ANOTHER CHARACTERIZATION
OF ROUND SPHERES

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ABSTRACT. A characterization of geodesic spheres in the simply connected space forms in terms of the ratio of the Gauss-Kronecker curvature and the (usual) mean curvature is given: An immersion of $n$ dimensional compact oriented manifold without boundary into the $n+1$ dimensional Euclidean space, hyperbolic space or open half sphere is a totally umbilic immersion if the mean curvature $H_1$ does not vanish and the ratio $H_n/H_1$ of the Gauss-Kronecker curvature $H_n$ and $H_1$ is constant.

1. Introduction

Let $M^n$ be an immersed submanifold of $N^{n+1}$ and let $H_k$ denote the $k$-th mean curvature function of $M^n$, that is, $H_k$ is the $k$-th elementary symmetric polynomial of principal curvatures of $M^n$ divided by $\binom{n}{k}$. For instance, $H_1$ is the usual mean curvature and $H_n$ is the Gauss-Kronecker curvature.

In [5], we obtained the following characterization of round spheres in the simply connected space forms in terms of the mean curvature functions $H_k$:

**Theorem A.** Let $N^{n+1}$ be one of the Euclidean space $\mathbb{R}^{n+1}$, the hyperbolic space $\mathbb{H}^{n+1}$ or the open half sphere $S^{n+1}_+$ and $\phi : M^n \to N^{n+1}$ be an isometric immersion of a compact oriented $n$-dimensional manifold without boundary $M^n$. If $H_{k-1}$ does not vanish and the ratio
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$H_k / H_{k-1}$ of two consecutive mean curvatures is a constant for some $k = 2, \ldots, n$, then $\phi(M^n)$ is a geodesic hypersphere.

While in the known characterizations of round spheres we need to assume that a mean curvature function is constant joint with some extra global conditions, for example, convexity [10], star-shapedness [4], or embeddedness [1], [6], [7], [8], [9], the above theorem shows that the mean curvature function itself is enough to characterize round spheres (cf. [3]).

In this note, we consider the other extreme case and prove the following theorem:

**Theorem B.** Let $N^{n+1}$ be one of the Euclidean space $\mathbb{R}^{n+1}$, the hyperbolic space $H^{n+1}$ or the open half sphere $S^{n+1}_+$ and $\phi : M^n \to N^{n+1}$ be an isometric immersion of a compact oriented $n$-dimensional manifold without boundary $M^n$. If $H_1$ does not vanish and the ratio $H_n / H_1$ of the Gauss-Kronecker curvature and the usual mean curvature is a constant, then $\phi(M^n)$ is a geodesic hypersphere.

We cannot expect the same result for the whole sphere $S^{n+1}$. For example, $H_1$ and $H_2$ of the torus

$$S^1(a) \times S^1(b) \subset S^3, \quad a^2 + b^2 = 1, \quad a \neq b$$

are nonzero constants.

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**2. Proof**

We use the hyperboloid model for $H^{n+1}$ and the usual embedding of $S^{n+1}$ into $\mathbb{R}^{n+2}$. Let $\eta$ denote a unit normal field on $M^n$. We use the following Minkowski formula (for proof, see [6]) where $(\ , \ )$ denotes the usual Euclidean inner product on $\mathbb{R}^{n+1}$ (on $\mathbb{R}^{n+2}$) when $N^{n+1}$ is $\mathbb{R}^{n+1}$ (when $N^{n+1}$ is $S^{n+1}_+$) and the Lorentzian inner product on $\mathbb{R}^{n+2}$ when $N^{n+1}$ is $H^{n+1}$.

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**Lemma A.** Set $H_0 = 1$. Then the following identities hold for every $k = 1, \ldots, n$.

1. When $N^{n+1}$ is $\mathbb{R}^{n+1}$,
   \[
   \int_M (H_{k-1} + H_k \langle \phi, \eta \rangle) \, dM = 0.
   \]

2. When $N^{n+1}$ is $\mathbb{H}^{n+1}$,
   \[
   \int_M (H_{k-1} \langle \phi, p \rangle + H_k \langle \eta, p \rangle) \, dM = 0 \text{ for any } p \in \mathbb{R}^{n+2}.
   \]

3. When $N^{n+1}$ is $S^{n+1}_+$,
   \[
   \int_M (H_{k-1} \langle \phi, p \rangle - H_k \langle \eta, p \rangle) \, dM = 0 \text{ for any } p \in \mathbb{R}^{n+2}.
   \]

We also use the following inequalities for higher order mean curvatures:

**Lemma B.** Suppose that all the principal curvatures are positive. Then, for every $k = 1, 2, \ldots, n$, the followings hold:

1. Every $k$-th mean curvature function $H_k$ is positive.
2. The equality $H_1 H_{n-1} = H_n$ holds only at umbilical points.
3. $H_k / H_{k-1} \leq H_{k-1} / H_{k-2}$.
4. For every $l < k$, $H_k / H_l \leq H_{k-1} / H_{l-1}$.

**Proof.** (1) is clear.

(2) is the equality case for the arithmetic–geometric mean inequality. For (3), see, for example, Section 12 of [2].

From (3), we have
\[
H_k / H_{k-1} \leq H_{k-1} / H_{k-2} \leq \cdots \leq H_{l+1} / H_l \leq H_l / H_{l-1}.
\]

Hence (4) holds.

Now, assume $H_n / H_1 = \alpha$ for a constant number $\alpha$.  

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(2.1) **Proof when** $N^{n+1} = \mathbb{R}^{n+1}$: Since $M^n$ is compact, one can find a point in $M^n$ where all the principal curvatures are positive. Then $H_n, H_1$ are positive at that point. Since $H_n/H_1$ is constant on $M^n$ and since $H_1$ does not vanish on $M^n$ by assumption, $H_1$ and $H_n$ are positive on $M^n$. Then $\alpha > 0$ and from the inequality (4) of Lemma B, we have

\[
(*) \quad 0 < \alpha = H_n/H_1 \leq H_{n-1} \ (= H_{n-1}/H_0).
\]

Since $H_n = \alpha H_1$, we have by Lemma A,

\[
0 = \int_M (H_{n-1} + H_n \langle \phi, \eta \rangle) \, dM = \int_M (H_{n-1} + \alpha H_1 \langle \phi, \eta \rangle) \, dM,
\]

that is,

\[
(1) \quad \int_M H_{n-1} \, dM = \int_M (-\alpha H_1 \langle \phi, \eta \rangle) \, dM.
\]

On the other hand, since $\alpha$ is constant, we also have by Lemma A,

\[
\int_M \alpha (1 + H_1 \langle \phi, \eta \rangle) \, dM = 0,
\]

that is,

\[
(2) \quad \int_M \alpha \, dM = \int_M (-\alpha H_1 \langle \phi, \eta \rangle) \, dM.
\]

From (1) and (2), we have

\[
\int_M (H_{n-1} - \alpha) \, dM = 0.
\]

Since we have from (*),

\[
H_{n-1} - \alpha \geq 0,
\]

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it follows that

$$H_{n-1} = \alpha = H_n / H_1$$

everywhere on $M^n$. Now, by (2) of lemma B, every point is an umbilical point, that is, $\phi(M^n)$ is a geodesic hypersphere.

(2.2) **Proof when $N^{n+1} = H^{n+1}$**: At a point of $M^n$ where the distance function of $H^{n+1}$ attains its maximum, all the principal curvatures are positive. Then (*) also holds in this case and $H_1, H_n$ are positive on $M^n$. Since $H_n = \alpha H_1$, we have

$$0 = \int_M (H_{n-1} \langle \phi, p \rangle + H_n \langle \eta, p \rangle) \, dM$$

$$= \int_M (H_{n-1} \langle \phi, p \rangle + \alpha H_1 \langle \eta, p \rangle) \, dM,$$

that is,

$$\int_M H_{n-1} \langle \phi, p \rangle \, dM = \int_M (-\alpha H_1 \langle \eta, p \rangle) \, dM.$$

Since $\alpha$ is constant, it also holds that

$$\int_M \alpha (\langle \phi, p \rangle + H_1 \langle \eta, p \rangle) \, dM = 0,$$

then, it follows that

$$\int_M (H_{n-1} - \alpha) \langle \phi, p \rangle \, dM = 0.$$

Now, if we take $p = (1, 0, \ldots, 0) \in \mathbb{R}^{n+2}$, then the sign of $\langle \phi, p \rangle$ does not change on $M^n$. Since $H_{n-1} - \alpha \geq 0$ from (*), we have $H_{n-1} - \alpha = 0$ everywhere on $M^n$. Then every point is an umbilical point as in (2.1). Hence $\phi(M^n)$ is a geodesic hypersphere.

(2.3) **Proof when $N^{n+1} = S^{n+1}_+$**: Let $c \in S^{n+1}$ be the centre of $S^{n+1}_+$. Then at a point of $M^n$ where the height function $\langle \phi, c \rangle$ attains its minimum, all the principal curvatures are positive because $M^n$ lies in
the open half sphere with the centre c. Then (*) holds and the equality in (*) holds only at umbilical points. Proceeding as in (2.2), we have

$$\int_M (H_{n-1} - \alpha) \langle \phi, p \rangle \, dM = 0.$$  

Since $M^n$ lies in the open half sphere, for $p = c$, $\langle \phi, c \rangle$ is positive on $M^n$. Then, since $H_{n-1} - \alpha \geq 0$ by (*), it follows that $H_{n-1} - \alpha = 0$ everywhere on $M^n$. Now arguing in the same way as above we can see that $\phi(M^n)$ is a geodesic hypersphere.

References


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