FIBREWISE INFINITE SYMMETRIC PRODUCTS AND M-CATEGORY

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ABSTRACT. Using a base-point free version of the infinite symmetric product we define a fibrewise infinite symmetric product for any fibration $E \to B$. The construction works for any commutative ring $R$ with unit and is denoted by $R_f(E) \to B$. For any pointed space $B$ let $G_i(B) \to B$ be the $i$-th Ganea fibration. Defining $M_{\text{cat}}(B) := \inf\{i \mid R_f(G_i(B)) \to B \text{ admits a section}\}$ we obtain an approximation to the Lusternik–Schnirelmann category of $B$ which satisfies e.g. a product formula. In particular, if $B$ is a 1-connected rational space of finite rational type, then $M_{\text{cat}}(B)$ coincides with the well-known (purely algebraically defined) $M$-category of $B$ which in fact is equal to $\text{cat}(B)$ by a result of K. Hess. All the constructions more generally apply to the Ganea category of maps.

0. Introduction

Let $\mathcal{S}$ be the category of simplicial sets. It carries the structure of a closed model category where the weak equivalences are the maps which turn into homotopy equivalences by realization and where the cofibrations are the injective maps [16]. Any map $Y \to X$ factors as a weak equivalence followed by a fibration which we call 'the' associated fibration.

We recall the Ganea construction [10]. Given a map $p : Y \to X$ with $X$ pointed by $*$ we let $G_0(p) : G_0(p,Y) \to X$ be the associated fibration; suppose the fibration $G_i(p_0) : G_i(p,Y) \to X$ with fibre $F_i$ is defined, $i \geq 0$, then let $G_{i+1}(p,Y) \to X$ be the map $\pi : G_i(p,Y) \cup_{p_i} C(F_i) \to X$ where $C(F_i)$ is the cone on $F_i$ and $\pi | G_i(p,Y) = G_i(p), \pi | C(F_i) = *$ and

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define \( G_{i+1}(p) : G_{i+1}(p, Y) \to X \) as the associated fibration. Recall that
\[
\text{cat}(X) := \inf \{ i \mid G_i(*) \to X \text{ admits a section} \}. \]
In this absolute case we write \( G_i(X) \to X \) for the fibration \( G_i(*) \to X \).

Let now \( p : E \to B \) be a map of 1-connected rational spaces of finite type over \( \mathbb{Q} \). Let \( M_B \) be a Sullivan model of \( B \) and \( M_B \otimes_{\tau} M' \) a \( KS \)-extension modeling \( p \) (the index \( \tau \) should remind that the differential on the tensor product is twisted).

We say that \( p \) has an \( M \)-section, if there is an \( M_B \)-module map
\[
r : M_B \otimes_{\tau} M' \to M_B \text{ with } r \circ p^* = \text{id}. \]
Define \( M \text{-cat}(p) := \inf \{ i \mid G_i(p) \text{ admits an } M \text{-section} \} \). In particular, \( M \text{-cat}(B) := M \text{-cat}(\ast \to B) \) has been introduced in [12] as an algebraic approximation to \( \text{cat}(B) \).
According to [13] one has \( M \text{-cat}(B) = \text{cat}(B) \).

Let \( R \) be a non-trivial commutative ring with unit. For \( X \in S \) let \( R \otimes X \) be the free module generated by \( X \) and let \( R(X) \subset R \otimes X \) be the affine subspace consisting of linear combinations of simplices with coefficient sum 1. If \( R = \mathbb{Z} \), then \( R(X) \) is a basepoint free version of the infinite symmetric product on \( X \) (see [2]). Given a fibration \( Y \to X \) in \( S \) there is a fibrewise version of \( R(-) \) denoted by \( R_f(Y) \to X \) according to [7].

Our main result says that \( M \text{-cat}(p) \leq k \), if and only if \( Q_f(G_k(p, E)) \to B \) admits a section.

By [13] this implies that for \( B \) a simply connected rational space of finite rational type \( G_k(B) \to B \) admits a section if and only if \( Q_f(G_k(B)) \to B \) admits one.

Thus for any \( R \) as above, any map \( g : Y \to X \) we may define an invariant \( M_R \text{-cat}(g) := \inf \{ i \mid R_f(G_i(g, Y)) \to X \text{ admits a section} \} \). In section 4 we will comment this notion and prove e.g. a product formula.

In section 1 we recall the construction of \( R_f(Y) \to X \). In section 2 we study an algebraic construction serving as a bridge to the spatial construction (over \( \mathbb{Q} \)). In section 3 we prove the main result.

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1. Construction of $R_f(Y) \rightarrow X$

Let $R$ be a non-trivial commutative ring with unit 1. For $X \in \mathcal{S}$ let $R \otimes X$ be the free module generated by $X$; in particular, $R \otimes X$ is a simplicial abelian group. According to [2] we consider $R(X) \subseteq R \otimes X$, $(R(X))_n := \{\sum \lambda_i x_i \mid x_i \in X_n, \sum \lambda_i = 1, \lambda_i \neq 0 \text{ for only finitely many indices } i\}$. Given a base point $* \in X$ the composition $R(X) \rightarrow R \otimes X \rightarrow R \otimes X/R \otimes *$ is a weak equivalence. (Moreover $R(X)$ can be identified with kernel $(R \otimes X \rightarrow R \otimes *)$).

There is a natural map $X \rightarrow R(X)$, $x \mapsto 1x$, hence $R(\_)$ is a coaugmented functor $\mathcal{S} \rightarrow \mathcal{S}$ in the sense of [7]. It takes weak equivalences to weak equivalences and its value on a point space is the point space. Therefore there is a fibrewise version of the functor by [7]. We recall its definition:

Let $\pi : Y \rightarrow X$ be a fibration. Denote by $I$ the simplex category of $X$ [4]; its objects are the simplices $\sigma : \Delta[n] \rightarrow X$ ($\Delta[n]$ the standard $n$–simplex), the morphisms $(\Delta[n], \sigma) \rightarrow (\Delta[m], \tau)$ are the order preserving maps $\alpha : \Delta[n] \rightarrow \Delta[m]$ with $\alpha^*(\tau) = \sigma$. Let $\tilde{X}$ denote the functor $I \rightarrow \mathcal{S}$ given on objects by $\tilde{X}((\Delta[n], \sigma)) = \Delta[n]$; let $\tilde{Y} : I \rightarrow \mathcal{S}$ be the functor with $\tilde{Y}((\Delta[n], \sigma)) := \sigma^*(Y)$, the pullback of $Y \rightarrow X$ by $\sigma$. We then have canonical weak equivalences $\text{hocolim} \tilde{X} \cong \text{colim} \tilde{X} \cong X$ and $\text{hocolim} \tilde{Y} \cong \text{colim} \tilde{Y} \cong Y$.

We now form the following diagram:

```
\begin{center}
\begin{tikzpicture}
  \node (Y) at (0,0) {$Y$};
  \node (X) at (0,-1) {$X$};
  \node (Yh) at (-1,2) {$\text{hocolim} \tilde{Y}$};
  \node (Xh) at (-1,0) {$\text{hocolim} \tilde{X}$};
  \node (R_Yh) at (-2,1) {$\text{hocolim} R \circ \tilde{Y}$};
  \node (R_Xh) at (-2,0) {$\text{hocolim} R \circ \tilde{X}$};
  \node (P) at (0,1) {$P$};

  \draw[->] (Y) to (Yh);
  \draw[->] (X) to (Xh);
  \draw[->] (Yh) to (P);
  \draw[->] (Yh) to (Y);
  \draw[->] (Xh) to (X);
  \draw[->] (R_Yh) to (R_Xh);
  \draw[->] (R_Yh) to (P);
  \draw[->] (R_Xh) to (P);

  \node (pi) at (1,0) {$\pi$};
  \draw[->] (P) to (pi);
  \draw[->] (Y) to (pi);

  \node (sim) at (0.5,0.5) {$\cong$};
  \node (sim2) at (0.5,1.5) {$\cong$};
\end{tikzpicture}
\end{center}
```

Here $R \circ \tilde{Y} : I \rightarrow \mathcal{S}$ is the composition of the functors $\tilde{Y}$ and $R(\_)$ (similarly for $R \circ \tilde{X}$). The lower left square is a homotopy pullback diagram. The upper triangle is a homotopy pushout defining $R_f(Y) \rightarrow$
X. The homotopy fibre of \( R_f(Y) \to X \) over a component of \( X \) is weakly equivalent to any \( R(\sigma^*(Y)) \), \( \sigma \) a simplex of the component.

2. The Universal \( A \)-algebra of an \( A \)-module

Let \( R \) be a commutative field throughout this section.

Denote by \( Mod \) the category of graded (in degrees \( \geq 0 \)) \( R \)-modules. Let \( dMod \) be the category of graded differential modules over \( R \), the differential having degree 1.

Let \( Alg \) be the category of differential commutative associative graded algebras over \( R \) with unit.

For any \( R \)-module \( M \in Mod \) we denote by \( \Lambda(M) \) the free \( R \)-algebra on \( M \). If \( M \in dMod \), then \( \Lambda(M) \) is given the unique algebra differential extending the differential on \( M \).

Given \( A \in Alg \). An \( A \)-module is an object \( M \in dMod \) together with a structure map \( A \otimes_R M \to M \) satisfying the usual properties (comp. [11]).

**Definition 1.** Let \( A \in Alg \). An \( A \)-algebra is an algebra \( B \in Alg \) together with a morphism (in \( Alg \)) \( j : A \to B \).

Note that an \( A \)-algebra is an \( A \)-module in the obvious way.

**Proposition 1.** Given an \( A \)-module \( M \), there exists an \( A \)-algebra \( U_M \) and an \( A \)-module map \( \iota : M \to U_M \) such that for any \( A \)-algebra \( B \) and \( A \)-module map \( \alpha : M \to B \) there exists a unique \( A \)-algebra morphism \( a : U_M \to B \) with \( a \circ \iota = \alpha \).

**Proof.** Denote by \( am \) the image of \( a \otimes m \in A \otimes M \) under the structure map \( A \otimes M \to M \).

Set \( U_M := (A \otimes \Lambda(M))/W \) where \( W \) is the differential ideal generated by \( \{a \otimes m - 1 \otimes am \mid a \in A, m \in M\} \). Define \( \iota : M \to U_M \) by \( \iota(m) := [1 \otimes m] \) (where \([ \_ ]\) means coset with respect to \( W \)).

We note that \( U_M \) is an \( A \)-algebra with \( A \to U_M \) given by \( A \to A \otimes \Lambda(X) \to A \otimes \Lambda(X)/W \). The map \( \iota \) is an \( A \)-module map; for given \( a \in A, m \in M \), \( \iota(am) = [1 \otimes am] = [a \otimes m] = a[1 \otimes m] = a\iota(m) \).

To check the universal property is straightforward. \( \square \)
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We shall need a variant of this construction.

**Definition 2.** An $R$–module $M \in \text{Mod}$ is pointed, if there is a distinguished element -- which we call $1_M$ -- in $M^0$. In case $M \in d\text{Mod}$ we require $d(1_M) = 0$.

An algebra $A$ is always pointed by its unit element $1_A$.

**Definition 3.** Given a pointed $R$–module $M$, let $\hat{\Lambda}(M)$ be the algebra $\Lambda(M)/J$ where $J$ is the differential ideal generated by $1 - 1_M$ (the unit element of $\Lambda(M)$).

**Remark 1.** The functor $\hat{\Lambda}(-)$ is left adjoint to the forgetful from $\text{Alg}$ to the category of pointed modules.

The universal construction of Proposition 1 has an obvious analogue in the pointed category. We give the construction as a definition.

**Definition 4.** Let $M$ be a pointed $A$–module. Then we set $\hat{U}_M := (A \otimes \hat{\Lambda}(M))/W$ where $W$ is the differential ideal generated by $\{a \otimes [m] - 1 \otimes [am] | a \in A, [m] \text{ coset of } m \in M \text{ in } \hat{\Lambda}(M)\}$.

**Definition 5.** An $A$–module $M$ is called semi–free [9], if there exists a free $R$–module $X = \bigoplus_{i=0}^{\infty} X_i$ (each $X_i$ a graded module) such that $M \cong A \otimes X$ and $d_M(1 \otimes X_i) \subseteq A \otimes (\bigoplus_{j < i} X_j)$, all $i$.

The semi–free module $M$ is pointed, if it is pointed by an element $1_A \otimes 1_X$ with $1_X \in X_0^0$; the module $X \in \text{Mod}$ is pointed by $1_X$.

**Proposition 2.** Let $M = A \otimes X$ be a semi–free $A$–module (resp. a pointed semi–free $A$–module). Then $U_M \cong A \otimes_r \Lambda(X)$ (resp. $\hat{U}_M \cong A \otimes_r \hat{\Lambda}(X)$). (Note: The symbol $(\_)_r$ indicates that the differential is twisted; it is the unique algebra differential extending the differential on $M$).

**Proof.** We consider only the first case. The $A$–module map $A \otimes X \to A \otimes_r \Lambda(X)$ induces an $A$–algebra morphism $U_M \cong (A \otimes \Lambda(A \otimes X))/W \to A \otimes_r \Lambda(X)$ which is surjective. Since any element in $A \otimes \Lambda(A \otimes X)$ is equivalent modulo $W$ to one in the subset $A \otimes_r \Lambda(X)$, this map is also injective. \qed
REMARK 2. Let $M \in d\text{Mod}$ be pointed and augmented, i.e., there is given a module map $c : M \rightarrow R$ with $c(1_M) = 1_R$. Let $\tilde{M} := \ker(c)$. Then $\hat{\Lambda}(M) \cong \Lambda(\tilde{M})$.

COROLLARY. Let $M_B \xrightarrow{\rho} M_B \otimes_r M'$ be a $KS$-extension modeling the fibration $p : E \rightarrow B$ (as in the Introduction). Then an $M_B$-module map $r : M_B \otimes_r M' \rightarrow M_B$ with $r \circ \rho^* = \text{id}$ exists, if and only if an algebra map $\rho : M_B \otimes_r \hat{\Lambda}(M') \rightarrow M_B$ with $\rho \circ (\iota \circ \rho^*) = \text{id}$ exists where $\iota : M_B \otimes_r M' \rightarrow M_B \otimes_r \hat{\Lambda}(M')$ is the canonical inclusion.

3. The Main Result

Let $p : E \rightarrow B$ be a fibration of 1-connected rational spaces of finite type over $\mathbb{Q}$. Let $M_B$ be a Sullivan model of $B$ and let the $KS$-extension $M_B \rightarrow M_B \otimes_r M'$ model $p$.

THEOREM. The $KS$-extension $M_B \rightarrow M_B \otimes_r \hat{\Lambda}(M')$ is a model of the fibration $\mathbb{Q}_f(E) \rightarrow B$.

To give the proof we have to recall a few facts from rational homotopy theory.

Denote by $\Delta$ the usual category of standard simplices $\Delta[n]$. For each $n$ let $A^n_\bullet$ be the algebra of polynomial differential forms on $\Delta[n]$ with rational coefficients. The collection $A^n_\bullet$, $n = 0, 1, 2, \ldots$ constitutes a simplicial object in $\text{Alg}$ to be denoted by $A^\bullet$. We recall that $A^\bullet$ is a cohomology theory in the sense of [3] and that $Z^n(A^\bullet)$ is in a canonical way an Eilenberg–MacLane space $K(\mathbb{Q}, n)$. (Here $Z^n(A^\bullet)$ is the module of cycles of degree $n$ in the differential algebra $A^\bullet$).

Given $X \in \mathcal{S}$ one defines $A^\bullet(X) := \{f : X \rightarrow A^\bullet \ | \ f \text{ simplicial}\}$; for $M \in \text{Alg}$ we set $\|M\| := \{g : M \rightarrow A^\bullet \ | \ g \text{ differential algebra map}\}$, so that $\|M\|_n := \{g : M \rightarrow A^n_\bullet \ | \ g \text{ map of differential algebras}\}$.

If $M_X \xrightarrow{\kappa} A^\bullet(X)$ is a cofibrant model, then we obtain a simplicial map $X \xrightarrow{\kappa} \|M_X\|$ by $\sigma \mapsto \sigma^* \circ \kappa$, where $\sigma \in X_n$ is viewed as a simplicial map $\sigma : \Delta[n] \rightarrow X$. If $X$ is 1-connected rational of finite type over $\mathbb{Q}$, this map $\kappa$ is a weak equivalence (see [1]).

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**Proposition 3.** Let $X \in S$ be 1-connected rational of finite type over $Q$. Let $M_X \xrightarrow{\kappa} A^*(X)$ be a cofibrant model. Then there is a canonical weak equivalence $Q(X) \to \|\Lambda(M_X)\|$. 

**Proof.** (a) We first find a weak equivalence

$$Q \otimes X \to \|\Lambda(M_X)\|.$$ 

We look at the following diagram:

\[ 
\begin{array}{ccc}
\|\Lambda(M_X)\| & \xleftarrow{\bar{\kappa}} & Q \otimes X \\
\|M_X\| & \xleftarrow{\bar{\kappa}} & X.
\end{array}
\]

Observe that $\|\Lambda(M_X)\| = \text{Alg}(\Lambda(M_X), A^*_\ast) = \text{Cochain}(M_X, A^*_\ast)$ is a simplicial abelian group. (Here $\text{Cochain}(\_ , \_ )$ denotes the set of cochain morphisms). Hence there is a unique morphism $\bar{\kappa}$ of simplicial abelian groups making the diagram commute. We want to show that $\bar{\kappa}$ induces isomorphisms of homotopy groups. Recall that $\pi_* (Q \otimes X)$ is canonically identified with $H_*(X, Q)$. On the other hand $\text{Cochain}(M_X, A^*_\ast)$ is weakly equivalent to $\text{Cochain}(\bigoplus_{i \geq 0} H^i(X, Q), A^*_\ast) = \prod_{i \geq 0} \text{Cochain}(H^i(X, Q), Z^i(A^*_\ast))$. 

Recall that $Z^i(A^*_\ast)$ is a $K(Q, i)$; therefore $\pi_n(\text{Cochain}(H^i(X, Q), Z^i(A^*_\ast)))$ is canonically isomorphic to $H_i(X, Q)$ for $n = i$ and is zero for $n \neq i$. Moreover, the composition $X \to \|M_X\| \to \|\Lambda(M_X)\|$ identifies $H_*(X, Q)$ with $\pi_* (\|\Lambda(M_X)\|)$. Hence $\bar{\kappa}$ is a weak equivalence.

(b) Choose a base point $* \in X$. Then $M_X$ inherits an augmentation $M_X \to A^*(X) \to A^*(\ast)$; let $\bar{M}$ be the augmentation ideal. We now consider the following diagram:

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\[
\begin{array}{c}
\text{Alg}(\Lambda(M_X), A^*_\tau) \cong \text{Alg}(\hat{\Lambda}(M_X), A_\tau^*) \leftarrow \leftarrow \quad R(X) \\
\text{Cochain}(M_X, A^*_\tau) \quad \kappa \quad \text{Cochain}(M_X, A^*_\tau) \leftarrow \leftarrow \quad R \otimes X \\
\end{array}
\]

We only need to check that \( \kappa \) induces a map \( R(X) \rightarrow \text{Alg}(\hat{\Lambda}(M_X), A^*_\tau) \).

A linear combination \( \Sigma \lambda_i \sigma_i \) of simplices in \( (R \otimes X)_n \) with \( \Sigma \lambda_i = 1 \) is mapped to the element given by \( \Sigma \lambda_i (\sigma_i \circ \kappa) | M_X \) in \( \text{Cochain} (M_X, A^*_\tau) \). Obviously the corresponding algebra map \( \Lambda(M_X) \rightarrow A^*_\tau \) vanishes on the element \( (1 - 1_{M_X}) \).

Proof of the Theorem. Let \( Q - S_1 \) be the category of 1-connected rational spaces of finite type over \( Q \). We first choose a functorial cofibrant model construction \( Q - S_1 \rightarrow \text{Alg}, X \mapsto M_X \). For the arrows \( M_X \rightarrow M_Y \) we then choose a functorial \( KS \)-model to be denoted by \( M_X \rightarrow M_X \otimes M_Y \). (We drop the index \( \tau \) reminding that the differential on the tensor product is twisted). These choices are possible according to \([1]\).

Let \( E \rightarrow B \) be a fibration, \( E, B \in Q - S_1 \). As in section 1 let \( I \) be the simplex category of \( B \) and \( \widetilde{B}, \bar{E} \) the corresponding functors.

For an \( n \)-simplex \( \sigma \in B_n \) we denote the \( KS \)-model of \( \sigma^*(E) \rightarrow \Delta[n] \) by \( M_B(\sigma) \rightarrow M_B(\sigma) \otimes M_E(\sigma) \). To perform the fibrewise construction of \( R \) on \( E \rightarrow B \) we may as well perform it on \( \| M_B \otimes M_E(\sigma) \| \rightarrow \| M_B \| \) using the maps \( \| M_B(\sigma) \otimes M_E(\sigma) \| \rightarrow \| M_B(\sigma) \| \) as building blocks. So the essential part of the construction is given by the diagram

\[
\begin{array}{c}
E \quad \cong \quad \| M_B \otimes M_E(\sigma) \| \quad \cong \quad \text{hocolim} \quad \| M_B(\sigma) \otimes M_E(\sigma) \| \quad \cong \quad \text{hocolim} \quad R(\| M_B(\sigma) \otimes M_E(\sigma) \|) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
B \quad \cong \quad \| M_B \| \quad \cong \quad \text{hocolim} \quad \| M_B(\sigma) \| \quad \cong \quad \text{hocolim} \quad R(\| M_B(\sigma) \|) \\
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\end{array}
\]
and it suffices to show that the upper row of the following diagram is constituted by weak equivalences:

\[
\begin{array}{ccc}
\| M_B \otimes \hat{A}(M'_E) \| & \xleftarrow{(3)} & \text{hocolim} \| M_B(\sigma) \otimes \hat{A}(M'_E(\sigma)) \| \xrightarrow{(2)} \text{hocolim} \| \hat{A}(M_B(\sigma) \otimes M'_E(\sigma)) \| \\
\downarrow & & \downarrow \\
\| M_B \| & \xrightarrow{\cong} & \text{hocolim} \| M_B(\sigma) \| \xrightarrow{\cong} \text{hocolim} \| \hat{A}(M_B(\sigma)) \| \\
\end{array}
\]

\[
\xrightarrow{\cong} \text{hocolim} R(\| M_B(\sigma) \otimes M'_E(\sigma) \|) \\
\xrightarrow{\cong} \text{hocolim} R(\| M_B(\sigma) \|)
\]

The weak equivalences on the right are provided by Proposition 3. The arrows (1),(2) map the homotopy fibres of the three vertical morphism to the left, \(\| \hat{A}(M'_E) \|, \| M_B(\sigma) \otimes \hat{A}(M'_E(\sigma)) \| \) and \(\| \hat{A}(M_B(\sigma) \otimes M'_E(\sigma)) \|\) resp. by weak equivalences. Hence (1),(2) are weak equivalences as required.

\[ \square \]

4. Comments

First we give the definition of the invariant \(M_R\)-cat mentioned in the introduction in more detail.

**Definition 6.** Let \( R \) be a commutative ring with unit. Let \( X, Y \) be pointed spaces and \( \pi : Y \to X \) a map.

(a) We say that \( \pi \) has an \( M_R \)-section, if \( R_f(\pi', Y') \to X \) has a section (where \( \pi' : Y' \to X \) is the associated fibration).

(b) \( M_R\)-\text{cat} \( (\pi) := \inf \{ k \mid G_k(\pi) : G_k(\pi, Y) \to X \text{ has an } M_R\text{-section } \} \).

(c) \( M_R\)-\text{cat} \( (B) := M_R\)-\text{cat} \( (* \to B) \).

Several facts have to be mentioned:

**Warning 1.** For fields \( K \) there exists already a notion \( M_K\)-\text{cat} \( (B) \) ([12,15]). We do not know whether it coincides with the one given here for \( K \neq Q \).

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**Warning 2.** Let $g : Y \to X$ be a map in $Q - S_1$ with model $M_X \to M_Y$. There is then a notion of category of the $M_X$-module $M_Y$ [15]. We do not think that it coincides with $M_{Q-\text{cat}}(g)$ in general. But of course this is true for $g : * \to X$.

**Warning 3.** There is another notion of category of a map which we can transfer into our setting: Let $B$ be pointed and $g : E \to B$ a map. Then define $\bar{M}_{R-\text{cat}}(g) := \inf \{ k \mid g \text{ factors over } R_f(G_k(B)) \to B \}$. For $R = Q$ and $g$ in $Q - S_1$ this invariant coincides with $M_{Q-\text{cat}}(g)$ as studied in [15]. But this may not be true for the corresponding invariant $M_{K-\text{cat}}(g)$ of [15] for $K \neq Q$.

**Proposition 4.** Let $B_1$, $B_2$ be pointed spaces. Then $M_{R-\text{cat}}(B_1 \times B_2) \leq M_{R-\text{cat}}(B_1) + M_{R-\text{cat}}(B_2)$.

Proof. Let $M_{R-\text{cat}}(B_1) \leq k$, $M_{R-\text{cat}}(B_2) \leq \ell$. Then $G_k(B_1) \to B_1$ and $G_\ell(B_2) \to B_2$ have $M_R$-sections. The next Proposition implies that $G_k(B_1) \times G_\ell(B_2) \to B_1 \times B_2$ has an $M_R$-section. Hence the statement follows from the existence of a map $G_k(B_1) \times G_\ell(B_2) \to G_{k+\ell}(B_1 \times B_2)$ over $B_1 \times B_2$ (by [14] or [8]).

**Proposition 5.** Let $E_1 \to B_1$, $E_2 \to B_2$ be fibrations with $M_R$-sections. Then $E_1 \times E_2 \to B_1 \times B_2$ has an $M_R$-section.

Proof. Let $I_1$, $I_2$ and $I$ be the simplex categories of $B_1$, $B_2$ resp. $B_1 \times B_2$. Let the functors $\tilde{B}_1, \tilde{E}_1, \tilde{B}_2, \tilde{E}_2$, $\tilde{B}_1 \times \tilde{B}_2$ and $\tilde{E}_1 \times \tilde{E}_2$ be defined as in Section 1. Note that we have canonical natural transformations $I \to I_1$, $I \to I_2$. Thus we obtain morphisms

$$\hocolim \tilde{E}_1 \times \tilde{E}_2 \to \hocolim \tilde{E}_1 \times \tilde{E}_2 \to (\hocolim \tilde{E}_1) \times (\hocolim \tilde{E}_2)$$

$I \to I_1 \times I_2 \quad I_1 \times I_2 \quad I_1 \quad I_2$

and

$$\hocolim \tilde{B}_1 \times \tilde{B}_2 \to \hocolim \tilde{B}_1 \times \tilde{B}_2 \to (\hocolim \tilde{B}_1) \times (\hocolim \tilde{B}_2)$$

$I \to I_1 \times I_2 \quad I_1 \times I_2 \quad I_1 \quad I_2$.

The two arrows to the right are weak equivalences because of the Fubini Theorem [4], hence the arrows to the left are weak equivalences, because

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there are canonical weak equivalences \( \text{hocolim} \widetilde{E}_1 \times \widetilde{E}_2 \simeq E_1 \times E_2 \simeq (\text{hocolim} \widetilde{E}_1) \times (\text{hocolim} \widetilde{E}_2) \) and similarly \( \text{hocolim} B_1 \times B_2 \simeq B_1 \times B_2 \simeq (\text{hocolim} \widetilde{B}_1) \times (\text{hocolim} \widetilde{B}_2) \).

The essential step in the construction of \( R_f(E_1 \times E_2) \to B_1 \times B_2 \) consists in applying the functor \( R(\cdot) \) to the 'building blocks' in a suitable hocolim representation. For these building blocks we now take the collection of arrows given by the natural transformation \( \widetilde{E}_1 \times \widetilde{E}_2 \to \widetilde{B}_1 \times \widetilde{B}_2 \). We are thus lead to the diagram:

\[
\begin{array}{ccc}
\text{hocolim} R \circ (\widetilde{E}_1 \times \widetilde{E}_2) & \xrightarrow{\alpha} & \text{hocolim} (R \circ \widetilde{E}_1) \times (R \circ \widetilde{E}_2) \\
I_1 \times I_2 & \downarrow & \downarrow \\
\text{hocolim} R \circ (\widetilde{B}_1 \times \widetilde{B}_2) & \xleftarrow{\alpha} & \text{hocolim} (R \circ \widetilde{B}_1) \times (R \circ \widetilde{B}_2) \\
I_1 \times I_2
\end{array}
\]

\[
\begin{array}{ccc}
R \circ \widetilde{E}_1 & \xrightarrow{\beta} & \text{hocolim} R \circ \widetilde{E}_2 \\
I_1 & \downarrow & \downarrow \\
R \circ \widetilde{B}_1 & \xleftarrow{\beta} & \text{hocolim} R \circ \widetilde{B}_2 \\
I_1
\end{array}
\]

**Remark 3.** Why do we not have the corresponding result of Proposition 4 for maps? In fact, there is a result, but it is more complicated.

Suppose that \( p_1: E_1 \to B_1, p_2: E_2 \to B_2 \) have \( M_{R-\text{cat}}(p_1) \leq k \) resp. \( M_{R-\text{cat}}(p_2) \leq \ell \). Suppose that \( \text{cat}(E_1), \text{cat}(E_2) \) are finite. Then there is a map [8] from \( G_k(p_1) \times G_\ell(p_2) \) towards \( G_{n(k,\ell)}(p_1 \times p_2) \) over \( B_1 \times B_2 \), where \( n(k,\ell) = k + \ell + \max\{ \text{cat}(E_1), \text{cat}(E_2) \} \). Therefore we only obtain \( M_{R-\text{cat}}(p_1 \times p_2) \leq M_{R-\text{cat}}(p_1) + M_{R-\text{cat}}(p_2) + \max\{ \text{cat}(E_1), \text{cat}(E_2) \} \).

**Remark 4.** The constructions of this section can similarly be done for any coaugmented functor \( T: S \to S \) that admits a fibrewise version
and a natural transformation $T(X) \times T(Y) \to T(X \times Y)$, $X, Y \in \mathcal{S}$, compatible with the coaugmentations.

References


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