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GOTTLIEB GROUPS ON LENS SPACES

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Abstract. In this paper we compute Gottlieb groups for generalized lens spaces. Then we apply this result to compute Gottlieb groups for total spaces of a principal torus bundle over a lens space.

1. Introduction

The Gottlieb group, $G_n(X)$, of a connected topological space X consists of all $\alpha \in \pi_n(X, x_0)$ such that there is an associated map $A : S^n \times X \to X$ and a homotopy commutative diagram

$$\begin{array}{ccc} S^n \times X & \xrightarrow{A} & X \\ \uparrow & & \nearrow \alpha \vee 1_X \\ S^n \vee X & & \end{array}$$

This group $G_n(X)$ is also characterized by $G_n(X) = \omega_{\#}(\pi_n(X^X, 1_X)) \subset \pi_n(X, x_0)$, where $\omega : X^X \to X$ is an evaluation map at $x_0 \in X$. Thus $G_n(X)$ is also called an *evaluation subgroup* of $\pi_n(X, x_0)$.

Gottlieb extensively studied $G_1(X)$ in [1], and $G_n(X)$ for $n \geq 2$ in [2]. He has shown that if X is an H-space , then $G_n(X) = \pi_n(X, x_0)$ for all n. He also computes $G_n(X)$ when X is an n-dimensional sphere S^n;

$$G_n(S^n) = \begin{cases} 0, & \text{for } n \text{ even} \\ Z, & \text{for } n = 1, 3, 7 \\ 2Z, & \text{for } n \text{ odd and } n \neq 1, 3, 7. \end{cases}$$

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Recently, Lee, Kim and Woo [3,5] introduced the notion of generalized evaluation subgroups and G-sequences and made some improvement on computing the evaluation subgroups.

The purpose of this paper is to give a computation of $G_{2n+1}(X)$ when X is a $(2n+1)$-dimensional generalized lens space $L_{2n+1}(p;q_1,\cdots,q_n)$ as follows

THEOREM. *Let $L_{2n+1}(p;q_1,\cdots,q_n)$ be a $(2n+1)$-dimensional generalized lens space. Then we have*

$$G_{2n+1}(L_{2n+1}(p;q_1,\cdots,q_n)) = \begin{cases} Z, & \text{for } n=0,1,3, \\ 2Z, & \text{for any other } n. \end{cases}$$

For $n=0$, we have $L_1(p)\simeq S^1$ and $G_1(L_1(p)) = G_1(S^1) = \pi_1(S^1) \simeq Z$ follows easily since S^1 is a compact Lie group.

For $n=1$, we have a 3-dimensional lens space $L_3(p;q)$ and the result follows from the following corollary of George Lang Jr. [4]. Let Y be a Lie group, G a finite subgroup of Y. Then $G_n(Y/G) = \pi_n(Y/G)$ for $n>1$. Here we take $Y=S^3$, a compact Lie group and $G=Z_p(g)$ to be a cyclic subgroup of order p of S^3. Then $S^3/Z_p(g) = L_3(p;q)$ and Lang's corollary implies $G_3(L_3(p;q)) = \pi_3(L_3(p;q)) \simeq Z$.

In the next section, we first show that $G_7(L_7(p;q_1,q_2,q_3)) = Z$ and then we prove the general case, that is, $G_{2n+1}(L_{2n+1}(p;q_1,\cdots,q_n)) = 2Z$.

Before proving our theorem we would like to introduce lens space for reader's convenience.

Let S^{2n+1} be a $(2n+1)$-dimensional unit sphere in Euclidean $(2n+2)$-space defined in terms of $(n+1)$ complex coordinates $z=(z_0,z_1,\cdots,z_n)$ satisfying $z_0\bar{z_0}+\cdots+z_n\bar{z_n}=1$. Let $p\geq 2$ be a fixed integer, and $q_1,\cdots,q_n$ be n integers relatively prime to p. We define an action α on S^{2n+1} by $\alpha(g,(z_0,z_1,\cdots,z_n)) = (e^{2\pi i/p}z_0, e^{2\pi iq_1/p}z_1,\cdots,e^{2\pi iq_n/p}z_n)$. Then g generates a fixed point free cyclic group $Z_p(g)$ of rotations of S^{2n+1} of order p. The orbit space $S^{2n+1}/Z_p(g) = L_{2n+1}(p;q_1,\cdots,q_n)$ is an orientable $(2n+1)$-dimensional manifold called a *lens space*. Let $L_{2n+1}(p;q_1,\cdots,q_n) = L_{2n+1}(p;\mathbf{q})$, where $\mathbf{q}=(q_1,\cdots,q_n)$. If $\pi : S^{2n+1}\to L_{2n+1}(p;\mathbf{q})$ is a projection map and $g^l\in Z_p(g)$, then $\pi g^l(z) = \pi(z)$ for $z\in S^{2n+1}$ and $l=1,\cdots,p$. Thus $Z_p(g)$ is a group of deck transformations since π is a covering map, and we have $\pi_1(L_{2n+1}(p;\mathbf{q})) \simeq Z_p(g)$ and $\pi_i(L_{2n+1}(p;\mathbf{q})) = \pi_i(S^{2n+1})$ for $i \geq 2$.

Topological classifications are given as follows: Two lens spaces $L_{2n+1}(p;q_1,.,q_n)$ and $L_{2n+1}(p;q_1',.,q_n')$ are homeomorphic if and only if there is a number b and there are numbers $\epsilon_j \in \{-1,1\}$ such that a $(q_1,\cdots,q_n)$ is a permutation of $(\epsilon_1 bq_1',\cdots\epsilon_n bq_n')$ mod p.

Homotopy classifications are given as follows: Two lens spaces $L_{2n+1}(p;q_1,.,q_n)$ and $L_{2n+1}(p;q_1',.,q_n')$ have the same homotopy type if and only if $q_1q_2\cdots q_n = \pm k^n q_1'q_2'\cdots q_n'$ for some integer k relatively prime to p. Thus $L_3(5,1)$ is not homotopic to $L_3(5,2)$ while they have the same homotopy groups.

Since $L_{2n+1}(p;q_1,.,q_n)$ and $L_{2n+1}(p;q_1',.,q_n')$ have the same homotopy groups and our proof of Theorem 3.1 shows that $G_{2n+1}(L_{2n+1}(p;q_1,.,q_n))$ is independent of $(q_1,.,q_n)$, we simply write $L_{2n+1}(p)$ for $L_{2n+1}(p;q_1,.,q_n)$.

For more information on lens spaces readers are referred to [6,7].

2. Proof of Theorem

In order to prove our theorem we need following two lemmas from [2].

LEMMA 2.1. *Let $p:\tilde{X}\to X$ be a covering map. If $k>1$, then $p_{\#}^{-1}(G_k(X)) \subseteq G_k(\tilde{X})$. In other works, if we identify $\pi_k(X)$ with $\pi_k(\tilde{X})$ under the isomorphism $p_{\#}$, then $G_k(X)\subseteq G_k(\tilde{X})$.*

Thus it follows that $G_{2n+1}(L_{2n+1}(p)) \subseteq Z$ for $n=0,1,3$ and $G_{2n+1}(L_{2n+1}(p)) \subseteq 2Z$ for other n.

LEMMA 2.2. *For any fibration $F \xrightarrow{i} E \xrightarrow{\pi} B$, $d(\pi_{n+1}(B)) \subseteq G_n(F)$ where $d:\pi_{n+1}(B)\to\pi_n(F)$ arises from the homotopy exact sequence of the fibration.*

Proof of theorem. For the cases of $n=0$ and $n=1$ are given in the introduction. Next we show $G_7(L_7(p)) \simeq Z$. We already know that $G_7(L_7(p)) \subseteq Z$ from lemma 2.1 since S^7 is a universal covering space of $L_7(p)$. Thus all we need to show is that $G_7(L_7(p)) \supseteq Z$. Steenrod constructs a fiber bundles S^{15} over S^8 with S^7 as fiber with bundle group $O(8)$ in [11,20.6]. Then we have $Z_p(g) \subset SO(8) \subset O(8)$, where $Z_p(g)$ is a cyclic group of order p acting freely on S^7 and so does act freely on S^{15} as well. Then their orbit spaces are $L_7(p)$ and $L_{15}(p)$ respectively and we have $\{L_{15}(p), \pi, S^8\}$ as a fiber space with $L_7(p)$ as fiber. It gives us a fiber homotopy exact sequence

$$\longrightarrow \pi_8(L_{15}(p)) \xrightarrow{\pi_\#} \pi_8(S^8) \xrightarrow{\partial} \pi_7(L_7(p)) \xrightarrow{i_\#} \pi_7(L_{15}(p)) \longrightarrow$$

and gives us $0 \longrightarrow Z \xrightarrow{\partial} Z \longrightarrow 0$. Here ∂ becomes an isomorphism and it follows $Z = \partial(Z) \subseteq G_7(L_7(p))$ from Lemma 2.2. Thus we have $G_7(L_7(p)) \simeq Z$.

Now we prove $G_{2n+1}(L_{2n+1}(p)) \simeq 2Z$ for $n \neq 0, 1, 3$. Let us consider the Stiefel manifold $V_{2n+3,2}$. It may be interpreted as the space of unit tangent vectors on S^{2n+2} [11]. Then $V_{2n+3,2}$ may be considered as the tangent bundle over S^{2n+2} with fiber $S^{2n+1} = V_{2n+2,1}$ with bundle group of $SO(2n+2)$. Then$Z_p(g) \subset SO(2n+2)$, a cyclic group of order p acts freely on S^{2n+1} and does act freely on $V_{2n+3,2}$ as well. This action induces a fibration

$$L_{2n+1}(p) \simeq S^{2n+1}/Z_p(g) \longrightarrow V_{2n+3,2}/Z_p(g) \xrightarrow{\pi} S^{2n+2}$$

and induces the following fiber homotopy exact sequence

$$\longrightarrow \pi_{2n+2}(V_{2n+3,2}/Z_p(g)) \xrightarrow{\pi_\#} \pi_{2n+2}(S^{2n+2}) \xrightarrow{\partial} \pi_{2n+1}(L_{2n+1}(p)) \xrightarrow{i_\#}$$
$$\pi_{2n+1}(V_{2n+3,2}/Z_p(g)) \xrightarrow{\pi_\#} 0.$$

Note that $V_{2n+3,2}$ is a p-fold covering space over $V_{2n+3,2}/Z_p(g)$ and we have

$$\pi_{2n+1}(V_{2n+3,2}) \simeq \pi_{2n+1}(V_{2n+3,2}/Z_p(g)) \simeq Z_2 \quad [11, 25.6].$$

Then the fiber homotopy exact sequence gives us

$$\longrightarrow Z \xrightarrow{\partial} Z \xrightarrow{i_\#} Z_2 \xrightarrow{\pi_\#} 0.$$

Since $i_{\#}$ is an epimorphism, exactness of the sequence implies that the image of ∂ is a subgroup of $\pi_{2n+1}(L_{2n+1}(p)) \simeq Z$ of index 2. Hence we obtain $\partial(u_{2n+2}) = 2u_{2n+1}$, where u_{2n+2} and u_{2n+1} are generators of $\pi_{2n+2}(S^{2n+2})$ and $\pi_{2n+1}(L_{2n+1}(p))$ respectively. Then we have $\partial\pi_{2n+2}(S^{2n+2}) \subset G_{2n+1}(L_{2n+1}(p))$ by Lemma 2.3. Thus we have $2Z \subset G_{2n+1}(L_{2n+1}(p)) \subset G_{2n+1}(S^{2n+1}) \simeq 2Z$ and we conclude $G_{2n+1}(L_{2n+1}(p)) = 2Z$ for $n \neq 0, 1, 3$. □

COROLLARY 2.3. *If* $p = 2$ *in our theorem, we have* $L_{2n+1}(2) = RP(2n+1)$, $(2n+1)$*-dimensional real projective space, and*

$$G_{2n+1}(RP(2n+1)) = \begin{cases} Z, & \text{for } n = 0, 1, 3 \\ 2Z, & \text{for any other } n. \end{cases}$$

This corollary is a part of Theorem 3.4 given in [10], and we have $G_{2n}(RP(2n)) = 0$ for n even [10].

REMARK. Let H be a finite group acting freely on S^{2n+1}. Then the orbit space S^{2n+1}/H is called a spherical orbit space. Thus a lens space is a special case of a spherical orbit space. Recently, Oprea [8] has shown that $G_1(S^{2n+1}/H) = Z(H)$, the center of H. It will be an interesting problem to find out what is $G_{2n+1}(S^{2n+1}/H)$. Note that from lemma 1 we know that $G_{2n+1}(S^{2n+1}/H) \subseteq Z$ for $n = 1, 3$ and $G_{2n+1}(S^{2n+1}/H) \subseteq 2Z$ for other n's. We suspect that the equality hold for both cases. For $n = 0$, H must be a finite cyclic subgroup of S^1 and $S^1/H \simeq S^1$ and $G_1(S^1/H) = \pi_1(S^1) = Z$. For $n = 1$, and if H is a finite cyclic or a finite binary polyhedral group , then the result follows from the Lang Jr's corollary which says that $G_n(S^3/H) = \pi_n(S^3/H) = \pi_n(S^3)$ for $n > 1$.

Let $\{E, \pi, CP(n)\}$ be a principal circle bundle over $2n$-dimensional complex projective space $CP(n)$. The bundle classifications are given by $[CP(n), CP(\infty)] = H^2(CP(n); Z) \simeq Z$. The topological classifications are given by $E \simeq CP(n) \times S^1$ for $0 \in Z$ and $E = S^{2n+1}$ for $\pm 1 \in Z$ for two extreme cases. For other cases we have $E = L_{2n+1}(|i|; 1, \cdots, 1) = L_{2n+1}(|i|)$, $(2n+1)$-dimensional lens space for $i \in Z$.

COROLLARY 2.4. *Let $i \neq 0$ be an integer and $E = L_{2n+1}(|i|; 1, \cdots, 1))$. Then we have*

$$G_{2n+1}(E) = \begin{cases} Z, & \text{for } n = 0, 1, 3, \\ 2Z, & \text{for all other } n. \end{cases}$$

For $i = 0 \in Z$, we have

$$G_{2n+1}(CP(n) \times S^1) = G_{2n+1}(CP(n)) \supseteq n!Z.$$

The first part follows from our theorem and the second part follows from [4].

Let M^{n+1} be a total space of a principal torus T^{n-2} bundle over a lens space $L_3(p; q)$, $n \geq 3$.

COROLLARY 2.5.

$$G_i(M^{n+1}) = \begin{cases} Z_k \oplus Z^{n-2}, & \text{for } i = 1 \\ \pi_i(L_3(k, q)), & \text{otherwise} \end{cases}$$

for some positive integer $k \leq p$.

Proof. It is known that the total space M^{n+1} must be $L_3(k, q) \times T^{n-2}$ for some positive integer $k \leq p$ [9]. Then for $i = 1$ we have $G_1(L_3(k; q) \times T^{n-2}) = G_1(L_3(k; q)) \oplus G_1(T^{n-2}) = Z_k \oplus Z^{n-2}$. For other i's we have $G_i(L_3(k; q) \times T^{n-2}) = G_i(L_3(k; q)) \oplus G_i(T^{n-2}) = G_i(L_3(k; q)) = \pi_i(L_3(k; q))$. □

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