EQUIVARIANT HOMOTOPY EQUIVALENCES
AND A FORGETFUL MAP

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ABSTRACT. We consider the forgetful map from the group of equivariant self equivalences to the group of non-equivariant self equivalences. A sufficient condition for this forgetful map being a monomorphism is obtained. Several examples are given.

1. Introduction

Let \((P, q, B, G)\) be a principal \(G\)-bundle over \(B\). Then \(G\) acts on \(P\) freely and \(P/G = B\). Let \(\text{aut}_G(P)\) be the space of unbased \(G\)-equivariant homotopy equivalences from \(P\) to \(P\) of the total space \(P\) of the given principal \(G\)-bundle. We work in the category of connected CW-complexes.

The path component of \(\text{aut}_G(P)\) forms a group where the multiplication is given by the composition of maps. That is, we define

\[
\mathcal{F}_G(P) = \pi_0(\text{aut}_G(P)).
\]

Two maps \(f, g\) in \(\text{aut}_G(P)\) are in the same path component if there is a \(G\)-equivariant homotopy between \(f\) and \(g\). We call this group the group of unbased \(G\)-equivariant self equivalences.

On the other side, we consider the space of unbased self homotopy equivalences from \(P\) to \(P\), which is denoted by \(\text{aut}(P)\). Two maps are in the same path component if there is a homotopy between them. By the same consideration as above, we define

\[
\mathcal{F}(P) = \pi_0(\text{aut}(P)).
\]

We call this group the group of unbased self homotopy equivalences.

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The author has been interested in the natural map which forgets a $G$-action on $P$, that is, a forgetful homomorphism

(1.1) \[ F : \mathcal{F}_G(P) \to \mathcal{F}(P) \]

and raised the following problem in 1988 (see [5, p. 206, Problem 13]):

When is the homomorphism $F$ of (1.1) a monomorphism?

This problem seems to be difficult, because the generators of $\mathcal{F}_G(P)$ are not known in general, even if we have the group structure of $\mathcal{F}_G(P)$ (see [3], [7]).

An example where $F$ is not a monomorphism, is given by [8] in 1996 as in the following.

Let $(G/T, q, BT, G)$ be a principal $G$-bundle, where $G$ is a compact connected Lie group, which is not a torus and $T$ is a maximal torus of $G$. It is shown that $\mathcal{F}_G(G/T)$ is an infinite group. But $\mathcal{F}(G/T)$ is a finite group. So

\[ F : \mathcal{F}_G(G/T) \to \mathcal{F}(G/T) \]

cannot be a monomorphism.

On the contrary, many examples where $F$ might be a monomorphism, could be found in [3], [7] by the calculations of the group of $G$-equivariant self equivalences.

For example, let $(E_G, q, B_G, G)$ be a universal principal $G$-bundle, then $\mathcal{F}_G(E_G) = 1$ (see [7, Example 3.1]). So

\[ \mathcal{F}_G(E_G) \to \mathcal{F}(E_G) \]

is a monomorphism.

In this note we consider the sufficient condition that the homomorphism $F$ of (1.1) is a monomorphism and several examples are given.

2. Bundle Map Theory

Let $f$ be a $G$-equivariant map from $P$ to $P$. Then one has the induced map on $\tilde{f}$ on $B$ such that $qf = \tilde{f}q$.

$$
\begin{array}{ccc}
P & \xrightarrow{f} & P \\
\downarrow{q} & & \downarrow{q} \\
B & \xrightarrow{\tilde{f}} & B
\end{array}
$$
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One has naturally the following map

$$\Phi : aut_G(P) \to aut(B), \Phi(f) = \tilde{f}.$$  

This construction determines a Serre fibration with fibre the space $I_G(P)$ of unbased bundle equivalences over $B$ (cf. [3], [4], [7]):

$$(2.1) \quad I_G(P) \to aut_G(P) \to aut(B)$$  

By the Gottlieb's theorem [1],

$$(2.2) \quad \pi_0(I_G(P)) = \pi_1(map(B, B_G), k),$$  

where $k : B \to B_G$ is a classifying map and $B_G$ is a classifying space.

So we have the following exact sequence of groups and homomorphisms by (2.1) and (2.2) (see [3], [4], [7])

$$(2.3) \quad \pi_1(map(B, B_G), k) \to \mathcal{F}_G(P) \to \mathcal{F}_k(B) \to 0,$$

where $\mathcal{F}_k(B) = \{f \in \mathcal{F}(B); k \tilde{f} \simeq k\}$.

3. $K(\pi, n)$-action

Let $\mathcal{R}_q(P)$ be the group of homotopy classes of $q$-retracting equivalences, that is, an element $f$ of $\mathcal{F}(P)$ for which there is an element $\tilde{f}$ of $\mathcal{F}(B)$ satisfying $qf \simeq \tilde{f}q$ (see [6, p. 645]). $\mathcal{R}_q(P)$ is a subgroup of $\mathcal{F}(P)$ and $F(\mathcal{F}_G(P))$ is a subgroup of $\mathcal{R}_q(P)$.

If the structure group of the principal $G$-bundle $(P, q, B, G)$ is an Eilenberg-MacLane space $K(\pi, n) = G(n \geq 1)$

$$(3.1) \quad \begin{array}{ccc}
P & \longrightarrow & E_{K(\pi, n)} \\
q \downarrow & & \downarrow \\
B & \overset{k}{\longrightarrow} & B_{K(\pi, n)} 
\end{array}$$

then $B_{K(\pi, n)} = K(\pi, n + 1)$ and

$$\pi_1(map(B, B_{K(\pi, n)}), k) = \pi_1(map(B, K(\pi, n + 1)), *) = H^n(B, \pi).$$

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Therefore if $H^n(B; \pi) = 0$, we have the following diagram by (3.1) and (2.3)

\[
\begin{array}{ccc}
\mathcal{F}_{K(\pi,n)}(P) & \xrightarrow{F} & \mathcal{R}_q(P) \\
\cong & \searrow & \\
\mathcal{F}_k(B) & \xrightarrow{i} & \mathcal{F}(B),
\end{array}
\]

where $i$ is an inclusion and $F' : \mathcal{F}_k(B) \to \mathcal{R}_q(P)$.

**Theorem 3.3.** Let $(P, q, B, K(\pi, n))$ be a principal $K(\pi, n)$-bundle, we assume $H^n(B, \pi) = 0$. The forgetful map $F : \mathcal{F}_{K(\pi,n)}(P) \to \mathcal{R}_q(P) \subset \mathcal{F}(P)$ is a monomorphism, if a map $G : \mathcal{R}_q(P) \to \mathcal{F}(B)$ which commutes the following diagram (3.4) exists.

\[
\begin{array}{ccc}
\mathcal{F}_{K(\pi,n)}(P) & \xrightarrow{F} & \mathcal{R}_q(P) \\
\cong & \searrow & \downarrow G \\
\mathcal{F}_k(B) & \xrightarrow{i} & \mathcal{F}(B)
\end{array}
\]

**Proof.** Since $GF' = i$ and $i$ is a monomorphism (inclusion), $F'$ is a monomorphism. $\mathcal{F}_{K(\pi,n)}(P)$ is isomorphic to $\mathcal{F}_k(B)$. So $F$ is a monomorphism. \hfill \Box

Let $\mathcal{E}(B)$ be the group of based self homotopy equivalences of $B$, and we denote by $\mathcal{E}_\#(B)$ the subgroup of $\mathcal{E}(B)$ consisting of classes that induce the identity automorphism of all homotopy groups.

**Corollary 3.5.** Let $(P_k, q, B, K(\pi, n))$ be a principal $K(\pi, n)$-bundle, where $\pi_i(B) = 0(i \leq n)$, $\pi$ is a free abelian group and $\mathcal{E}_\#(B) = 1$. Then $F : \mathcal{F}_{K(\pi,n)}(P_k) \to \mathcal{F}(P_k)$ is a monomorphism for any $k$.

**Proof.** By the homotopy exact sequence of the given principal bundle, we have $\pi_i(B) = \pi_i(P_k)(i \geq n + 2)$, $\pi_{n+1}(B) = \pi_{n+1}(P) \oplus \pi'$ ($\pi'$ is a free subgroup of $\pi$) and $\pi_i(B) = 0(i \leq n)$. $H^n(B, \pi) = 0$, since $\pi_i(B) = 0(i \leq n)$. Since $\mathcal{E}_\#(B) = 1$, the induced map on the base

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space $B$ is determined uniquely for the given self homotopy equivalence in $\mathcal{R}_q(P)$. So a map $G : \mathcal{R}_q(P) \to \mathcal{F}(B)$ which commutes the diagram (3.4) exists. 

4. Examples

**Example 4.1.** For the trivial bundle $(T^n \times B, q, B, T^n)$, where $T^n$ is an $n$-dimensional torus ($n \geq 1$) and $\pi_1(B) = 0$,

$$F : \mathcal{F}_{T^n}(T^n \times B) \to \mathcal{F}(T^n \times B)$$

is a monomorphism.

**Proof.** For a given homotopy equivalence $f : T^n \times B \to T^n \times B$, we define $\tilde{f} : B \to B$ by $q \pi_1(i : B \to S^1 \times B, i(b) = (\ast, b))$. Since $\pi_1(B) = 0$, $\tilde{f} : B \to B$ induces an isomorphism on homotopy groups. So $\tilde{f}$ is a homotopy equivalence. Therefore $G : \mathcal{R}_q(T^n \times B) = \mathcal{F}(T^n \times B) \to \mathcal{F}(B)$ exists. Since $K(Z^n, 1) = T^n$ and $H^1(B; Z^n) = 0$ by assumption. The result follows by Theorem 3.3. 

**Example 4.2.** For the trivial bundle $(K(\pi, n) \times B, q, B, K(\pi, n))$ with $\pi_i(B) = 0$ ($i \leq n$),

$$F : \mathcal{F}_{K(\pi, n)}(K(\pi, n) \times B) \to \mathcal{F}(K(\pi, n) \times B)$$

is a monomorphism.

**Proof.** Since $\pi_i(B) = 0$ ($i \leq n$), $H^n(B, \pi) = 0$. The proof is similar to example 4.1.

**Example 4.3.** Let $(P_k, q, B, T^n)$ be a principal $T^n$-bundle ($n \geq 1$) with $\pi_1(B) = 0$ and $E_\ast(B) = 1$. For example, take $B = P^m(C)(m \geq 1)$ (complex projective space) (see [2, p.32]). Then

$$F : \mathcal{F}_{T^n}(P_k) \to \mathcal{F}(P_k)$$

is a monomorphism for any $k$.

**Proof.** This is obtained from Corollary 3.5 for $n=1$.

**Example 4.4.** Let $(P, q, B, K(\pi, n))$ be a principal $K(\pi, n)$-bundle with $\pi_i(B) = 0$ ($i \geq n$) and $H^n(B; \pi) = 0$. Then

$$F : \mathcal{F}_{K(\pi, n)}(P) \to \mathcal{F}(P)$$

is a monomorphism.

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Proof. Since $\pi_1(B) = 0 (i \geq n)$, it is easy to see that a map $G : \mathcal{F}(P) \to \mathcal{F}(B)$ exists by the elementary homotopy theory (e.g. Postnikov tower) and the diagram (3.4) is commutative. So the result is obtained from Theorem 3.3. □

Problem 1. For any $S^1$-bundle $(P_k, q, B, S^1)$ with $\pi_1(B) = 0$, is the forgetful map

$$F : \mathcal{F}_{S^1}(P_k) \to \mathcal{F}(P_k)$$

a monomorphism?

References


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