., a singleton).

Received February 26, 1998.

1991 Mathematics Subject Classification: 41A65, 46B20.

Key words and phrases: Chebyshev, proximinal, pointwise proximinal, proximity map, strongly measurable.

*This studies were supported in part by the Basic Research Institute Program, BRSI-98-1412, Moe, Korea and the Sogang University Research Grant in 1999.

Let K be a proximinal subset of X. The set-valued map $P_K : X \to 2^K$ thus defined is called the metric projection onto K and the kernel of the metric projection P_K is the set

$$\begin{aligned}\ker P_K &:= \{x \in X : 0 \in P_K(x)\} \\ &= \{x \in X : \|x\| = d(x, K)\}.\end{aligned}$$

A map $p : X \to K$ which associates with each element of X one of its best approximations in K is called a proximity map.

Many authors investigated the proximinality of $L_p(S, G)$ in $L_p(S, X)$ when (S, Ω, μ) is a finite measure space [1], [3], [4], [5], [6], [8] and [10]. You and Guo [11] proved the equivalent relation between the proximinality of $L_p(S, G)$ in $L_p(S, X)$ and the pointwise proximinality of $L(S, G)$ in $L(S, X)$. From this result, they improved those in [3], [4], [5] and [8] for a σ-finite measure space. In particular, they proved that if G is a separable and proximinal subspace of X then $L_p(S, G)$ is proximinal in $L_p(S, X)$ $(0 < p < \infty)$. In order to prove the above result, they used the measurable selection theorem due to Himmelberg and Vleck [2] (see also Theorem 2.7).

In section 3, we improve the results of Khalil and Saidi [6] and Saidi [10]. To do this, we also use the measurable selection theorem due to Himmelberg and Vleck [2].

2. Preliminaries

DEFINITION 2.1. Let X be a Banach space. A function from S to X is called *strongly measurable* if it is the pointwise limit of a sequence of simple Borel measurable functions from S to X, equivalently, f is Borel measurable and has a separable range.

Denote by $L(S, X)$ the space of all strongly measurable functions from (S, Ω, μ) to X. Functions equal a.e. are identified. For $1 \le p < \infty$, $L_p(S, X)$ is the Banach space (for $0 < p < 1$, Frechet space) consisting of strongly measurable functions $f : S \to X$ such that $\int \|f(s)\|^p d\mu(s)$ is finite. For $p = \infty$, $L_\infty(S, X)$ is the Banach space of essentially bounded strongly measurable functions $f : S \to X$. For $f \in L_p(S, X)$,

$$\|f\|_p = \left(\int \|f(s)\|^p d\mu(s)\right)^{\frac{1}{p}} \quad 1 \le p < \infty,$$

and

$$\|f\|_\infty = \text{ess sup}_{s\in S}\|f(s)\|.$$

For $A \in \Omega$ and a strongly measurable function $f : S \to X$, we write I_A for the characteristic function of A and $I_A \otimes f$ denote the function defined by $(I_A \otimes f)(s) = I_A(s)f(s)$. In particular, for $A \in \Omega$ and $x \in X$, $(I_A \otimes x)(s) = I_A(s)x$.

DEFINITION 2.2. [11] Let $f \in L(S,X)$ and $D \subset L(S,X)$. $f_0 \in D$ is called a pointwise best approximation of f in D if, for all $g \in D$, we have $\|f(s) - f_0(s)\| \le \|f(s) - g(s)\|$ a.e.

In [11], You and Guo proved the following theorem.

THEOREM 2.3. [11] *Let $0 < p < \infty$. Then the following are equivalent:*

(i) *$L(S,G)$ is pointwise proximinal in $L(S,X)$;*
(ii) *$L_p(S,G)$ is proximinal in $L_p(S,X)$.*

THEOREM 2.4. *If $L(S,G)$ is pointwise proximinal in $L(S,X)$, then $L_\infty(S,G)$ is proximinal in $L_\infty(S,X)$.*

Proof. Let $f \in L_\infty(S,X)$. Then there exists $f_0 \in L(S,G)$ such that $\|f(s) - f_0(s)\| \le \|f(s) - q(s)\|$ a.e. for any $q \in L(S,G)$ and so $\|f(s) - f_0(s)\| \le \|f(s)-g(s)\|$ a.e. for any $g \in L_\infty(S,G)$. Since $0 \in G$, $\|f_0(s)\| \le 2\|f(s)\|$ a.e. Thus $f_0 \in L_\infty(S,G)$. Therefore $f_0 \in P_{L_\infty(S,G)}(f)$. □

By Theorem 2.3 and Theorem 2.4, we get the following corollary. This corollary is a generalization of Theorem 1.1 in [1] for a σ-finite measure space.

COROLLARY 2.5. *Let $0 < p < \infty$. If $L_p(S,G)$ is proximinal in $L_p(S,X)$ then $L_\infty(S,G)$ is proximinal in $L_\infty(S,X)$.*

Now we state some measurable selection theorems.

THEOREM 2.6. [9] *Let (M,d) be a separable metric space. If a multifunction $F : S \to 2^M$ having complete values is weakly measurable, in the sense that for each open subset O of M the set $F^{-1}(O) := \{s \in S : F(s) \cap O \ne \emptyset\}$ is measurable in S, then there exists a M-valued Borel measurable function $f : S \to M$ such that $f(s) \in F(s)$ for all $s \in S$.*

THEOREM 2.7. [2] *Let (M,d) be a separable metric space. Then a multifunction $F : S \to 2^M$ having complete values is compactly measurable, in the sense that for each compact subset K of M the set $F^{-1}(K) := \{s \in S : F(s) \cap K \neq \emptyset\}$ is measurable in S if and only if there exists a countable set $\{V_n : n \in \mathbb{N}\}$ of M-valued Borel measurable functions defined on (S, Ω, μ) such that $F(s) = \overline{\{V_n(s) : n \in \mathbb{N}\}}$ for all $s \in S$.*

3. Main Results

In [11], You and Guo proved the following theorem. In this article we want to replace proximinality by Chebyshevity.

THEOREM 3.1. [11] *Let $1 < p < \infty$. Then the following are equivalent:*

(i) *$L_1(S,G)$ is proximinal in $L_1(S,X)$;*
(ii) *$L_p(S,G)$ is proximinal in $L_p(S,X)$.*

Similar to the proof of Theorem 1.1 in [5], we have the following

THEOREM 3.2. *Let $1 \le p < \infty$. If $L_p(S,G)$ is proximinal in $L_p(S,X)$, then for any $f \in L_p(S,X)$*

$$d_p(f, L_p(S,G)) = \left(\int d(f(s),G)^p d\mu(s)\right)^{\frac{1}{p}}.$$

Proof. Case $1 : p = 1$.
This is a special case of Lemma 2.10 in [9].
Case $2 : 1 < p < \infty$.
Define a map $\Phi : L_p(S,X) \to L_1(S,X)$ by

$$\Phi(f)(s) = ||f||^{p-1} f(s)$$

for all $f \in L_p(S,X)$. Then Φ is 1-1. Take any $g \in L_1(S,X)$ and put $f(s) = ||g(s)||^{\frac{1}{p}-1} g(s)$ if $g(s) \neq 0$ and $f(s) = 0$ otherwise. Then $f \in L_p(S,X)$ and

$$\begin{aligned}\Phi(f)(s) &= ||f(s)||^{p-1} f(s) = ||f(s)||^{p-1} ||g(s)||^{\frac{1}{p}-1} g(s) \\ &= ||g(s)||^{1-\frac{1}{p}} ||g(s)||^{\frac{1}{p}-1} g(s) = g(s).\end{aligned}$$

Hence Φ is onto. Suppose that $L_p(S, G)$ is proximinal in $L_p(S, X)$. Let $f \in L_p(S, X)$. By Theorem 3.1, $L_1(S, G)$ is proximinal in $L_1(S, X)$ and so there exists $g \in L_p(S, G)$ such that

$$\|\Phi(f) - \Phi(g)\|_1 \leq \|\Phi(f) - \Phi(h)\|_1$$

for all $h \in L_p(S, G)$. By Case 1, for almost all $s \in S$, $\Phi(g)(s)$ is a best approximation of $\Phi(f)(s)$ in G and so for almost all $s \in S$,

$$\|\Phi(f)(s) - \Phi(g)(s)\| \leq \|\Phi(f)(s) - \|f(s)\|^{p-1}z\|$$

for any $z \in G$. Put $w(s) = \|f(s)\|^{1-p}\|g(s)\|^{p-1}g(s)$ if $f(s) \neq 0$ and $w(s) = 0$ otherwise. Then for almost all $s \in S$,

$$\|f(s) - w(s)\| \leq \|f(s) - z\|$$

for any $z \in G$. Since $\|w(s)\| \leq 2\|f(s)\|$ for a.e. $s \in S$, $w \in L_p(S, G)$ and so w is a best approximation of f in $L_p(S, G)$. Hence

$$d_p(f, L_p(S, G)) = \|f - w\|_p = \left(\int d(f(s), G)^p d\mu(s)\right)^{\frac{1}{p}}. \qquad \square$$

COROLLARY 3.3. *Let $1 \leq p < \infty$ and let $L_p(S, G)$ be proximinal in $L_p(S, X)$. Then $g \in L_p(S, G)$ is a best approximation of $f \in L_p(S, X)$ if and only if for almost all $s \in S$, $g(s)$ is a best approximation of $f(s)$ from G.*

Proof. The necessity is clear. Let $f \in L_p(S, X)$ and g be a best approximation of f in $L_p(S, G)$. Then by Theorem 3.2,

$$d_p(f, L_p(S, G)) = \left(\int d(f(s), G)^p d\mu(s)\right)^{\frac{1}{p}}$$

and so

$$\int \|g(s) - f(s)\|^p d\mu(s) = \int d(f(s), G)^p d\mu(s).$$

Since $d(f(s), G)^p \leq \|f(s) - g(s)\|^p$ for all $s \in S$, $\|g(s) - f(s)\| = d(f(s), G)$ for a.e. $s \in S$. Thus for almost all $s \in S$, $g(s)$ is a best approximation of $f(s)$. $\square$

COROLLARY 3.4. *Let $1 \leq p < \infty$ and $L_p(S, G)$ be proximinal in $L_p(S, X)$. If G is Chebyshev in X then $L_p(S, G)$ is Chebyshev in $L_p(S, X)$.*

THEOREM 3.5. *Let $1 \leq p < \infty$. If $L_p(S, G)$ is Chebyshev in $L_p(S, X)$, then G is a Chebyshev subspace of X.*

Proof. Clearly G is a proximinal subspace of X [7]. Let $x_0 \in X$ and $y \in P_G(x_0)$. Choose a measurable subset A such that $0 < \mu(A) < \infty$. For any $x \in X$ define $f_x : S \to X$ by

$$f_x(s) = (I_A \otimes x)(s)$$

for all $s \in S$. Since $\|x_0 - y\| \leq \|x_0 - z\|$ for all $z \in G$, we have

$$\|f_{x_0}(s) - f_y(s)\| \leq \|f_{x_0}(s) - g(s)\|$$

for all $s \in S$ and $g \in L_p(S, G)$. Since $L_p(S, G)$ is Chebyshev in $L_p(S, X)$, f_y is the unique best approximation of f_{x_0}. Thus y is the unique best approximation of x_0 in G. □

Next we can get some corollaries by Theorem 3.1, Corollary 3.4 and Theorem 3.5.

COROLLARY 3.6. *Let $1 < p < \infty$. Then the following are equivalent:*

(i) *$L_1(S, G)$ is Chebyshev in $L_1(S, X)$;*
(ii) *$L_p(S, G)$ is Chebyshev in $L_p(S, X)$.*

COROLLARY 3.7. *Let $1 < p < \infty$. Then the following are equivalent:*

(i) *$l_1(G)$ is Chebyshev in $l_1(X)$;*
(ii) *$l_p(G)$ is Chebyshev in $l_p(X)$.*

THEOREM 3.8. *Let G be a proximinal subspace of X. If, for every $f \in L(S, X)$, there exists a closed separable subspace Y of G such that*

$$P_G(f(s)) \cap Y \neq \emptyset \quad \text{for a.e. } s \in S,$$

then $L(S, G)$ is pointwise proximinal in $L(S, X)$.

Proof. Since $0 \in Y$, we may assume, without loss of generality, that

$$P_G(f(s)) \cap Y \neq \emptyset \text{ for all } s \in S.$$

Define $\phi : S \to 2^Y$ by

$$\phi(s) = P_G(f(s)) \cap Y$$

for all $s \in S$. Then each $\phi(s)$ is a nonempty closed subset of Y so it is a complete subset of Y.

Now we want to prove that ϕ is compactly measurable. Let B be a compact subset of Y. Since B is proximinal in X,

$$\begin{aligned}\phi^{-1}(B) &= \{s \in S : \phi(s) \cap B \neq \emptyset\} \\ &= \{s \in S : P_G(f(s)) \cap Y \cap B \neq \emptyset\} \\ &= \{s \in S : d(f(s), B) = d(f(s), G)\}.\end{aligned}$$

Since the subtraction and the norm in X are continuous and f is measurable, the mapping $s \to d(f(s), A)$ is measurable for any set A. Thus $\phi^{-1}(B)$ is measurable. Therefore, ϕ is compactly measurable. By Theorem 2.7, there exists a measurable function $g : S \to G$ such that $g(s) \in Y$ and $g(s) \in P_G(f(s))$ for all $s \in S$. Hence $L(S, G)$ is pointwise proximinal in $L(S, X)$. □

COROLLARY 3.9. *Let G be a proximinal subspace of X and $1 \leq p < \infty$. Then for every $f \in L_p(S, X)$, there exists a closed separable subspace Y of G such that*

$$P_G(f(s)) \cap Y \neq \emptyset \quad \text{for a.e. } s \in S$$

if and only if $L_p(S, G)$ is proximinal in $L_p(S, X)$.

Proof. ($\Rightarrow$) Similar to the proof of Theorem 3.8, we obtain a strongly measurable function $g : S \to G$ such that $g(s) \in Y$ and $g(s) \in P_G(f(s))$ for all $s \in S$. Since $g(s) \in P_G(f(s))$ for all $s \in S$, we have that $\|g(s)\| \leq 2\|f(s)\|$ for all $s \in S$ and so $g \in L_p(S, G)$. Moreover, g is a best approximation of f in $L_p(S, G)$.

($\Leftarrow$) Take any $f \in L_p(S, X)$ and let g be a best approximation of f in $L_p(S, G)$. Since $g(S)$ is separable, there is a countable subset $\{a_n\}$ of $g(S)$ such that $\{a_n\}$ is dense in $g(S)$. Let $Y = \overline{\text{span}\{a_n\}}$. Then Y is a closed separable subspace of G and $g(S) \subset Y$. By Corollary 3.3, for almost all $s \in S$, $g(s)$ is a best approximation of $f(s)$ in G. Thus

$$P_G(f(s)) \cap Y \neq \emptyset \quad \text{for a.e. } s \in S. \qquad □$$

REMARK. (1) We can easily show that the sufficiency of Corollary 3.9 holds for $0 < p < 1$ and $p = \infty$.

(2) In Theorem 4.1 of [10], Saidi proved that for a finite measure space (S, Ω, μ), $0 < p \leq \infty$ and a proximinal subspace G of X, $L_p(S, G)$ is proximinal in $L_p(S, X)$ if the following two conditions hold:

(i) There exists a positive constant $R \geq 1$ such that

$$\overline{B(0,1) + \ker P_G} \subset B(0,R) + \ker P_G$$

where $B(0,r) = \{g \in G : \|g\| \leq r\}$.

(ii) For $f \in L_p(S,X)$, there exists a closed separable subspace Y of G such that

$$P_G(f(s)) \cap Y \neq \emptyset \quad \text{for a.e. } s \in S.$$

(3) In Remark 4.1 of [10], Saidi proved that for a finite measure space (S, Ω, μ) and $1 \leq p < \infty$, if $L_p(S,G)$ is proximinal in $L_p(S,X)$, then for every $f \in L_p(S,X)$, there exists a closed separable subspace Y of G such that

$$P_G(f(s)) \cap Y \neq \emptyset \quad \text{for a.e. } s \in S.$$

THEOREM 3.10. *Let G be a proximinal subspace of X. If G or* $\ker P_G$ *is separable, then $L(S,G)$ is pointwise proximinal in $L(S,X)$.*

Proof. If G is separable, it follows from Theorem 3.4 in [11]. Now we suppose that $\ker P_G$ is separable. Let $f \in L(S,X)$ and define $\phi : S \to 2^{\ker P_G}$ by

$$\phi(s) = f(s) - P_G(f(s))$$

for all $s \in S$. Then each $\phi(s)$ is a nonempty closed subset of $\ker P_G$ so it is a complete subset of $\ker P_G$.

Now we want to prove that ϕ is compactly measurable: Let B be a compact subset of $\ker P_G$. Then

$$\begin{aligned} \phi^{-1}(B) &= \{s \in S : \phi(s) \cap B \neq \emptyset\} \\ &= \{s \in S : (f(s) - P_G(f(s))) \cap B \neq \emptyset\} \\ &= \{s \in S : f(s) \in B + G\} \\ &= f^{-1}(B + G). \end{aligned}$$

Since B is compact and G is closed, $B + G$ is closed. Thus $\phi^{-1}(B)$ is measurable. Therefore, ϕ is compactly measurable. By Theorem 2.7, there exists a measurable function $g : S \to \ker P_G$ such that $f(s) - g(s) \in P_G(f(s))$ for all $s \in S$. Let $\hat{g} = f - g \in L(S,G)$. Then

$$\|f(s) - \hat{g}(s)\| = \|f(s) - (f(s) - g(s))\| = d(f(s), G)$$

for all $s \in S$. Hence $L(S,G)$ is pointwise proximinal in $L(S,X)$. □

COROLLARY 3.11. *Let G be a proximinal subspace of X. If G or $\ker P_G$ is separable, then for $0 < p \leq \infty$, $L_p(S, G)$ is proximinal in $L_p(S, X)$.*

Proof. This follows from Theorem 2.3, Theorem 2.4 and Theorem 3.10. □

REMARK. Let G be a proximinal subspace of X. In Theorem 1.1 of [6], Khalil and Saidi proved that for a finite measure space (S, Ω, μ) and a proximinally null compact subspace G of X, if G or $\ker P_G$ is separable, then $L_1(S, G)$ is proximinal in $L_1(S, X)$.

References

[1] W. Deeb and R. Khalil, *Best approximation in $L(X, Y)$*, Math. Proc. Camb. Phil. Soc. **104** (1988), 527-531.

[2] C. J. Himmelberg, and F. S. Van Vleck, *Multifunctions on abstract measurable spaces and applications to stochastic decision theory,* Ann. Mat. Pure Appl. **101** (1974), 229-236.

[3] D. Hussein and R. Khalil, *Best approximation in tensor product spaces,* Soochow J. Math. **18** (1992), No. 4, 397-407.

[4] R. Khalil, *Best approximation in $L^p(I, X)$*, Math. Proc. Camb. Phil. Soc. **94** (1983), 277-279.

[5] R. Khalil and W. Deeb, *Best approximation in $L^p(\mu, X)$ II,* J. Approx. Theory **59** (1989), 296-299.

[6] R. Khalil and F. Saidi, *Best approximation in $L^1(I, X)$,* Proc. Amer. Math. Soc. **123** (1995), 183-190.

[7] M. B. Lee and S. H. Park, *Proximity maps for certain spaces*, Bull. Korean Math. Soc. **94** (1997), 259-271.

[8] W. A. Light, *Proximinality in $L_p(S, Y)$*, Rocky Mountain J. Math. **19** (1989), 251-259.

[9] W. A. Light and E. W. Cheney, *Approximation theory in tensor product spaces,* Lect. Notes in Math. Vol. 1169, Springer-Verlag, New York/Berlin, 1985.

[10] F. B. Saidi, *On the smoothness of the metric projection and its application to proximinality in $L^p(S, X)$,* J. Approx. Theory **83** (1995), 205-219.

[11] You Zhao-Yong and Guo Tie-Xin, *Point-wise best approximation in the space of strongly measurable functions with applications to best approximation in $L^p(\mu, X)$,* J. Approx. Theory **78** (1994), 314-320.

MUN BAE LEE, SUNG HO PARK, DEPARTMENT OF MATHEMATICS, SOGANG UNIVERSITY, SEOUL 121-742, KOREA

HYANG JOO RHEE, DEPARTMENT OF GENERAL STUDIES, DUKSUNG WOMEN'S UNIVERSITY, SEOUL 132-714, KOREA