Bull. Korean Math. Soc. **36** (1999), No. 3, pp. 609–619

LOCAL STRUCTURE OF TRAJECTORY FOR EXTREMAL FUNCTIONS

SUK YOUNG LEE

ABSTRACT. In this note we study more about the omitted arc for the extremal functions and its $\frac{\pi}{4}$-property based upon Schiffer's variational method and Brickman-Wilken's result. We give an example other than the Koebe function which is both a support point of S and the extreme point of HS. Furthermore, we discuss the relations between the support points and the Löwner chain.

1. Introduction

Let $\triangle$ be the open unit disk in the complex plane $\mathbb{C}$, and let $H(\triangle)$ denote the linear space of holomorphic functions in $\triangle$, endowed with the usual topology of local uniform convergence. A particular subset of $H(\triangle)$ is the class S which consists of all functions f which are univalent in $\triangle$ and normalized so that $f(0) = 0$ and $f'(0) = 1$.

For the study of linear extremal problems in S it is natural to consider two sets of functions, the support points of S and the extreme points of S.

We call $f \in S$ a support point of S if there exists a continuous linear functional J defined on $H(\triangle)$ which is non-constant on S and

$$ReJ(f) = \max_{g \in S} ReJ(g).$$

$f \in S$ is an extreme point of S provided for $0 < t < 1$, $g \in S$, $h \in S$,

$$f = tg + (1-t)h \text{ implies that } f = g = h.$$

Received August 10, 1998.

1991 Mathematics Subject Classification: Primary 30C45.

Key words and phrases: slit mapping, trajectories, support points, extreme points.

The author acknowledges support received from the Ministry of Education, ROK via 1997-98, BSRI-97-1424.

It is well known that all rotations of the Koebe functions

$$k_\theta(z) = \frac{z}{(1 - e^{i\theta}z)^2}$$

maps $\triangle$ onto the complement of a ray from $-\frac{1}{4}e^{-i\theta}$ to ∞. A single slit mapping is a slit mapping whose range is the complement of a single Jordan arc. Functions in S that map $\triangle$ onto the complement of a single Jordan arc are known to play a crucial role in the study of extremal problems for S.

Schiffer ([14]) showed that for a quite general functional J, any solution to $\{\max ReJ(g) : g \in S\}$ maps $\triangle$ onto the complement of a finite number of analytic Jordan arcs, and he determined a differential equation for the arcs in terms of parameters involving the extremal functions.

Goluzin ([7]) showed that if J has the special form

$$J(g) = \sum_{i=1}^{n} b_u g^{(i)}(0), \quad (n \geq 2),$$

then $\mathbb{C} \setminus f(\triangle)$ consists of finitely many arcs with the $\frac{\pi}{4}$-property; that is, the angle between the position vector and the tangent vector at any point on the slit is smaller in magnitude than $\frac{\pi}{4}$.

For the particular functional $J(f) = Re\ a_n$, where $f(z) = z + \sum_{k=1}^{\infty} a_k z_k$ Schiffer ([15]) verified that $J(f^2) \neq 0$ and any extremal function maps $\triangle$ onto the complement of a single analytic slit with an asymptotic direction at ∞ and this slit possesses the $\frac{\pi}{4}$-property.

Pfluger ([13]) generalized this result by showing that any extremal function for

$$\max_{g \in S} ReJ(f) \quad (J \text{ non-constant on } S)$$

maps $\triangle$ onto the complement of an analytic slit which has the $\frac{\pi}{4}$-property and an asymptotic direction at ∞. Brickman and Wilken ([3]) found a considerably simpler proof of this result.

In this note we study more about the omitted arc and $\frac{\pi}{4}$-property based upon Schiffer's variational method and Brickman and Wilken's result. We provide local structure of trajectories that the range of an extremal function is the complement of a Jordan analytic arc satisfying a certain differential equation. Furthermore, we discuss the relations between the support points and the Löwner chain.

2. Local Structure of Trajectories

According to the Schiffer's result, if Γ is the complement of the range of an extremal function, Γ consists of a collection of analytic arcs satisfying a differential equation of the form $Q(w)dw^2 > 0$, where Q is analytic on Γ. Such an expression $Q(w)dw^2$ is called a quadratic differential and the arcs for which $Q(w)dw^2 > 0$ are called its trajectories. The following Schiffer's variational method will give us much more precise information about the omitted arc.

LEMMA 1 (Schiffer). *Let J be a continuous functional on $H(\triangle)$, and let $f \in S$ be a point where $Re\{J\}$ attains its maximum value on S. Suppose that J has a Fréchet differential $l(\cdot\,;f)$ which is not constant on S. Then f maps the unit disk $\triangle$ onto the complement of a system of finitely many analytic arcs $w = w(t)$ satisfying the differential equation*

$$\frac{1}{w^2} l\left(\frac{f^2}{f-w}; f\right), \left(\frac{dw}{dt}\right)^2 > 0.$$

LEMMA 2 (Brickman and Wilken). *Each extreme point of S and each support point of S have the monotonic modulus property, i.e., it maps $\triangle$ onto the complement of an arc which extends to ∞ with increasing modulus.*

THEOREM 2.1 (Duren [5]). *Let J be a continuous linear functional on $H(\triangle)$ which is not constant on S and let f maximize $Re\{J\}$ on S. Then f maps $\triangle$ onto the complement of a single analytic arc Γ which satisfies the differential equation*

$$\frac{1}{w^2} J\left(\frac{f^2}{f-w}\right) dw^2 > 0. \tag{2.1}$$

At each point $w \in \Gamma$ except perhaps the finite tip, the tangent line makes angle of less than $\frac{\pi}{4}$ with the radical line from 0 to w.

Proof. Since a continuous linear functional is its own Fréchet differential, Lemma 1 shows that Γ consists of finitely many analytic arcs satisfying the differential equation (2.1). In fact, Γ is a single unbranched arc which extends to ∞ with monotonic modulus by Lemma 2.

Choose a point $w \in \Gamma$, not an endpoint, and consider the function

$$g = \frac{wf}{w-f}.$$

Observe that g belongs to S and maps $\triangle$ onto the complement of two disjoint arcs extending to ∞. Thus g is not a support point, and so

$$Re\{J(g)\} < Re\{J(f)\}. \tag{2.2}$$

Since J is linear, (2.2) is equivalent to

$$Re\left\{J\left(\frac{f^2}{f-w}\right)\right\} > 0, \quad w \in \Gamma, \tag{2.3}$$

where w is not an endpoint of Γ.

The inequality (2.3) has two consequences. First, the fact that $J(\frac{f^2}{f-w}) \neq 0$ assures that the quadratic differential has no singularities on Γ, except perhaps at the endpoints, so that Γ has no corners. In other words, Γ is a single analytic arc. Second, the inequality (2.3) may be combined with (2.1) to show that

$$Re\left\{\left(\frac{dw}{w}\right)^2\right\} > 0 \quad \text{on } \Gamma,$$

which is equivalent to $|arg\{\frac{dw}{w}\}| \leq \frac{\pi}{4}$. This completes the proof. □

REMARK. A recent result of Duren, Leung and Schiffer ([6]) shows that under very general conditions the omitted arc is a half line whenever it has a radial angle of $\pm\frac{\pi}{4}$ at its tip.

LEMMA 3 (Schiffer). *Let J be a continuous linear functional on $H(\triangle)$ which is not constant on S, and let f maximize $Re\{J\}$ on S. Then*

$$J(f^2) \neq 0.$$

THEOREM 2.2 (Duren [5]). *Let J be a continuous linear functional on $H(\triangle)$ which is not constant on S, and let f maximize $Re\{J\}$ on S. Then the arc Γ omitted by f is asymptotic to the half-line*

$$w = \frac{1}{3}\frac{J(f^3)}{J(f^2)} - J(f^2)t, \quad t \geq 0, \tag{2.4}$$

at ∞. Furthermore, the radial angle arg $\{\frac{dw}{w}\}$ of Γ tends to 0 at ∞.

Proof. Let Γ be parametrized by $w = w(t)$, $0 < t < \infty$, in such a way that $w(t) \to \infty$ as $t \to 0$ and the differential equation (2.1) takes the form

$$\frac{1}{w^2} J\left(\frac{f^2}{f-w}\right)\left(\frac{dw}{dt}\right)^2 = 1.$$

By Lemma 3, we have $J(f^2) \neq 0$. So the substitution $w = u^{-2}$ transforms Γ to an analytic curve

$$u = b_1 t + b_3 t^3 + \cdots$$

through the origin which satisfies

$$-4J\left(\frac{f^2}{1-fu^2}\right)\left(\frac{du}{dt}\right)^2 = 1,$$

or

$$(c_0 + c_1 b_1^2 t^2 + \cdots)(b_1^2 + 6b_1 b_3 t^2 + \cdots) = -\frac{1}{4},$$

where $c_n = J(f^{n+2})$, $n = 0, 1, 2, \cdots$. Equating coefficients, we obtain

$$c_0 b_1^2 = -\frac{1}{4}, \qquad c_1 b_1^4 + 6c_0 b_1 b_3 = 0. \tag{2.5}$$

On the other hand,

$$w = u^{-2} = b_1^{-2}t^{-2} - 2b_1^{-3}b_3 + O(t^2), \quad t \to 0.$$

Thus Γ is asymptotic to the line

$$w = \alpha + \beta t, \quad t \to \infty,$$

where $\alpha = -2b_1^{-3}b_3$ and $\beta = b_1^{-2}$. But the equations (2.5) give

$$b_1^{-2} = -4c_0, \qquad b_1^{-3}b_3 = -\frac{c_1}{6c_0}.$$

This proves that Γ approaches the half-line (2.4) near ∞. In particular, $\arg w \to \arg\{-J(f^2)\}$ as $w \to \infty$ along Γ.

Since $J\left(\frac{f^2}{f-w}\right) = -\frac{J(f^2)}{w} + O\left(\frac{1}{w^2}\right)$, it follows that $\arg J\left(\frac{f^2}{f-w}\right) \to 0$. Thus the differential equation

$$J\left(\frac{f^2}{f-w}\right)\left(\frac{dw}{w}\right)^2 > 0$$

shows that the radial angle $\arg\{\frac{dw}{w}\} \to 0$ as $w \to \infty$ along Γ. □

REMARK. Hengartner and Schober ([8]) used the monotone modulus property to show that for every support point $f \in S$, both $\frac{f(z)}{z}$ and $\log\left[\frac{f(z)}{z}\right]$ are univalent in Δ. The $\frac{\pi}{4}$-property was used in [9] to show that for every support point $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in S$, $|a_2| > 1$ and $|a_3| > \frac{3}{8}$.

Kirwan and Pell ([10]) improved these estimates to

$$|a_2| > \sqrt{2} \quad \text{and} \quad |a_3| > 1,$$

and they produced an example for which

$$|a_2| < 1.774.$$

The sharp lower bounds are unknown.

3. Examples

It is well known in [5] that the Koebe function

$$k_x(z) = \frac{z}{(1-xz)^2}, \qquad |x| = 1.$$

uniquely maximize ReJ_x over S, where $J_x g = \bar{x}g(0)$, $|x| = 1$.

Thus, the Koebe functions k_x are both support points of S and extreme points of HS, the closed convex hull of S.

Let C denote the subclass of S which consists of close-to-convex functions and let HC the closed convex hull of C.

It is known in [2] that the functions

$$f_{xy}(z) = \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2}, \quad |x| = |y| = 1, \quad x \neq y,$$

are both support points of C and extreme points of HC. If we set $y = 1$, $x \neq 1$ and $a = \frac{1}{2}(1+x)$, the tip of omitted arc Γ is

$$f_{xy}\left(\frac{1}{2a-1}\right) = -\frac{1}{4(1-a)} = -\frac{1}{4} - \frac{i}{4}\cot\frac{\theta}{2} \text{ when } x = e^{i\theta}, \ \ 0 < \theta < 2\pi.$$

Thus, with varying θ, the rays obtained consists of all rays through $w = -\frac{1}{2}$ and having the tip on the line $Re\, w = -\frac{1}{4}$. It is evident that if $\left|\cot\frac{\theta}{2}\right| > 1$, then the ray will not have strictly increasing modulus. Therefore, if $|\theta| < \frac{\pi}{2}$, then $\mathbb{C} \setminus f_{xy}(\triangle)$ does not have strictly increasing modulus and thus f_{xy} is not a support point of S.

However, if the omitted half-line Γ is oriented so that Γ is traversed from the tip P of Γ to ∞, a computation shows that $|arg(-\frac{x}{y})|$ is the angle between the tangent vector to Γ and the radius vector to Γ at P. It is easily seen that the radial angle $arg\{\frac{dw}{w}\}$ of Γ decreases monotonically to 0 as Γ traverses from P to ∞.

Thus, if $\frac{\pi}{4} < |arg(-\frac{x}{y})| < \pi$, then f_{xy} can be neither a support point of S nor an extreme point of HS because Γ fails to satisfy the $\frac{\pi}{4}$-property.

If $|arg(-\frac{x}{y})| < 0$, i.e., if $-x = y$, then f_{xy} is the Koebe function $k_y(z) = \frac{z}{(1-yz)^2}$. If $0 < |arg(-\frac{x}{y})| \leq \frac{\pi}{4}$, then Γ does not violate the $\frac{\pi}{4}$-property. So if $0 < |arg(-\frac{x}{y})| \leq \frac{\pi}{4}$, then the function

$$f_{xy}(z) = \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2}$$

is both a support point of S and an extreme point of HS.

4. The Support Points and the Löwner Chain

Suppose $f \in S$ maps Δ onto the complement of a Jordan arc which extends to ∞. By Löwner theory ([11]), f may be embedded in a family of mappings

$$\{f(z,t)|0 \leq t < \infty\},$$

called a Löwner chain, with the following properties;

(i) $f(z,0) = f(z)$

(ii) $f(z,t_1)$ is subordinate to $f(z,t_2)$ if $t_1 < t_2$.

(iii) $e^{-t}f(z,t) \in S, \quad 0 \leq t < \infty$.

(iv) $\frac{\partial f(z,t)}{\partial t} = z\frac{\partial f(z,t)}{\partial z}\frac{1+\eta(t)z}{1-\eta(t)z}$, where $|\eta(t)| = 1, \quad z \in \Delta$.

From (ii) it follows that for any $t \geq 0$. $f(z) = f[\phi(z,t),t]$ where $\phi(z,t) \in H(\Delta)$, is one-to-one in Δ and satisfies $\phi(0,t) = 0, \ \phi'(0,t) = e^{-t}$. $f(z,t)$ carries a point of the unit circle to the tip of the slit bounding $f(\Delta,t)$.

THEOREM 4.1. *If f is an extreme point of S and f is embedded in the Löwner chain $\{f(z,t), t \geq 0\}$, then for all $t \geq 0$, $e^{-t}f(z,t)$ is an extreme point of S.*

Proof. Suppose not, i.e., for some $t \geq 0$, suppose that

$$e^{-t}f(z,t) = sf_1(z) + (1-s)f_2(z), \quad (z \in \Delta)$$

where $0 < s < 1, \ f_1, f_2 \in S$ and $f_1 \neq f_2$. Then

$$f(z,t) = se^t f_1(z) + (1-s)e^t f_2(z).$$

By the subordination property of Löwner chain,

$$f(z) = f(z,0) = f[\phi(z,t),t] = se^t f_1[\phi(z,t),t] + (1-s)e^t f_2[\phi(z,t),t].$$

Since $e^t f_1[\phi(z,t),t]$ and $e^t f_2[\phi(z,t),t]$ are distinct functions in S, it contradicts the fact that $f(z)$ is an extreme point of S. □

LEMMA 4 (Duren [5]). *Each continuous linear functional J on the space $H(\triangle)$ has the form*

$$J(f) = \int\int_E f(z)d\mu(z), \quad f \in H(\triangle),$$

where μ is a complex-Borel measure supported on a compact subset E of $\triangle$.

THEOREM 4.2. *Let J be a continuous linear functional on $H(\triangle)$ which is not constant on S, and let f maximize $Re\{J\}$ on S. If f is embedded in the Löwner chain $\{f(z,t),\ t \geq 0\}$, then for all $t \geq 0$, $e^{-t}f(z,t)$ is a support point of S.*

Proof. Let $f \in S$ satisfy

$$ReJ(f) = \max_{g \in S} ReJ(g)$$

where J is non-constant on S. By Lemma 4, we can write

$$J(f) = \int\int_E f(z)d\mu(z)$$

where μ is a complex-Borel measure supported on a compact subset E of $\triangle$. Denote by $\{f(z,t)\}$ the Löwner chain associated with f. Then by the subordination property of Löwner chain,

$$J(f) = \int\int_E f(z)d\mu(z) = \int\int_E f(z,0)d\mu(z) = \int\int_E f[\phi(z,t),t]d\mu(z).$$

Setting $\zeta = \phi(z,t)$ we get $z = \phi^{-1}(\zeta,t)$. Thus,

$$(4.1)\qquad \begin{aligned} J(f) &= \int\int_{\phi(E,t)} f(\zeta,t)d\mu[\phi^{-1}(\zeta,t)] \\ &= \int\int_{\phi(E,t)} e^{-t}f(\zeta,t)e^{t}d\mu[\phi^{-1}(\zeta,t)]. \end{aligned}$$

Since $dv(\zeta,t) = e^{t}d\mu[\phi^{-1}(\zeta,t)]$ is a complex Borel measure supported on the compact set $\phi(E,t) \subset \triangle$, we may define a continuous functional J_t such that

$$J_t(g) = \int\int_{\phi(E,t)} g(\zeta)dv(\zeta,t).$$

It follows from (4.1) that

$$(4.2)\qquad J(f) = J_t[e^{-t}f(z,t)].$$

Moreover, if $g \in S$, then by another change of variable we see that

$$(4.3)\qquad J_t(g) = J\{e^{t}g[\phi(z,t)]\}$$

and $e^{t}g[\phi(z,t)] \in S$. From (4.2) and (4.3) we obtain

$$ReJ_t[e^{-t}f(z,t)] = \max_{g\in S} ReJ_t(g).$$

Since $e^{t}\phi(z,t)$ is a bounded univalent function, $e^{t}\phi(z,t)$ is not a support point of S. So

$$ReJ_t(f) = ReJ[e^{t}f(z,t)] < ReJ(f) = ReJ_t[e^{-t}f(z,t)]$$

and we conclude that J_t is non-constant on S. This completes the proof. □

References

[1] L. Brickman, *Extreme points of the set of univalent functions*, Bull. Amer. Math. Soc. **76** (1970), 372–374.

[2] L. Brickman, T. H. MacGregor and D. R. Wilken, *Convex hulls of some classical families of univalent functions*, Trans. Amer. Math. Soc. **156** (1971), 91-107.

[3] L. Brickman and D. R. Wilken, *Support point of the set of univalent functions*, Proc. Amer. Math. Soc. **42** (1974), 523-528.

[4] J. E. Brown, *A method for investigating geometric properties of support points and applications*, Trans. Amer. Math. Soc. **287** (1985), 285-291.

[5] P. Duren, *Univalent functions*, Springer-Verlag, New York, 1983.

[6] P. Duren, Y. Leung and M. Schiffer, *Support points with maximum radial angle*, Complex variables: Theory and Applications (1983), 263–277.

[7] G. W. Goluzin, *Geometric theory of functions of a complex variable*, Translations. Math. Mono., vol. 26, Amer. Math. Soc. Providence R. I., 1969.

[8] W. Hengartner and G. Schober, *Extreme points for some classes of univalent functions*, Trans. Amer. Math. Soc. **185** (1973), 265–270.

[9] W. Hengartner and G. Schober, *Some new properties of support points for compact families of univalent functions in the unit disk*, Michigan Math. J. **23** (1976), 207-216.

[10] W. Kirwan and R. Pell, *Extremal properties of a class of slit conformal mappings*, Michigan Math. J. **25** (1978), 223–232.

[11] K. Löwner, *Untersuchugen über Schilichte Konforme Abbildungen des Einheitskreises*, Math. Z. **3** (1923), 103–121.

[12] K. Pearce, *New support points of S and extreme points of HS*, Proc. Amer. Math. Soc. **81** (1981), 425–428.

[13] A. Pfluger, *Linear Extremal probleme bei Schlichten Functionen*, Ann. Acad. Sci. Fenn. Ser. AI **489** (1971), p. 32.

[14] M. Schiffer, *A method of variation within the family of simple functions*, Proc. London. Math. Soc. **44** (1938), 432-449.

[15] ______, *On the coefficient problem for univalent functions*, Tran. Amer. Math. Soc. **134** (1968), 95-101.

Department of Mathematics, Ewha Womans University, Seoul 120-750, Korea

E-mail: sylee@mm.ewha.ac.kr