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WEAKLY LAGRANGIAN EMBEDDING $S^m \times S^n$ INTO $\mathbb{C}^{m+n}$

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ABSTRACT. We investigate when the product of two smooth manifolds admits a weakly Lagrangian embedding. Assume M, N are oriented smooth manifolds of dimension m and n, respectively, which admit weakly Lagrangian immersions into $\mathbb{C}^m$ and $\mathbb{C}^n$. If m and n are odd, then $M \times N$ admits a weakly Lagrangian embedding into $\mathbb{C}^{m+n}$. In the case when m is odd and n is even, we assume further that $\chi(N)$ is an even integer. Then $M \times N$ admits a weakly Lagrangian embedding into $\mathbb{C}^{m+n}$. As a corollary, we obtain the result that $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}$, $k > 1$, admits a weakly Lagrangian embedding into $\mathbb{C}^{n_1+n_2+\cdots+n_k}$ if and only if some n_i is odd.

1. Introduction

In the previous paper ([2]), we have shown that $S^m \times S^n$ admits a weakly Lagrangian embedding into $\mathbb{C}^{m+n}$ under the condition that $n = 1$ or 3. It has been observed also that if both m, n are even then $S^m \times S^n$ does not admit any weakly Lagrangian embedding into $\mathbb{C}^{m+n}$. For the case when one of m, n is odd while it is neither 1 nor 3, the question was left open.

In this paper the remaining case has been resolved by the following (see section 3):

THEOREM 1.1. *Let M, N be smooth closed manifolds of dimension m and n, $m, n \geq 1$, respectively. Assume M, N admit weakly Lagrangian immersions into $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. If both m, n are odd, then $M \times N$ admits a weakly Lagrangian embedding into $\mathbb{C}^{m+n}$. In the case that m is odd and n is even, we assume further that $\chi(N)$ is*

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an even integer. Then $M \times N$ admits a weakly Lagrangian embedding into $\mathbb{C}^{m+n}$.

In fact, as an immediate consequence, we have:

COROLLARY 1.2. *If m is odd, then $S^m \times S^n$ admits a weakly Lagrangian embedding into $\mathbb{C}^{m+n}$.*

More generally the following holds:

COROLLARY 1.3. *$S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}$, $n_i \geq 1$, $i = 1, 2, \cdots, k$, $k \geq 2$, admits a weakly Lagrangian embedding into $\mathbb{C}^{n_1+n_2+\cdots+n_k}$ if and only if some n_i is odd.*

This result of course provides a complete answer to the problem of weakly Lagrangian embedding the product of more than one spheres into a complex plane of the same dimension. Note that the product admits a Lagrangian embedding if and only if one of the spheres is S^1: If M^m admits a Lagrangian immersion into $\mathbb{C}^m$ and N^n admits a Lagrangian embedding into $\mathbb{C}^n$ then $M^m \times N^n$ admits a Lagrangian embedding into $\mathbb{C}^{m+n}$ (6.2.2, [1]) and there is no closed *simply* connected Lagrangian submanifold in $\mathbb{C}^n$ ([5]).

Note that we now have examples in which neither M^m nor N^n admits weakly Lagrangian embedding into $\mathbb{C}^m$ or into $\mathbb{C}^n$ while $M \times N$ does into $\mathbb{C}^{m+n}$: According to Kawashima ([6]), S^n admits a weakly Lagrangian embedding into $\mathbb{C}^n$ if and only if n is 1 or 3. Consider the case when $M = S^m$ and $N = S^n$ where m is odd, $m > 3$ and $n = 2$ or $n > 3$. In the totally real embedding category such examples do exist (cf. [8]) while in the Lagrangian embedding category no such example has been known so far.

The main tool for the proof of Theorem 1.1 is Theorem 2.1 below, which provides a product formula for self-intersection numbers. In fact Theorem 2.1 is a part of a theorem in another paper ([3]). However for the convenience of the reader, we provide a sketch of the proof in section 4 below.

2. Basic notions and facts

Two subbundles η_0 and η_1 of a vector bundle ξ over a topological space M is said to be *homotopic* if there exists a subbundle $\tilde{\eta}$ of $\xi \times I$ such that $\tilde{\eta}|_{M\times\{0\}} = \eta_0$ and $\tilde{\eta}|_{M\times\{1\}} = \eta_1$.

A *symplectic form* on a vector bundle is a nondegenerate two form on it. A vector bundle of finite rank is referred to as a *symplectic vector bundle* if it is considered with a fixed symplectic two form. Note that a symplectic vector bundle should be of even rank. A subbundle η of a symplectic vector bundle ξ is a *Lagrangian subbundle* if 2 (rank η) = rank ξ and the restriction of the symplectic form to η is the zero form. A subbundle η of a symplectic vector bundle ξ is called a *weakly Lagrangian subbundle* if η is homotopic to a Lagrangian subbundle of ξ.

Now let $f : L \to M$ be an embedding (resp. immersion) of a smooth manifold L into a symplectic manifold M with a symplectic structure ω. We call f a *Lagrangian embedding* (resp. *immersion*) if the tangent bundle TL of L is a Lagrangian subbundle of the symplectic vector bundle f^*TM (with the symplectic form $f^*\omega$). Similarly, f is a *weakly Lagrangian embedding* (resp. *immersion*) if TL is a weakly Lagrangian subbundle of f^*TM.

We will consider $\mathbb{C}^n$ with the usual symplectic structure. A Lagrangian embedding or a weakly Lagrangian embedding will be understood as 'into $\mathbb{C}^n$' unless otherwise specified.

Note that the notion of weakly Lagrangian embedding (resp. immersion) is invariant under regular homotopy. That is, if f_0 and f_1 are embeddings (resp. immersions) homotopic through immersions and f_0 is a weakly Lagrangian embedding (resp. immersion), then f_1 is also such.

An immersion $f : M^m \to P^{2m}$ is referred to as *completely regular* if it has no triple points, that is, $|f^{-1}y| \leq 2$ for any $y \in P$, and is self-transverse. If $f : M \to P$ is a completely regular immersion, one may define the intersection number $I(f)$ of f as follows: (i) For the mod 2 intersection number, one defines $I(f) \in \mathbb{Z}_2$ as the number of the double points mod 2. (ii) Assume that M, P are oriented and m is even. Then one may define the integral intersection number as follows: Let $r = f(p) = f(p')$, $p \neq p'$, be a double point of f. Let $v = (v_1, v_2, \cdots, v_m)$, $v' = (v'_1, v'_2, \cdots, v'_m)$ be sequences of tangent vectors which represent the orientation of M at p and p', respectively. If the sequence of tangent vectors $(dfv, dfv') = (dfv_1, dfv_2, \cdots, dfv_m, dfv_1', dfv_2', \cdots, dfv_m')$ represents the orientation of P at r, write $\varepsilon_r = +1$ and, otherwise, write $\varepsilon_r = -1$. Note that ε_r remains unchanged even if we interchange p, p'. Define $I(f) = \sum_r \varepsilon_r \in \mathbb{Z}$, where r runs through all the double points of f.

If f, g are completely regular immersions which are regularly homotopic to each other, then we have $I(f) = I(g)$: According to J. Cerf ([4]), for generic regular homotopy, the double points vary continuously except at a finite set of points at each of which a pair of double points appear or disappear. If m is even, the two have opposite values for ε_r. Furthermore, since every immersion is regularly homotopic to a completely regular immersion, it follows that $I(f)$ is well-defined for any immersion f.

Now assume $m \geq 3$ and P is simply connected. Let $I(f)$ denote the mod 2 intersection number if the dimension of M is odd or M is unorientable and, in the remaining case, the integral intersection number. Then $I(f)$ vanishes if and only if the regular homotopy class of f can be represented by an embedding, which is a consequence of the Whitney trick (cf. [3]).

The following is a part of Theorem A, [3].

THEOREM 2.1. *Let $f : M^m \to P^{2m}$, $g : N^n \to Q^{2n}$ be immersions where M, N are closed smooth manifolds and P, Q, smooth manifolds. Then, for the mod 2 intersection numbers, we have*

$$I(f \times g) = \chi(\nu_f)I(g) + I(f)\chi(\nu_g) \in \mathbb{Z}_2 ,$$

where $\chi(\cdot)$ is the Euler characteristic in $\mathbb{Z}_2$-coefficients. Furthermore, assume M, N, P, Q are oriented, and both m, n are odd. Then, for the integral intersection numbers, we have

$$I(f \times g) = 0 \in \mathbb{Z} .$$

The sketch of proof for Theorem 2.1 will be provided in section 4.

3. Proofs

Proof of Theorem 1.1. If $m+n = 2$, then $M \times N$ admits a Lagrangian embedding. Therefore we may assume $m + n \geq 3$.

Let $f : M^m \to \mathbb{C}^m$, $g : N^n \to \mathbb{C}^n$ be weakly Lagrangian immersions. If m and n are odd, then, by Theorem 2.1, we have $I(f \times g) = 0$. This implies that $f \times g$ is regularly homotopic to an embedding. Now since being weakly Lagrangian is invariant under regular homotopy, $M \times N$ admits a weakly Lagrangian embedding into $\mathbb{C}^{m+n}$.

If m is odd, n is even, then $\chi(M) = 0$ (see Lemma 4.3 below). Furthermore, by assumption $\chi(N)$ is an even integer. Now note that $\nu_f \cong (-1)^{m(m-1)/2}TM$, $\nu_g \cong (-1)^{n(n-1)/2}TN$ (see [2]). Therefore, $I(f \times g) = \chi(\nu_f)I(g) + I(f)\chi(\nu_g) = 0 \in \mathbb{Z}_2$. We conclude that $f \times g$ is regularly homotopic to an embedding. Thus $M \times N$ admits a weakly Lagrangian embedding into $\mathbb{C}^{m+n}$. □

Proof of Corollary 1.2. For $n \geq 1$, S^n admits a (weakly) Lagrangian immersion into $\mathbb{C}^n$ with only one double point ([9], p. 26). Now our statement is an immediate consequence of Theorem 1.1. □

Proof of Corollary 1.3. Assume n_1 is odd without loss of generality. Since each S^{n_i}, $i = 1, 2, \cdots, k$, admits a Lagrangian immersion into $\mathbb{C}^{n_i}$, it follows that $S^{n_2} \times S^{n_3} \times \cdots \times S^{n_k}$ admits a Lagrangian immersion into $\mathbb{C}^{n_2+n_3+\cdots+n_k}$. Therefore we may apply Theorem 1.1 to conclude that $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}$ admits a weakly Lagrangian embedding into $\mathbb{C}^{n_1+n_2+\cdots+n_k}$.

On the other hand, if all n_i are even, then $\chi(S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}) \neq 0$. Thus $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}$ does not admit any weakly Lagrangian embedding into $\mathbb{C}^{n_1+n_2+\cdots+n_k}$ (see [2]). □

4. Sketch of proof for Theorem 2.1

This section is entirely devoted to a sketch of proof of Theorem 2.1.

We may assume f, g are completely regular immersions.

Then as the first step we consider the case each of f, g has only one double point. Furthermore, we assume that both ν_f, ν_g admit nowhere vanishing sections. Then the following holds, which is a special case of Theorem 2.1.

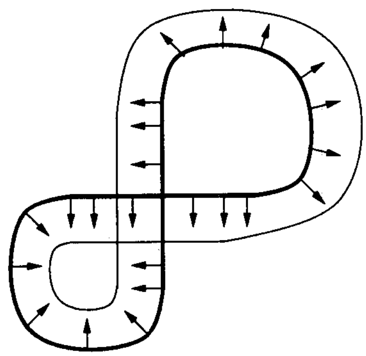
PROPOSITION 4.1. *The product $f \times g$ is regularly homotopic to a completely regular immersion with exactly two double points. Furthermore, assume M, N, P, Q are oriented and $m + n$ is even. Then the signs of the two double points differ from each other by multiplication by $(-1)^n$.*

The proof depends the following.

LEMMA 4.2. *There is a smooth regular homotopy $f_t : M \to P$, $t \in I$, such that $f_0 = f$ and the following conditions hold*

(i) *f_t is a completely regular immersion with exactly one double pair $\{p_t, p'_t\}$ for each t,*
(ii) *the map $I \times \{0,1\} \to M$ which sends $(t, 0)$ to p_t and $(t, 1)$ to p'_t is a smooth embedding,*
(iii) *'$f_t(x) = f_s(y)$, $(x, t) \neq (y, s)$' implies that '$(x, y) = (p_s, p'_t)$ or $(x, y) = (p'_s, p_t)$' and*
(iv) *f_t meets f_s transversely if $t \neq s$.*

We omit the proof of the above lemma which is rather technical. However the lemma itself and its proof is motivated by the simple one dimensional example, as illustrated below.



Proof of 4.1 *modulo* 4.2. Let f_t, $\{p_t, p'_t\}$ be as in Lemma 4.2 and also let $g_t : N \to Q$, $t \in I$, be a smooth regular homotopy for g satisfying the conditions of Lemma 4.2 with double pairs $\{q_t, q'_t\}$.

Choose a smooth function $\varphi : M \to I$ which is constantly 1 on a neighborhood of $\{p_t | t \in I\}$ and constantly 0 on a neighborhood of $\{p'_t | t \in I\}$. Likewise choose a smooth function $\psi : N \to I$ satisfying the same condition for the two sets $\{q_t | t \in I\}$, $\{q'_t | t \in I\}$.

Then we define a homotopy $\Lambda_t : M \times N \to V \times W$, $t \in I$, by

$$\Lambda_t(x, y) = (f_{t\psi(y)}(x), g_{t\varphi(x)}(y)) .$$

Then it is straightforward to see that Λ_t is a smooth homotopy through immersions such that $\Lambda_0 = f \times g$.

We must show that $\Lambda_1 = \Lambda$ has only two double points.

Assume $\Lambda(x, y) = \Lambda(x', y')$ and $(x, y) \neq (x', y')$. Then we have $x \neq x'$ or $y \neq y'$.

First consider the case $x \neq x'$. Then we have from $f_{\psi(y)}(x) = f_{\psi(y')}(x')$ that

$$(x, x') = (p_{\psi(y')}, p'_{\psi(y)}) \text{ or } (x, x') = (p'_{\psi(y')}, p_{\psi(y)}) .$$

Assume $(x, x') = (p_{\psi(y')}, p'_{\psi(y)})$, it follows that $\varphi(x) = 1, \varphi(x') = 0$ and that $g_1(y) = g_0(y')$, which means that (y, y') is (q_0, q'_1) or (q'_0, q_1). If $(y, y') = (q_0, q'_1)$, then $(x, x') = (p_0, p'_1)$ and, if $(y, y') = (q'_0, q_1)$, then $(x, x') = (p_1, p'_0)$. Thus we have in this case as the double pairs for Λ

$$\{(p_0, q_0), (p'_1, q'_1)\}, \ \{(p_1, q'_0), (p'_0, q_1)\} .$$

Assume $(x, x') = (p'_{\psi(y')}, p_{\psi(y)})$ and proceed similarly as in the above. Then we obtain the same two double pairs for Λ as in the above.

Now assume $y \neq y'$. Then, from $g_{\varphi(x)}(y) = g_{\varphi(x')}(y')$, we may easily infer that $\psi(y) \neq \psi(y')$. Then, from $f_{\psi(y)}(x) = f_{\psi(y')}(x')$, we conclude that $x \neq x'$. Thus this case reduces to the case when $x \neq x'$.

We conclude that $\{(p_0, q_0), (p'_1, q'_1)\}$, $\{(p_1, q'_0), (p'_0, q_1)\}$ are the only two double pairs for Λ.

That Λ is self-transverse follows from the fact that f_0 , f_1 are transverse to each other as well as g_0, g_1 together with the fact that φ, ψ are constant on each of some neighborhoods of $p_i, p'_i, q_i, q'_i, i = 0, 1$.

Finally we prove the last statement of the proposition.

Let $v_t = (v_{1,t}, v_{2,t}, \cdots, v_{m,t})$, $v'_t = (v'_{1,t}, v'_{2,t}, \cdots, v'_{m,t})$ and $w_t = (w_{1,t}, w_{2,t}, \cdots, w_{n,t})$, $w'_t = (w'_{1,t}, w'_{2,t}, \cdots, w'_{n,t})$ be sequences of vectors, continuously parameterized by $t \in I$, representing the given orientations of M and N at p_t, p'_t and at q_t, q'_t, respectively.

Let $\varepsilon(\omega)$ be 1 or -1 for each sequence ω of independent $2(m+n)$ tangent vectors in $T_{(x,y)}P \times Q$, $(x,y) \in P \times Q$, according to whether or not it represents the orientation of $P \times Q$, which is non other than the product orientation.

Then $\varepsilon(df_t v_0, dg_t w_0, df v'_t, dg w'_t)$, $\varepsilon(df v_t, dg_t w'_0, df_t v'_0, dg w_t)$ are constant for $t \in I$ and, by the usual sign convention, we have

$$\varepsilon(df v_0, dg w_0, df v'_0, dg w'_0) = (-1)^{2mn+n^2} \varepsilon(df v_0, dg w'_0, df v'_0, dg w_0)\ .$$

Note that $2mn + n^2 \equiv n \bmod 2$. Thus we conclude that

$$\varepsilon(df_1 v_0, dg_1 w_0, df v'_1, dg w'_1) = (-1)^n \varepsilon(df v_1, dg_1 w'_0, df_1 v'_0, dg w_1)\ .$$

Note that the left hand side of the equality is the intersection number of Λ at $\Lambda(p_0, q_0) = \Lambda(p'_1, q'_1)$ and the right hand side is the intersection number at $\Lambda(p_1, q'_0) = \Lambda(p'_0, q_1)$. These observations also proves the last statement of the proposition. □

Now we observe the following:

LEMMA 4.3. *Let m be a positive odd integer. Then any orientable vector bundle of rank m over an orientable manifold M^m admits a nowhere vanishing section.*

Proof. The Euler class of an oriented vector bundle of an odd rank is 2-torsion (cf. [7], p. 98). Since $H^m(M;\mathbb{Z})$ has no torsion, this means the Euler class of the bundle vanishes. However the Euler class is the exact obstruction for an oriented vector bundle in concern to admit a nowhere vanishing section. This completes the proof. □

Then Theorem 2.1 is an immediate consequence of the following:

PROPOSITION 4.4. *Let $f : M^m \to P^{2m}$, $g : N^n \to Q^{2n}$ be completely regular immersions with respective double points $r_1, r_2, \cdots \in P$, $s_1, s_2, \cdots \in Q$. Assume there are sections $\alpha : M \to \nu_f$ and $\beta : N \to \nu_g$ which meet the zero sections transversely, respectively, at $a_1, a_2, \cdots \in M$, $\{a_1, a_2, \cdots\} \cap f^{-1}\{r_1, r_2, \cdots\} = \emptyset$, and at $b_1, b_2, \cdots \in N$, $\{b_1, b_2, \cdots\} \cap g^{-1}\{s_1, s_2, \cdots\} = \emptyset$. Then*

(a) *$f \times g : M \times N \to P \times Q$ is regularly homotopic to a completely regular immersion Λ which has, as its double points, two for each of the ordered pairs (r_i, s_j) and one for each of (a_k, s_j), (r_i, b_l), all of which are distinct among themselves.*

Furthermore, assume both m, n are odd and M, N, P, Q are oriented. Then we have that

(b) *if $x_{i,j}, x_{i,j'}$ are the two double points of Λ corresponding to each of (r_i, s_j), we have $\varepsilon_{x_{i,j}} = -\varepsilon_{x'_{i,j}}$.*

In fact the proof of the above is a slight modification of Proposition 4.1 exploiting the following generalization of Lemma 4.2 for which we also omit the proof:

LEMMA 4.5. *Let $f : M \to P$, $r_1, r_2, \cdots \in P$, $\alpha : M \to \nu_f$, $a_1, a_2, \cdots \in M$ be as in 4.2. Then there is a smooth regular homotopy $f_t : M \to P$, $t \in I$, such that $f_0 = f$ and satisfying the following conditions*

(i) *f_t is a completely regular immersion with exactly one double pair $\{p_{i,t}, p'_{i,t}\}$ for each $t \in I$ and for each $i = 1, 2, \cdots$,*

(ii) *the map $I \times \{0, 1\} \times \{1, 2, \cdots\} \to M$ which sends $(t, 0, i)$ to $p_{i,t}$ and $(t, 1, i)$ to $p'_{i,t}$ is a smooth embedding,*

(iii) *'$f_t(x) = f_s(y)$, $(x, t) \neq (y, s)$' implies that '$(x, y) = (p_{i,s}, p'_{i,t})$ or $(x, y) = (p'_{i,s}, p_{i,t})$, for some $i = 1, 2, \cdots$, or $x = y = a_j$, for some $j = 1, 2, \cdots$,'*

(iv) *and f_t meets f_s transversely if $t \neq s$.*

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