REMARKS ON WEAK HYPERMODULES

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ABSTRACT. $H_v$-rings first were introduced by Vougiouklis in 1990. Then Darafsheh and the present author defined the $H_v$-ring of fractions $S^{-1}R$ of a commutative hyperring. The largest class of multi-valued systems satisfying the module-like axioms is the $H_v$-module. In this paper we define $H_v$-module of fractions of a hypermodule. Some interesting results concerning this $H_v$-module is proved.

1. Basic Definitions of Hyperstructures

The concept of a hyperstructure first was introduced by Marty in [2]. A hyperstructure is a set $H$ together with a function $\cdot : H \times H \rightarrow \mathcal{P}^*(H)$ called hyperoperation, where $\mathcal{P}^*(H)$ denotes the set of all non-empty subsets of $H$. If $A, B \subseteq H$, $x \in H$ then we define

$$A \cdot B = \bigcup_{a \in A, b \in B} a \cdot b, \quad x \cdot B = \{x\} \cdot B, \quad A \cdot x = A \cdot \{x\}.$$  

**DEFINITION 1.1.** A hyperstructure $(H, \cdot)$ is called a hypergroup if

(i) $x \cdot (y \cdot z) = (x \cdot y) \cdot z, \forall x, y, z \in H,$

(ii) $a \cdot H = H \cdot a = H, \forall a \in H.$

**DEFINITION 1.2.** A multivalued system $(R, +, \cdot)$ is a hyperring if

(i) $(R, +)$ is a hypergroup,

(ii) $(R, \cdot)$ is a semi hypergroup,

(iii) $(\cdot)$ is distributive with respect to $(+)$, i.e., for all $x, y, z$ in $R$ we have

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z), \quad (x + y) \cdot z = (x \cdot z) + (y \cdot z).$$

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A hyperring may be commutative with respect to $(+)$ or $(\cdot)$. If $R$ is commutative with respect to both $(+)$ and $(\cdot)$, then we call it a commutative hyperring. If there exists $u \in R$ such that $x \cdot u = u \cdot x = \{x\}$, $\forall x \in R$, then $u$ is called the scalar unit of $R$ and is denoted by 1.

**Definition 1.3.** $M$ is a left hypermodule over hyperring $R$ ($R$-hypermodule) if $(M, +)$ is a commutative hypergroup and there exists a map $\cdot : R \times M \rightarrow \mathcal{P}^*(M)$ denoted by $(r, m) \rightarrow rm$ such that for all $r_1, r_2$ in $R$ and $m_1, m_2$ in $M$, we have

\[
\begin{align*}
    r_1(m_1 + m_2) &= r_1m_1 + r_1m_2, \\
    (r_1 + r_2)m_1 &= r_1m_1 + r_2m_1, \\
    (r_1r_2)m_1 &= r_1(r_2m_1).
\end{align*}
\]

There are generalizations of the above hyperstructures (hypergroup, hyperring and hypermodule) where axioms are replaced by the weak ones. That is instead of the equality on sets one has non-empty intersections.

$H_v$-structures first introduced by Vougiouklis in the Fourth AHA congress (1990) [5]. In this paper we are interested in $H_v$-rings and $H_v$-modules.

**Definition 1.4.** A multivalued system $(R, +, \cdot)$ is called an $H_v$-ring if the following axioms hold:

(i) $(R, +)$ is an $H_v$-group, i.e.,

\[
(x + y) + z \cap x + (y + z) \neq \emptyset, \quad \forall x, y, z \in R,
\]

\[
a + R = R + a = R, \quad \forall a \in R,
\]

(ii) $(x \cdot y) \cdot z \cap x \cdot (y \cdot z) \neq \emptyset, \quad \forall x, y, z \in R,$

(iii) $(\cdot)$ is weak distributive with respect to $(+)$, i.e., for all $x, y, z \in R$,

\[
x \cdot (y + z) \cap (x \cdot y + x \cdot z) \neq \emptyset,
\]

\[
(x + y) \cdot z \cap (x \cdot z + y \cdot z) \neq \emptyset.
\]

**Definition 1.5.** $M$ is a left $H_v$-module over an $H_v$-ring $R$ if $(M, +)$ is a weak commutative $H_v$-group and there exists a map $\cdot : R \times M \rightarrow \mathcal{P}^*(M)$ denoted by $(r, m) \rightarrow rm$ such that for all $r_1, r_2$ in $R$ and $m_1, m_2$ in $M$, we have

\[
\begin{align*}
    r_1(m_1 + m_2) \cap (r_1m_1 + r_1m_2) &\neq \emptyset, \\
    (r_1 + r_2)m_1 \cap (r_1m_1 + r_2m_1) &\neq \emptyset, \\
    (r_1r_2)m_1 \cap r_1(r_2m_1) &\neq \emptyset.
\end{align*}
\]
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**Definition 1.6.** Let $M_1$ and $M_2$ be two $H_v$-modules over an $H_v$-ring $R$. A mapping $f : M_1 \rightarrow M_2$ is called an $R - H_v$-homomorphism if, \( \forall x, y \in M_1 \) and \( \forall r \in R \), the following relations hold:

\[
f(x + y) \cap (f(x) + f(y)) \neq \emptyset, \quad f(rx) \cap rf(x) \neq \emptyset.
\]

$f$ is called an inclusion $R$-homomorphism if, \( f(x+y) \subseteq f(x)+f(y) \), \( f(rx) \subseteq rf(x) \).

Finally $f$ is called a strong $R$-homomorphism if, \( f(x + y) = f(x) + f(y) \), \( f(rx) = rf(x) \).

If there exists a strong one to one homomorphism from $M_1$ onto $M_2$, then $M_1$ and $M_2$ are called isomorphic.

In this paper, we shall work over a commutative hyperring $R$ with scalar unit, and we shall assume that $M$ is an $R$-hypermodule. We remark that according to [1], a non-empty subset $S$ of $R$ is a strong multiplicatively closed subset (s.m.c.s) if the following conditions hold:

(i) \( 1 \in S \),
(ii) \( a \cdot S = S \cdot a = S, \forall a \in S \).

Darafsheh and the present author in [1] defined the $H_v$-ring of fractions $S^{-1}R$ of a commutative hyperring. The construction of $S^{-1}R$ can be carried through with an $R$-hypermodule $M$ in place of the hyperring $R$.

In section 2 of this paper we introduce the set of fractions $S^{-1}M$ and define addition and multiplication by elements of $S^{-1}R$, then we get that $S^{-1}M$ is an $S^{-1}R - H_v$-module as well as some interesting results with this respect.

**2. $H_v$-module of Fractions**

Let $X$ be the set of all ordered pairs $(m, s)$ where $m \in M$, $s \in S$. For $A \subseteq M$ and $B \subseteq S$, we denote the set \( \{(a, b) \mid a \in A, b \in B\} \) by $(A, B)$.

The relation $\sim$ is defined on $P^*(X)$ as follows:

$(A, B) \sim (C, D)$ iff there exists a subset $T$ of $S$ such that $T \cdot (B \cdot C) = T \cdot (D \cdot A)$.

**Lemma 2.1.** $\sim$ is an equivalence relation on $P^*(X)$. 

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If we restrict the relation \( \sim \) on \( X \), and identify \( (m, x) \in X \) with the subset \( \{(m, x)\} \) of \( X \), then we obtain the following two lemmas.

**Lemma 2.2.** For \( (m_1, s_1), (m_2, s_2) \in X \), we have \( (m_1, s_1) \sim (m_2, s_2) \) iff there exists \( T \subseteq S \) such that \( T \cdot (s_1 \cdot m_2) = T \cdot (s_2 \cdot m_1) \).

**Lemma 2.3.** \( \sim \) is an equivalence relation on \( X \).

The equivalence class containing \( (m, s) \) is denoted by \( [m, s] \) and we let \( S^{-1}M \) to be the set of all the equivalence classes.

In \( \mathcal{P}^*(X) \), the equivalence class containing \( (A, B) \) is denoted by \( [[A, B]] \).

We define
\[
\langle \langle A, B \rangle \rangle = \bigcup_{(C, D) \in [[A, B]]} \{ [c, d] \mid c \in C, d \in D \}.
\]

**Lemma 2.4.** For all \( m \in M, s \in S \), we have \( \langle \langle m, s \rangle \rangle = \langle \langle sm, ss \rangle \rangle \).

Now we define addition and multiplication by elements of \( S^{-1}R \), as follows:
- \( [m_1, s_1] \oplus [m_2, s_2] = \bigcup \{ [a, b] \mid a \in A, b \in B \} = \langle \langle s_1 m_2 + s_2 m_1, s_1 s_2 \rangle \rangle \), where the union is over \( (A, B) \in [[s_1 m_2 + s_2 m_1, s_1 s_2]] \)
- \( [r, s] \odot [m_1, s_1] = \bigcup \{ [a, b] \mid a \in A, b \in B \} = \langle \langle rm_1, ss_1 \rangle \rangle \), where the union is over \( (A, B) \in [[rm_1, ss_1]] \)

In both cases \( [m_1, s_1], [m_2, s_2] \in S^{-1}M \) and \( [r, s] \in S^{-1}R \).

**Theorem 2.5.** \( \oplus \) and \( \odot \) defined above are independent of the choices of representatives \( [m_1, s_1], [m_2, s_2] \) and \( [r, s] \) and that \( S^{-1}M \) satisfies the axioms of an \( S^{-1}R - H_v \)-module.

If we define
\[
r \odot [m_1, s_1] = \langle \langle rm_1, s_1 \rangle \rangle
\]
then \( S^{-1}M \) becomes an \( R - H_v \)-module.

**Proof.** The proof is straightforward and omitted. \( \square \)

**Definition 2.6.** Hypersubmodule \( U \) of \( M \) is called a hyperisolated submodule if it satisfies the following axiom:
For all \( A \subseteq U, B \subseteq S \) if \( (X, Y) \in [[A, B]] \) then \( X \subseteq U \).
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**Lemma 2.7.** Let $U$ be a hyperisolated submodule of $M$ then the set $S^{-1}U = \{ [u, s] \mid u \in U, \ s \in S \}$ is an $R - H_0$-submodule of $S^{-1}M$.

**Proof.** First, we prove that $(S^{-1}U, \oplus)$ is an $H_0$-subgroup of $(S^{-1}M, \oplus)$. For every $[m_1, s_1], [m_2, s_2] \in S^{-1}U$, we have

$$[m_1, s_1] \oplus [m_2, s_2] = \bigcup_{(A, B) \in [[s_2 m_1 + s_1 m_2, s_1 s_2]]} \{ [m, s] \mid m \in A, \ s \in B \}.$$

Since $m_1, m_2 \in U$ then we get $s_2 m_1 + s_1 m_2 \subseteq U$ and since $U$ is a hyperisolated submodule of $M$, then $A \subseteq U$. Therefore $[m_1, s_1] \oplus [m_2, s_2] \subseteq S^{-1}U$.

Now we prove the equality $S^{-1}U = [m_1, s_1] \oplus S^{-1}U$, for all $[m_1, s_1] \in S^{-1}U$. Suppose $[m, s] \in S^{-1}U, \ m \in U$. Since $s, s_1 \in S$, by definition of $S$ there exists $s_2 \in S$ such that $s \in s_1 s_2$. And since $U$ is a hypersubmodule, we have $s_2 m_1 + (s_1 + 1)U = U$. Now $m \in U$ and $s_2 m_1 + (s_1 + 1)U = U$ imply that there exists $m_2 \in U$ such that $m \in s_2 m_1 + s_1 m_2 + m_2$ hence $m \in s_2 m_1 + s_1 (m_2 + s_3 m_2)$ where $1 \in s_3 s_1$. So there exists $x \in m_2 + s_3 m_2$ such that $m \in s_2 m_1 + s_1 x$, therefore $[m, s] \in [m_1, s_1] \oplus [x, s_2]$ implying $S^{-1}U \subseteq [m_1, s_1] \oplus S^{-1}U$.

It remains to prove that $R \circ (S^{-1}U) \subseteq S^{-1}U$. To do this suppose that $[u, s] \in S^{-1}U$ and $r \in R$, then

$$r \circ [u, s] = \bigcup_{(A, B) \in [[ru, s]]} \{ [x, y] \mid x \in A, \ y \in B \}.$$ 

Since $u \in U$ we have $ru \subseteq U$ and since $U$ is a hyperisolated submodule of $M$ we get $A \subseteq U$. Consequently $r \circ [u, s] \subseteq S^{-1}U$. Therefore the lemma is proved. □

**Theorem 2.8.** Let $M_1$ and $M_2$ be two $R$-hypermodules and let $f : M_1 \rightarrow M_2$ be a strong $R$-homomorphism. Then the map $S^{-1}(f) : S^{-1}M_1 \rightarrow S^{-1}M_2$ defined by $S^{-1}(f)[m, s] = [f(m), s]$, is an $S^{-1}R - H_0$-homomorphism.

**Proof.** Suppose that $[m_1, s_1], [m_2, s_2] \in S^{-1}M_1$ and $[r, s] \in S^{-1}R$. First we show that $S^{-1}(f)$ is well-defined. If $[m_1, s_1] = [m_2, s_2]$ then there exists $T \subseteq S$ such that $T \cdot (s_1 \cdot m_2) = T \cdot (s_2 \cdot m_1)$ which implies $f(T \cdot (s_1 \cdot m_2)) = f(T \cdot (s_2 \cdot m_1))$ and so $T \cdot (s_1 \cdot f(m_2)) = T \cdot (s_2 \cdot f(m_1))$ or
[f(m_1), s_1] = [f(m_2), s_2]. Therefore S^{-1}(f) is well-defined.
Moreover, S^{-1}(f) is an S^{-1}R - H_v-homomorphism because, we have
\[
S^{-1}(f)([m_1, s_1] \oplus [m_2, s_2])
= S^{-1}(f) \left( \bigcup \{ [a, b] \mid a \in A, b \in B \} \right) \tag{1}
= \bigcup \{ S^{-1}(f)([a, b]) \mid a \in A, b \in B \} \tag{2}
= \bigcup \{ [f(a), b] \mid a \in A, b \in B \} \tag{3}
\]
and
\[
S^{-1}(f)([m_1, s_1] \oplus S^{-1}(f)([m_2, s_2])
= [f(m_1), s_1] \oplus [f(m_2), s_2]
= \bigcup \{ [a, b] \mid a \in A, b \in B \} \tag{4}
= \bigcup \{ [a, b] \mid a \in A, b \in B \} \tag{5}
\]
Therefore we have
\[
\{ [f(a), b] \mid a \in s_1 m_2 + s_2 m_1, b \in s_1 s_2 \}
\subseteq S^{-1}(f)([m_1, s_1] \oplus [m_2, s_2]),
\]
\[
\{ [a, b] \mid a \in f(s_1 m_2 + s_2 m_1), b \in s_1 s_2 \}
\subseteq S^{-1}(f)([m_1, s_1]) \oplus S^{-1}(f)([m_2, s_2]).
\]
And so
\[
S^{-1}(f)([m_1, s_1] \oplus [m_2, s_2]) \cap S^{-1}(f)([m_1, s_1]) \oplus S^{-1}(f)([m_2, s_2]) \neq \emptyset.
\]
Similarly, we get
\[
\{ [f(a), b] \mid a \in r m_1, b \in s s_1 \} \subseteq S^{-1}(f)([r, s] \odot [m_1, s_1]),
\]
\[
\{ [a, b] \mid a \in r f(m_1), b \in s s_1 \} \subseteq [r, s] \odot S^{-1}(f)([m_1, s_1]).
\]
And so
\[
S^{-1}(f)([r, s] \odot [m_1, s_1]) \cap [r, s] \odot S^{-1}(f)([m_1, s_1]) \neq \emptyset,
\]
which proves that S^{-1}(f) is an S^{-1}R - H_v-homomorphism. \qed

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**Lemma 2.9.** The natural mapping $\Psi : M \rightarrow S^{-1}M$ defined by $\Psi(m) = [m, 1]$ is an inclusion $R$-homomorphism.

**Proof.** For every $m_1, m_2 \in M$, we have

$$\Psi(m_1 + m_2) = \{[\alpha, 1] \mid \alpha \in m_1 + m_2\} \subseteq \bigcup_{(A,B)\in[[m_1+m_2,1]]} \{[a, b] \mid a \in A, b \in B\}$$

$$= [m_1, 1] \oplus [m_2, 1]$$

$$= \Psi(m_1) \oplus \Psi(m_2).$$

And for every $r \in R$ and $m \in M$ we have

$$\Psi(rm) = \{[\alpha, 1] \mid \alpha \in rm\} \subseteq \langle\langle rm, 1\rangle\rangle = r \circ \Psi(m).$$

Therefore $\Psi$ is an inclusion $R$-homomorphism.

**Definition 2.10.** Let $M_1$ and $M_2$ be two $R$-hypermodules and $f, g : M_1 \rightarrow M_2$ be $R$-homomorphisms. We define the mapping $S^{-1}(f + g)$ from $S^{-1}M_1$ into $P^*(S^{-1}M_2)$ as follows:

$$S^{-1}(f + g)([m, s]) = \langle\langle f(m) + g(m), s\rangle\rangle.$$

**Proposition 2.11.** Let $M_1$, $M_2$ and $M_3$ be $R$-hypermodules and $f, g : M_1 \rightarrow M_2$ and $h : M_2 \rightarrow M_3$ be $R$-homomorphisms. Then

(i) $S^{-1}(f + g) = S^{-1}(f) + S^{-1}(g),$

(ii) $S^{-1}(ho f) = S^{-1}(h)oS^{-1}(f),$

(iii) $S^{-1}(id_{M_1}) = id_{S^{-1}M_1}.$

**Proposition 2.12.** Let $C_1$ be the category of $R$-hypermodules and strong $R$-homomorphisms and let $C_2$ be the category of $R - H_v$-modules and $R - H_v$-homomorphisms. Then the mapping $S^{-1} : C_1 \rightarrow C_2$ is a functor.

**Definition 2.13.** Let $M_1$, $M_2$ and $M_3$ be $R$-hypermodules ($R - H_v$-modules respectively) and let $U$ be a subhypermodule ($H_v$-submodule) of $M_3$. The sequence of strong $R$-homomorphisms $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ is said to be $U$-exact if $\text{Im}f = g^{-1}(U)$.
THEOREM 2.14. Let $U$ be an $H_v$-isolated submodule of $M_3$ and the sequence $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ be $U$-exact. Then the sequence $S^{-1}M_1 \xrightarrow{S^{-1}(f)} S^{-1}M_2 \xrightarrow{S^{-1}(g)} S^{-1}M_3$ is $S^{-1}U$-exact, i.e., $\text{Im}S^{-1}(f) = (S^{-1}(g))^{-1}(S^{-1}U)$.

Proof. Suppose that $[m, s] \in \text{Im}S^{-1}(f)$ then there exists $m_1 \in M_1$ such that $[m, s] = [f(m_1), s]$. Since $f(m_1) \in \text{Im}f = g^{-1}(U)$, there exists $u \in U$ such that $f(m_1) \in g^{-1}(u)$ and so $[f(m_1), s] \in \{ [x, s] \mid x \in g^{-1}(u) \}$ which implies $[f(m_1), s] \in (S^{-1}(g))^{-1}([u, s])$. Therefore $[f(m_1), s] \in (S^{-1}(g))^{-1}(S^{-1}U)$.

Now, if $[u, t] \in (S^{-1}(g))^{-1}(S^{-1}U)$, then $[u, t] \in (S^{-1}(g))^{-1}([u, s] \mid u \in U, s \in S \}$ which implies $[u, t] \in \{ [x, s] \mid g(x) \in U, s \in S \}$. Therefore for some $x$ where $g(x) \in U$ we have $[u, t] = [x, s]$. From $g(x) \in U$ we get $x \in \text{Im}f$ and so $[x, s] \in \text{Im}S^{-1}(f)$. Therefore $[u, t] \in \text{Im}S^{-1}(f)$. \qed

3. The Fundamental Relations $\gamma^*$ and $\epsilon^*$

Consider the left $H_v$-module $M$ over an $H_v$-ring $R$. The relation $\gamma^*$ is the smallest equivalence relation on $R$ such that the quotient $R/\gamma^*$ is a ring. $\gamma^*$ is called the fundamental equivalence relation on $R$ and $R/\gamma^*$ is called the fundamental ring, see [3], [4]. The fundamental relation $\epsilon^*$ on $M$ over $R$ is the smallest equivalence relation such that $M/\epsilon^*$ is a module over the ring $R/\gamma^*$, see [4].

According to [4], if $\mathcal{U}$ denotes the set of all expressions consisting of finite hyperoperations of either on $R$ and $M$ or the external hyperoperation applied on finite subsets of $R$ and $M$. Then a relation $\epsilon$ can be defined on $M$ whose transitive closure is the fundamental relation $\epsilon^*$. The relation $\epsilon$ is defined as follows: for all $x, y \in M$,

$$x \epsilon y \iff \{ x, y \} \subseteq u, \text{ for some } u \in \mathcal{U}.$$ 

Suppose $\gamma^*(r)$ is the equivalence class containing $r \in R$ and $\epsilon^*(x)$ is the equivalence class containing $x \in M$. On $M/\epsilon^*$ the sum $\circ$ and the external product $\Box$ using the $\gamma^*$ classes in $R$, are defined as follows: for all $x, y \in M$ and for all $r \in R$,

$$\epsilon^*(x) \circ \epsilon^*(y) = \epsilon^*(c), \quad \forall c \in \epsilon^*(x) + \epsilon^*(y),$$

$$\gamma^*(r) \Box \epsilon^*(x) = \epsilon^*(d), \quad \forall d \in \gamma^*(r) \cdot \epsilon^*(x).$$ 

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Now, we will prove two theorems concerning the fundamental relations $\gamma^*$ and $\epsilon^*$.

Let $\epsilon^*_s$ be the fundamental equivalence relation on $S^{-1}M$ and $U_s$ be the set of all expressions consisting of finite hyperoperations of either on $S^{-1}R$ and $S^{-1}M$ or of external hyperoperation. In this case $S^{-1}M/\epsilon^*_s$ is an $S^{-1}R/\gamma^*_s$-module.

**THEOREM 3.1.** $S^{-1}M/\epsilon^*_s$ is an $R/\gamma^*$-module.

**Proof.** We can define

$$\gamma^*(r) * \epsilon^*_s([m, s]) = \gamma^*_s([r, 1]) \Box \epsilon^*_s([m, s]).$$

Then it is clear that $S^{-1}M/\epsilon^*_s$ is an $R/\gamma^*$-module. $\square$

**THEOREM 3.2.** There is an $R/\gamma^*$-homomorphism $h : M/\epsilon^* \rightarrow S^{-1}M/\epsilon^*_s$.

**Proof.** We define $h(\epsilon^*(m)) = \epsilon^*_s([m, 1])$. First we prove that $h$ is well defined. If $\epsilon^*(m_1) = \epsilon^*(m_2)$ then $m_1 \epsilon^* m_2$ which holds iff $\exists x_1, \ldots, x_{m_1}, u, \ldots, u_{m_2} \in U$ with $x_1 = m_1, x_{m_1} = m_2$ such that $\{x_i, x_{i+1}\} \subseteq u, i = 1, \ldots, m$ which implies $\{[x_i, 1], [x_{i+1}, 1]\} \subseteq \langle \langle u, 1 \rangle \rangle \subseteq U_s$. Therefore $[m_1, 1] \epsilon^*_s[m_2, 1]$ and so $\epsilon^*_s([m_1, 1]) = \epsilon^*_s([m_2, 1])$. Thus $h$ is well-defined.

$h$ is a homomorphism because

$$h(\epsilon^*(a) \circ \epsilon^*(b)) = h(\epsilon^*(c)) = \epsilon^*_s([c, 1]), \quad \forall c \in \epsilon^*(a) + \epsilon^*(b),$$

$$h(\epsilon^*(a)) \circ h(\epsilon^*(b)) = \epsilon^*_s([a, 1]) \circ \epsilon^*_s([b, 1]) = \epsilon^*_s([d, s]),$$

$$\forall[d, s] \in \epsilon^*_s([a, 1]) \oplus \epsilon^*_s([b, 1]),$$

setting $d = c \in a + b, s = 1$. So it is proved that $h(\epsilon^*(a) \circ \epsilon^*(b)) = h(\epsilon^*(a)) \circ h(\epsilon^*(b))$.

And also we have

$$h(\gamma^*(r) \Box \epsilon^*(m)) = h(\epsilon^*(a)) = \epsilon^*_s([a, 1]), \quad \forall a \in \gamma^*(r) \epsilon^*(m),$$

$$\gamma^*(r) * h(\epsilon^*(m)) = \gamma^*(r) * \epsilon^*_s([m, 1]) = \gamma^*_s([r, 1]) \Box \epsilon^*_s([m, 1]) = \epsilon^*_s([b, s]),$$

$$\forall [b, s] \in \gamma^*_s([r, 1]) \oplus \epsilon^*_s([m, 1]).$$

Therefore, since we can take $b = a \in r \cdot m$ and $s = 1$, we get

$$h(\gamma^*(r) \Box \epsilon^*(m)) = \gamma^*(r) * h(\epsilon^*(m))$$

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Hence $h$ is a homomorphism of modules.

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