THE CONVERGENCE OF FINITE DIFFERENCE APPROXIMATIONS FOR SINGULAR TWO-POINT BOUNDARY VALUE PROBLEMS

H. Y. Lee, J. M. Seong, and J. Y. Shin

Abstract. We consider two finite difference approximations to a singular boundary value problem arising in the study of a nonlinear circular membrane under normal pressure. It is shown that the rates of convergence are $O(h)$ and $O(h^2)$, respectively. An iterative scheme is introduced which converges to the solution of the finite difference equations. Finally the numerical experiments are given.

1. Introduction

In the study of a nonlinear circular membrane under normal pressure [3,4], the following singular boundary value problem arises:

\begin{align}
- y'' - \frac{3}{x} y' - \frac{2}{y^2} &= 0, \quad 0 < x < 1, \\
y'(0) &= 0, \quad y'(1) + (1 - v)y(1) = 0, \quad 0 < v < 1,
\end{align}

where $v$, $0 < v < 1$, is a constant. The existence of a unique positive solution of (1.1) has been discussed by [2,3,4,9]. Numerical solutions of this problem can be obtained by the iterative method [2] and numerical techniques [4] on the integral equation equivalent to (1.1). It is mentioned in [4] that because of singularity and the nonlinearity, difficulties are encountered if numerical solutions of (1.1) are attempted by finite difference methods. In [8], a finite difference method to a class of singular boundary value problem is introduced.

When the boundary condition at $x = 1$ is $y(1) = \lambda(>0)$ instead of $y'(1) + (1 - v)y(1) = 0$, the unique existence of a positive solution and a

Received March 28, 1998.
1991 Mathematics Subject Classification: 65L12, 65L10.
Key words and phrases: A finite difference approximation, a singular boundary value problem, a rate of convergence.
numerical solution are studied by [2,3,4,7,8,9]. In [7], a finite difference approximation to (1.1) is introduced whose rate of convergence is $O(h^2)$ and which may avoid the above difficulties stated in [4]. And the global error estimate $O(h^2)$ is better than one in [8].

In this paper, motivated by the method in [7], two finite difference approximations to (1.1), scheme I and scheme II, are considered. The rates of convergence are $O(h)$ and $O(h^2)$, respectively and both methods may avoid the difficulties stated in [4]. To obtain the solution of each finite difference equation, an iterative technique is introduced which converges monotonically to the solution of the finite difference equation. In section 2, some preliminaries are given. In section 3, two finite difference approximations, scheme I and scheme II, are introduced, and an iterative technique is given which converges monotonically to the solution of the finite difference equations. In section 4, we prove analytically that the rates of convergence of the scheme I and the scheme II are $O(h)$ and $O(h^2)$, respectively. The rates of convergence of scheme I and scheme II are verified numerically in section 5.

2. Preliminaries

To discuss the behavior of the solution of (1.1) at $x = 0$, we begin with the following lemma whose proof is straightforward.

**Lemma 2.1.** Let $f \in C[0,1]$ and $f' \in C(0,1]$. If $\lim_{x \to 0^+} f'(x)$ exists, then

$$f_+(0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x \to 0^+} f'(x),$$

which implies that $f'(x)$ continuous at $x = 0$.

It was shown in [9] that there exists a unique solution $Y \in C^2(0,1) \cap C^1[0,1]$ of (1.1). Thus the following lemma is obtained from Lemma 2.1 and the fact that

$$Y'(x) = -\frac{1}{x^3} \int_0^x \frac{2s^3}{Y^2(s)} \, ds.$$

**Lemma 2.2** [7]. Let $Y$ be a positive solution of (1.1). Then

(1) $Y''(0)$ exists and $Y''(x)$ is continuous at $x = 0$. 


(2) \( Y^{(3)}_+(0) = 0 \) exists and \( Y^{(3)}(x) \) is continuous at \( x = 0 \).
(3) \( Y^{(4)}_+(0) \) exists and \( Y^{(4)}(x) \) is continuous at \( x = 0 \).

REMARK. Lemma 2.2 implies that if \( Y \) is a positive solution of (1.1) then \( Y \in C^4[0,1] \).

3. Finite difference approximations

3.1. Scheme I

Let \( N \) be a positive integer, \( h = \frac{1}{N} \), \( x_i = i \cdot h \), \( i = 0, 1, 2, \cdots, N \), and let \( y_i \) be the approximation of \( Y(x_i) \), \( i = 0, 1, 2, \cdots, N \). Consider the following finite difference approximation (scheme I):

\[
-8 \cdot \frac{y_{i+1} - y_i}{h^2} - \frac{2}{y_i^3} = 0,
-4 \cdot \frac{y_{i+2} - 2y_{i+1} + y_i}{h^2} - \frac{2}{y_i^3} = 0,
(3.1.1) \quad - \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \cdot \frac{3}{x_i} \cdot \frac{y_{i+1} - y_{i-1}}{2h} - \frac{2}{y_i^3} = 0,
\]

\[
- \frac{y_{N-1} - y_N}{h} + (1 - v)y_N = 0.
\]

Let

\[
L_1 = \begin{bmatrix}
8 & -8 & 0 & \cdots & 0 \\
-4 & 8 & -4 & 0 & \cdots & 0 \\
0 & -1 + \frac{3h}{2x_2} & 2 & -1 - \frac{3h}{2x_2} & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & -1 + \frac{3h}{2x_{N-1}} & 2 & -1 - \frac{3h}{2x_{N-1}} & \cdots \\
0 & \cdots & 0 & -1 & 1 + h(1 - v) &
\end{bmatrix},
\]

\( \tilde{y} = (y_0, y_1, \cdots, y_{N-1}, y_N)^t \).
and
\[ N_1 \tilde{y} = \left( -\frac{2h^2}{y_0^2}, -\frac{2h^2}{y_1^2}, \ldots, -\frac{2h^2}{y_{N-1}^2}, 0 \right)^t, \]

where \(N_1 \tilde{y}\) and \(\tilde{y}\) are column vectors. Now we have a nonlinear system

\[(3.1.2)\]
\[ L_1 \tilde{y} + N_1 \tilde{y} = 0, \]

where \(0\) is the zero vector. To solve the nonlinear system (3.1.2), we use Newton’s method. So, for \(m = 0, 1, 2, \ldots, \) we have

\[(3.1.3) \quad \tilde{y}^{(m+1)} = \tilde{y}^{(m)} - \left( L_1 + N_1 \tilde{y}^{(m)} \right)^{-1} \left( L_1 \tilde{y}^{(m)} + N_1 \tilde{y}^{(m)} \right), \]

where \(N_1 \tilde{y}\) is the diagonal matrix, \[
\text{diag} \left[ \frac{4h^2}{y_0^3}, \frac{4h^2}{y_1^3}, \ldots, \frac{4h^2}{y_{N-1}^3}, 0 \right].
\]

Therefore, from (3.1.3), we derive

\[(3.1.4) \quad L_1 \tilde{y}^{(m+1)} + \left[ N_1 \tilde{y}^{(m)} \right] \tilde{y}^{(m+1)} = \left[ N_1 \tilde{y}^{(m)} \right] \tilde{y}^{(m)} - N_1 \tilde{y}^{(m)} \]

and

\[(3.1.5) \quad L_1 \tilde{y}^{(m+1)} + N_1 \tilde{y}^{(m+1)}
= N_1 \tilde{y}^{(m+1)} - N_1 \tilde{y}^{(m)} - N_1 \tilde{y}^{(m)} \left[ \tilde{y}^{(m+1)} - \tilde{y}^{(m)} \right]
= \frac{1}{2} N_1'' \xi^{(m)} \left( \left( y_j^{(m+1)} - y_j^{(m)} \right)^2 \right),
\]

where \(N_1'' \tilde{y}\) is the diagonal matrix, \[
\text{diag} \left[ -\frac{12h^2}{y_0^4}, -\frac{12h^2}{y_1^4}, \ldots, -\frac{12h^2}{y_{N-1}^4}, 0 \right],
\]

and \(\xi^{(m)}\) is between \(y_j^{(m+1)}\) and \(y_j^{(m)}\).

**Theorem 3.1.1** [1].

(i) The M-matrix \(L_1\) is an inverse positive matrix.

(ii) The matrix \(L_1 + N_1 \tilde{y}\) is an inverse positive matrix for any \(\tilde{y} > 0\).
Proof. (i) Let $D_i$ be the $(i + 1)$-th leading principal minor of $L_1$. Then we obtain
\[D_0 = 8, \quad D_1 = 32, \quad D_2 = \left(1 + \frac{3h}{2x_2}\right) D_1, \quad \cdots,\]
\[D_{N-1} = \left(1 + \frac{3h}{2x_{N-1}}\right) D_{N-2}, \quad D_N = h(1 - v) \left(1 + \frac{3h}{2x_{N-1}}\right) D_{N-2},\]
which imply that the $M$-matrix $L_1$ is an inverse positive matrix.

(ii) Since $L_1$ is an $M$-matrix and $N_1'\tilde{y}$ is a nonnegative diagonal matrix for any $\tilde{y} > 0$, $L_1 + N_1'\tilde{y}$ is an inverse positive matrix. \qed

**Lemma 3.1.1.** If $u$ satisfies $L_1u + N_1u \geq 0$ and $l$ satisfies $L_1l + N_1l \leq 0$, then
\[l \leq u,\]
where $0 < u_i$ and $0 < l_i$ for $i = 0, 1, 2, \cdots, N$.

**Proof.** From the assumptions on $u$ and $l$, we have
\[0 \leq L_1u + N_1u - L_1l - N_1l = L_1(u - l) + N_1(u - l) = (L_1 + N_1'\xi)(u - l),\]
where $\xi_i$ lies between $l_i$ and $u_i$. Since $L_1 + N_1'\xi$ is inverse positive, $u - l \geq 0$, which completes the proof. \qed

**Lemma 3.1.2.** If $u$ satisfies $L_1u + N_1u \geq 0$, $y^{(0)} > 0$, $L_1y^{(0)} + N_1y^{(0)} \leq 0$, and $\{y^{(m)}\}$ is given by (3.1.3) or (3.1.4), then
\[y^{(0)} \leq y^{(1)} \leq y^{(2)} \leq \cdots \leq y^{(m)} \leq \cdots \leq u \quad \text{for} \quad m = 0, 1, 2, \cdots,\]
where $0 < u_i$ for $i = 0, 1, 2, \cdots, N$.

**Proof.** It is obvious from (3.1.3), (3.1.5) and Lemma 3.1.1. \qed

Let
\[l(x) = -\frac{(1 - v)^2}{4(2 - v)^2} \frac{3}{8} \left(x^2 - \frac{2 - v}{1 - v}\right),\]
\[l_i = l(ih), i = 0, 1, 2, \cdots, N,\]
\[l = (l_0, l_1, l_2, \cdots, l_N)\]
where $h = \frac{1}{N}$. Then it is easy to show that $1$ satisfies $L_1 1 + N_1 1 \leq 0$. And let

\[
u(x) = -\left[\frac{(1-v)^2}{16}\right]^\frac{1}{2}(x^2 - \frac{3-v}{1-v}),
\]

\[u_i = u(ih), i = 0, 1, 2, \ldots, N,
\]

\[u = (u_0, u_1, u_2, \ldots, u_N)^t.
\]

Then it is also easy to show that $\mathbf{u}$ satisfies $L_1 \mathbf{u} + N_1 \mathbf{u} \geq 0$.

**Theorem 3.1.2.** The system of equations (3.1.2) has a unique positive solution.

*Proof.* The system of equations (3.1.2) has a positive solution from Lemma 3.1.2 and the above remark. Suppose that $\mathbf{y}$ and $\mathbf{w}$ are positive solutions of the system of equations (3.1.2) and $\mathbf{z} = \mathbf{y} - \mathbf{w}$. Then we have

\[L_1 \mathbf{z} + N_1 \mathbf{y} - N_1 \mathbf{w} = \mathbf{0}.
\]

So we obtain

\[(L_1 + N_1 \xi) \mathbf{z} = \mathbf{0},
\]

where $\xi_i$ is between $y_i$ and $w_i$. Since $L_1 + N_1 \xi$ is an inverse positive matrix, $\mathbf{z} = \mathbf{0}$ and hence $\mathbf{y} = \mathbf{w}$. \qed

### 3.2. Scheme II

Using the same notations as given in the beginning of Section 3.1, we consider the following finite difference approximation (scheme II):

\[\begin{align*}
- 8 \cdot \frac{y_1 - y_0}{h^2} - \frac{2}{y_0^2} &= 0, \\
- 4 \cdot \frac{y_2 - 2y_1 + y_0}{h^2} - \frac{2}{y_1^2} &= 0, \\
(3.2.1) \quad - \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{3}{x_i} \cdot \frac{y_{i+1} - y_{i-1}}{2h} - \frac{2}{y_i^2} &= 0, \\
&\quad i = 2, 3, \ldots, N - 1, \\
- \frac{2y_N - 2y_{N-1}}{h^2} + \frac{2}{h} (1-v)y_N + \frac{3}{x_N} (1-v)y_N - \frac{2}{y_N^2} &= 0.
\end{align*}\]
Let $L_2$ be the same matrix as the matrix $L_1$ in section 3.1 except the $n$-th row and let the $n$-th row of $L_2$ be $0, 0, \ldots, 0, -2, 2 + 2h(1 - v) + 3h^2(1 - v)$. Let
\[ \tilde{y} = (y_0, y_1, \cdots, y_{N-1}, y_N)^t, \]
and
\[ N_2\tilde{y} = \left( \begin{array}{ccccc} -\frac{2h^2}{y_0^2}, & -\frac{2h^2}{y_1^2}, & \cdots, & -\frac{2h^2}{y_{N-1}^2}, & -\frac{2h^2}{y_N^2} \end{array} \right)^t, \]
where $N_2\tilde{y}$ and $\tilde{y}$ are column vectors. Now we have a nonlinear system
\[(3.2.2) \quad L_2\tilde{y} + N_2\tilde{y} = 0,\]
where $0$ is the zero vector. So, for $m = 0, 1, 2, \cdots$, we have
\[(3.2.3) \quad \tilde{y}^{(m+1)} = \tilde{y}^{(m)} - \left( L_2 + N_2'\tilde{y}^{(m)} \right)^{-1} \cdot \left( L_2\tilde{y}^{(m)} + N_2\tilde{y}^{(m)} \right).\]
where $N_2'\tilde{y}$ is the diagonal matrix, diag $\left[ \frac{4h^2}{y_0^3}, \frac{4h^2}{y_1^3}, \frac{4h^2}{y_2^3}, \cdots, \frac{4h^2}{y_N^3} \right]$. Therefore, from (3.2.3), we derive
\[(3.2.4) \quad L_2\tilde{y}^{(m+1)} + \left[ N_2'\tilde{y}^{(m)} \right] \tilde{y}^{(m+1)} = \left[ N_2'\tilde{y}^{(m)} \right] \tilde{y}^{(m)} - N_2\tilde{y}^{(m)}\]
and
\[(3.2.5) \quad L_2\tilde{y}^{(m+1)} + N_2\tilde{y}^{(m+1)} = N_2\tilde{y}^{(m+1)} - N_2\tilde{y}^{(m)} - N_2'\tilde{y}^{(m)} \left[ \tilde{y}^{(m+1)} - \tilde{y}^{(m)} \right] = \frac{1}{2} N_2''\xi^{(m)} \left( \left( y_j^{(m+1)} - y_j^{(m)} \right)^2 \right),\]
where $N_2''\tilde{y}$ is the diagonal matrix, diag $\left[ -\frac{12h^2}{y_0^4}, -\frac{12h^2}{y_1^4}, \cdots, -\frac{12h^2}{y_N^4} \right]$, and $\xi_j^{(m)}$ is between $y_j^{(m+1)}$ and $y_j^{(m)}$. 
THEOREM 3.2.1 [1].

(i) The $M$-matrix $L_2$ is an inverse positive matrix.
(ii) The matrix $L_2 + N_2\tilde{y}$ is an inverse positive matrix for any $\tilde{y} > 0$.

Proof. (i) Let $D_i$ be the $(i + 1)$-th leading principal minor of $L_2$. Then we obtain

\[
D_0 = 8, \quad D_1 = 32, \quad D_2 = \left(1 + \frac{3h}{2x_2}\right) D_1, \quad \cdots,
\]

\[
D_{N-1} = \left(1 + \frac{3h}{2x_{N-1}}\right) D_{N-2},
\]

\[
D_N = \{2h(1-v) + 3h^2(1-v)\} \cdot \left(1 + \frac{3h}{2x_{N-1}}\right) D_{N-2},
\]

which imply that the $M$-matrix $L_2$ is an inverse positive matrix.

(ii) The proof is the same as that of Theorem 3.1.1. $\square$

LEMMA 3.2.1. If $u$ satisfies $L_2u + N_2u \geq 0$ and $1$ satisfies $L_21 + N_21 \leq 0$, then

\[
1 \leq u,
\]

where $0 < u_i$ and $0 < l_i$ for $i = 0, 1, 2, \cdots, N$.

Proof. The proof is similar to that of Lemma 3.1.1. $\square$

LEMMA 3.2.2. If $u$ satisfies $L_2u + N_2u \geq 0$, $y^{(0)} > 0$, $L_2y^{(0)} + N_2y^{(0)} \leq 0$, and $\{y^{(m)}\}$ is given by (3.2.3) or (3.2.4), then

\[
y^{(0)} \leq y^{(1)} \leq y^{(2)} \leq \cdots \leq y^{(m)} \leq \cdots \leq u, \quad \text{for} \quad m = 0, 1, 2, \cdots,
\]

where $0 < u_i$ for $i = 0, 1, 2, \cdots, N$.

Proof. It is obvious from (3.2.3), (3.2.5), and Lemma 3.2.1. $\square$

Let

\[
l(x) = -\frac{(1-v)^2}{4(2-v)^2}\frac{1}{2}(x^2 - \frac{2-v}{1-v}),
\]

\[
l_i = l(ih), i = 0, 1, 2, \cdots, N,
\]

\[
l = (l_0, l_1, l_2, \cdots, l_N)^t,
\]
where $h = \frac{1}{N}$. Then it is easy to show that $l$ satisfies $L_2 l + N_2 l \leq 0$. And let
\[
u(x) = -\left[\frac{(1 - \nu)^2}{16}\right]^{\frac{1}{2}} \left(x^2 - \frac{3 - \nu}{1 - \nu}\right),
\]
u_i = u(ih), i = 0, 1, 2, \cdots, N,
u = (u_0, u_1, u_2, \cdots, u_N)^t.
Then it is also easy to show that $u$ satisfies $L_2 u + N_2 u \geq 0$.

**Theorem 3.2.3.** The system of equations (3.2.2) has a unique positive solution.

**Proof.** The proof is similar to that of Theorem 3.1.3. \hfill \square

4. The convergence of finite difference approximations

4.1. Scheme I

**Lemma 4.1.1** [5,7]. Let $Q(x_i) = Q_i$ and $E(x_i) = E_i$ be discrete functions defined on $x_0$, $x_1$, $x_2$, $\cdots$, $x_N$. Assume that there exists an $\omega > 0$ such that
\[Q_i \leq -\omega < 0, \quad i = 0, 1, 2, \cdots, N - 1.
Set $C = \max \left(\frac{4}{\omega}, \frac{1}{1 - \nu}\right)$. At the grid points $x_0$, $x_1$, $x_2$, $\cdots$, $x_N$ define a difference operator $L_1 h$ by
\[
L_1 h E_0 = 8 \cdot \frac{E_1 - E_0}{h^2} + Q_0 E_0,
\]
\[
L_1 h E_1 = 4 \cdot \frac{E_2 - 2E_1 + E_0}{h^2} + Q_1 E_1,
\]
\[
L_1 h E_i = \frac{E_{i+1} - 2E_i + E_{i-1}}{h^2} + \frac{3}{x_i} \cdot \frac{E_{i+1} - E_{i-1}}{2h} + Q_i E_i,
\]
i = 2, 3, $\cdots$, N - 1,
\[
L_1 h E_N = \frac{E_N - E_{N-1}}{h} + (1 - \nu)E_N.
Then
\[|E_i| \leq C \cdot \max_{0 \leq j \leq N} |L_1 h E_j|, \quad i = 0, 1, 2, \cdots, N.
\]
Proof. Note that $C \geq 1$. If $\max |E_i|$ occurs for $i = N$, then

$$|E_N| \leq \frac{1}{1 - v} |L^h_1 E_N|.$$ 

Suppose that $\max |E_i|$ occurs for one of $i = 0, 1, 2, \cdots, N - 1$. Then from the proof of Lemma 4.1 in [7], we have

$$\max_{0 \leq j \leq N-1} |E_j| \leq \frac{4}{\omega} \cdot \max_{0 \leq j \leq N-1} |L^h_1 E_j|.$$ 

Thus the proof is completed. \qed

**Theorem 4.1.1.** Let $Y(x) \in C^4[0,1]$ be an analytic solution of the boundary value problem (1.1). Let $y_i$, $i = 0, 1, 2, \cdots, N$, be numerical solutions of $L^h_1 y + N_1 y = 0$ and $E_i = Y(x_i) - y_i$ be errors. Then

$$|E_i| \leq C M_4 h,$$

where

$$M_4 = \sup_{[0,1]} \left| \frac{d^4 Y}{dx^4} \right| \quad \text{and } C \text{ is a constant.}$$

Proof. By the mean value theorem and Taylor theorem, we obtain

$$0 = 4Y''(x_0) + \frac{2}{[Y(x_0)]^2}$$

$$= 8 \cdot \frac{Y(x_1) - Y(x_0)}{h^2} + \frac{2}{[Y(x_0)]^2} - Y^{(4)}(\xi_0) \cdot \frac{h^2}{3},$$

where $x_0 < \xi_0 < x_1$. For $x_1$, we have

$$0 = Y''(x_1) + \frac{3}{x_1} \cdot Y'(x_1) + \frac{2}{[Y(x_1)]^2}$$

$$= 4Y''(x_1) + 3 \left( Y''(\xi_0) - Y''(x_1) \right) + \frac{2}{[Y(x_1)]^2}$$

$$= 4 \cdot \frac{Y(x_2) - 2Y(x_1) + Y(x_0)}{h^2} - \frac{h^2}{6} \left[ Y^{(4)}(\eta_0) + Y^{(4)}(\eta_1) \right]$$

$$+ 3Y^{(4)}(\xi_2) \cdot \xi_1 (\xi_0 - x_1) + \frac{2}{[Y(x_1)]^2},$$
where $x_0 < \eta_0 < x_1 < \eta_1 < x_2, x_0 < \xi_0 < \xi_1 < x_1, \text{ and } x_0 < \xi_2 < \xi_1.$

And for $i = 2, 3, 4, \ldots, N - 1,$

(4.1.7)

$$0 = Y''(x_i) + \frac{3}{x_i} \cdot Y'(x_i) + \frac{2}{[Y(x_i)]^2}$$

$$= \frac{Y(x_{i+1}) - 2Y(x_i) + Y(x_{i-1})}{h^2} + \frac{3}{x_i} \cdot \frac{Y(x_{i+1}) - Y(x_{i-1})}{2h}$$

$$- \frac{h^2}{24} \left[ Y^{(4)}(\eta_0) + Y^{(4)}(\eta_1) \right] - \frac{3}{x_i} \cdot \frac{h^2}{12} \left[ Y^{(3)}(\xi_0) + Y^{(3)}(\xi_1) \right]$$

$$+ \frac{2}{[Y(x_i)]^2}$$

$$= \frac{Y(x_{i+1}) - 2Y(x_i) + Y(x_{i-1})}{h^2} + \frac{3}{x_i} \cdot \frac{Y(x_{i+1}) - Y(x_{i-1})}{2h}$$

$$- \frac{h^2}{4} \left[ 2Y^{(4)}(\xi_4) + \frac{Y^{(4)}(\xi_2)}{x_i}(\xi_0 - x_i) + \frac{Y^{(4)}(\xi_3)}{x_i}(\xi_1 - x_i) \right]$$

$$+ \frac{2}{[Y(x_i)]^2} - \frac{h^2}{24} \left[ Y^{(4)}(\eta_0) + Y^{(4)}(\eta_1) \right],$$

where $x_{i-1} < \eta_0 < x_i < \eta_1 < x_{i+1}, x_{i-1} < \xi_0 < \xi_2 < x_i < \xi_3 < \xi_1 < x_{i+1}, x_0 < \xi_4 < x_i.$ And for $x_N,$ we obtain

(4.1.8)

$$0 = Y'(x_N) + (1 - v)Y(x_N)$$

$$= \frac{Y(x_N) - Y(x_{N-1})}{h} + (1 - v)Y(x_N) + \frac{h}{2} Y''(\xi_0).$$

where $x_{N-1} < \xi_0 < x_N.$ From (3.1.1), (4.1.1), and (4.1.5) we obtain

$$L_1^hE_0 = 8 \cdot \frac{E_1 - E_0}{h^2} + Q_0E_0 = Y^{(4)}(\xi_0) \cdot \frac{h^2}{3}.$$
And, from (4.1.3) and (4.1.7), we have

\[ L^h_i E_i = \frac{E_{i+1} - 2E_i + E_{i-1}}{h^2} + \frac{3}{x_i} \cdot \frac{E_{i+1} - E_{i-1}}{2h} + Q_i E_i \]

\[ = \frac{h^2}{2} \left[ 2Y^{(4)}(\xi_4) + \frac{Y^{(4)}(\xi_2)}{x_i}(\xi_0 - x_i) + \frac{Y^{(4)}(\xi_3)}{x_i}(\xi_1 - x_i) \right] \]

\[ + \frac{h^2}{24} \left[ Y^{(4)}(\eta_0) + Y^{(4)}(\eta_1) \right], \quad i = 2, 3, \ldots, N - 1. \]

From (4.1.4) and (4.1.8)

\[ L^h_i E_N = \frac{E_N - E_{N-1}}{h} + (1 - v)E_N = -\frac{h}{2} Y''(\xi_0), \]

where \( F(y) = \frac{2}{y^2}, \quad Q_i = F'(\mu_i) = -\frac{4}{\mu_i^3} \leq -\omega < 0, \) and \( \mu_i \) lies between \( Y(x_i) \) and \( y_i. \) Let \( M_4 = \sup_{[0,1]} \left| \frac{d^4Y}{dx^4} \right|. \) Then we obtain

\[ |L^i E_0| \leq \frac{h^2}{3} M_4, \]

\[ |L^i E_1| \leq \frac{h^2}{3} M_4 + 3h^2 M_4, \]

\[ |L^i E_i| \leq \frac{h^2}{12} M_4 + 2h^2 M_4, \quad i = 2, 3, \ldots, N - 1 \]

\[ |L^i E_N| \leq \frac{h}{2} M_4. \]

Thus, by Lemma 4.1.1, we have

\[ |E_i| \leq CM_4 h, \quad \text{for} \quad i = 0, 1, 2, \ldots, N, \]

which completes the proof. \( \square \)
4.2. Scheme II

**Lemma 4.2.1.** Let \( Q(x_i) = Q_i, E(x_i) = E_i \) be discrete functions defined on \( x_0, x_1, x_2, \cdots, x_N \). Assume that there exists an \( \omega > 0 \) such that
\[
Q_i \leq -\omega < 0, \quad i = 0, 1, 2, \cdots, N.
\]
Set \( C = \max \left( \frac{4}{\omega}, \frac{1}{2(1-v)} \right) \). At the grid points \( x_0, x_1, x_2, \cdots, x_N \) define a difference operator \( L_2^h \) by
\[
(4.2.1) \quad L_2^h E_i = L_1^h E_i, \quad i = 0, 1, 2, \cdots, N-1,
\]
and
\[
(4.2.2) \quad L_2^h E_N = \frac{2E_{N-1} - 2E_N}{h^2} - \frac{2}{h} (1-v) E_N - \frac{3}{x_N} (1-v) E_N + Q_N E_N.
\]
Then
\[
|E_i| \leq C \cdot \max \left[ \max_{0 \leq j \leq N-1} |L_2^h E_j|, \frac{h}{2(1-v)} |L_2^h E_N| \right], \quad i = 0, 1, 2, \cdots, N.
\]

**Proof.** Note that \( C \geq 1 \). If \( \max |E_i| \) occurs for \( i = N \), then
\[
|E_N| \leq \frac{h}{2(1-v)} |L_2^h E_N|.
\]
Suppose that \( \max |E_i| \) occurs for one of \( i = 0, 1, 2, \cdots, N-1 \). Then since \( L_1^h E_i = L_2^h E_i, i = 0, 1, 2, \cdots, N-1 \), the remaining part of the proof is the same as that of Lemma 4.1.1. \( \square \)

Since \( Y(x) \in C^4[0,1] \) and \( Y(1) > 0 \), we may extend the positive solution of (1.1) to the interval \([0, 1 + \delta]\), for sufficiently small \( \delta > 0 \). So we have the following theorem whose proof is the same as that of Theorem 4.1.1.

**Theorem 4.2.1.** Let \( Y(x) \in C^4[0,1 + \delta] \), be an analytic solution of the boundary value problem (1.1) for sufficiently small \( \delta > 0 \). Let \( y_i \) be numerical solutions of \( L_2^h \dot{y} + N_2 \ddot{y} = 0 \) and \( E_i = Y(x_i) - y_i \) be errors, where \( i = 0, 1, 2, \cdots, N \). Then
\[
|E_i| \leq C \tilde{M}_4 h^2,
\]
where
\[
\tilde{M}_4 = \sup_{[0,1+\delta]} \left| \frac{d^4 Y}{dx^4} \right| \quad \text{and} \quad C \text{ is a constant.}
\]
Proof. From (1.1), by the mean value theorem and Taylor theorem, we obtain

$$0 = Y''(x_N) + \frac{3}{x_N} \cdot Y'(x_N) + \frac{2}{[Y(x_N)]^2}$$

$$= Y(x_{N+1}) - 2Y(x_N) + Y(x_{N-1}) - (1 - v) \frac{3}{x_N} Y(x_N)$$

$$+ \frac{2}{[Y(x_N)]^2} - \frac{h^2}{24} \left[Y^{(4)}(\eta_0) + Y^{(4)}(\eta_1)\right],$$

where $x_{N-1} < \eta_0 < x_N < \eta_1 < x_{N+1}$. And we obtain

$$\frac{Y(x_{N+1}) - Y(x_{N-1})}{2h} + (1 - v)Y(x_N) - \frac{h^2}{6} [Y^{(3)}(\xi_0) + Y^{(3)}(\xi_1)] = 0,$$

where $x_{N-1} < \xi_0 < x_N < \xi_1 < x_{N+1}$. By substituting (4.2.4) into (4.2.3), we get

$$0 = \frac{2Y(x_{N-1}) - 2Y(x_N)}{h^2} - \frac{2}{h} (1 - v)Y(x_N) + \frac{h}{3} [2Y^{(4)}(\xi_4)$$

$$+ Y^{(4)}(\xi_2)(\xi_0 - x_N) + Y^{(4)}(\xi_3)(\xi_1 - x_N)] + \frac{2}{[Y(x_N)]^2}$$

$$- \frac{h^2}{24} [Y^{(4)}(\eta_0) + Y^{(4)}(\eta_1)] - (1 - v) \frac{3}{x_N} Y(x_N),$$

where $x_N < \xi_3 < x_{N+1}$, $x_0 < \xi_4 < x_N$. Therefore we have

$$L^2 E_N = \frac{2E_{N-1} - 2E_N}{h^2} - \frac{2}{h} (1 - v)E_N - \frac{3}{x_N} (1 - v)E_N + Q_N E_N$$

$$= - \frac{h}{3} \left[2Y^{(4)}(\xi_4) + Y^{(4)}(\xi_2)(\xi_0 - x_N) + Y^{(4)}(\xi_3)(\xi_1 - x_N)\right]$$

$$+ \frac{h^2}{24} [Y^{(4)}(\eta_0) + Y^{(4)}(\eta_1)],$$

where $F(y) = \frac{2}{y^2}$, $Q_i = F'(\mu_i) = -\frac{4}{\mu_i^3} \leq -\omega < 0$, $\mu_i$ lies between $Y(x_i)$ and $y_i$. Since $L^2 E_i = L^2 E_i$, $i = 0, 1, 2, \cdots, N - 1$, we obtain from the proof of Theorem 4.1.1

$$|L^2 E_i| \leq C \tilde{M} h^2, i = 0, 1, 2, \cdots, N - 1$$
and from (4.2.5) we get

\[ |L_2^h E_N| \leq \left[ \frac{2h^2}{12} + \frac{4}{3} h \right] \tilde{M}_4, \]

where \( \tilde{M}_4 = \sup_{[0,1+\delta]} \left| \frac{d^4 Y}{dx^4} \right| \) for sufficiently small \( \delta > 0 \). Thus, by Lemma 4.2.1, we have

\[ |E_i| \leq C\tilde{M}_4 h^2, \quad \text{for} \quad i = 0, 1, 2, \ldots, N, \]

which completes the proof. \( \square \)

5. Numerical experiments

5.1. Scheme I

The scheme I, proposed in section 3.1, has been implemented on an IBM PC. In the computation, we use

\[ \max_{j=0,1,\ldots,N-1} \left| y^{(k+1)}(x_j) - y^{(k)}(x_j) \right| \leq \text{TOL} = 1.0 \times 10^{-12} \]

to stop the iteration when we solve the nonlinear system (3.1.2) by Newton's method (3.1.3) or (3.1.4). In table 1, we report the values of \( \delta_{\text{max}}(N) \) and \( \delta_{\text{min}}(N) \) for \( N = 10, 20, 40, 80 \) and \( v = 0.1 \), where

\[ \delta_{\text{max}}(N) = \max_{j=0,1,\ldots,N} \left| \frac{y_{2N}(x_j) - y_N(x_j)}{y_{4N}(x_j) - y_{2N}(x_j)} \right|, \]

\[ \delta_{\text{min}}(N) = \min_{j=0,1,\ldots,N} \left| \frac{y_{2N}(x_j) - y_N(x_j)}{y_{4N}(x_j) - y_{2N}(x_j)} \right| \]

and \( y_N \) represents the solution of the nonlinear system (3.1.2) for the given \( N \). And in table 2, the value of \( \delta_{\text{max}}(N) \) and \( \delta_{\text{min}}(N) \) are given for \( N = 10, 20, 40, 80 \) and \( v = 0.9 \). From table 1 and table 2, we see numerically that Theorem 4.1.1 is valid.
Table 1. $\delta_{\text{max}}(N)$, $\delta_{\text{min}}(N)$ for $N = 10$, 20, 40, 80 and $v = 0.1$

<table>
<thead>
<tr>
<th>N</th>
<th>$\delta_{\text{max}}(N)$</th>
<th>$\delta_{\text{min}}(N)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>2.109554</td>
<td>2.059469</td>
</tr>
<tr>
<td>20</td>
<td>2.055978</td>
<td>2.033158</td>
</tr>
<tr>
<td>40</td>
<td>2.028407</td>
<td>2.017376</td>
</tr>
<tr>
<td>80</td>
<td>2.014309</td>
<td>2.008876</td>
</tr>
</tbody>
</table>

Table 2. $\delta_{\text{max}}(N)$, $\delta_{\text{min}}(N)$ for $N = 10$, 20, 40, 80 and $v = 0.9$

<table>
<thead>
<tr>
<th>N</th>
<th>$\delta_{\text{max}}(N)$</th>
<th>$\delta_{\text{min}}(N)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>2.036283</td>
<td>2.034866</td>
</tr>
<tr>
<td>20</td>
<td>2.017864</td>
<td>2.017137</td>
</tr>
<tr>
<td>40</td>
<td>2.008664</td>
<td>2.008500</td>
</tr>
<tr>
<td>80</td>
<td>2.004415</td>
<td>2.004234</td>
</tr>
</tbody>
</table>

5.2. Scheme II

The scheme II, proposed in section 3.2, has also been implemented on an IBM PC. In the computation, we use

$$\max_{j=0,1,\ldots,N-1} \left| y^{(k+1)}(x_j) - y^{(k)}(x_j) \right| \leq \text{TOL} = 1.0 \times 10^{-12}$$

to stop the iteration when we solve the nonlinear system (3.2.2) by Newton's method (3.2.3) or (3.2.4). In table 3 we report the values of $\delta_{\text{max}}(N)$ and $\delta_{\text{min}}(N)$ for $N = 10$, 20, 40, 80 and $v = 0.1$, where

$$\delta_{\text{max}}(N) = \max_{j=0,1,\ldots,N} \frac{|y_{2N}(x_j) - y_N(x_j)|}{|y_{4N}(x_j) - y_{2N}(x_j)|},$$

$$\delta_{\text{min}}(N) = \min_{j=0,1,\ldots,N} \frac{|y_{2N}(x_j) - y_N(x_j)|}{|y_{4N}(x_j) - y_{2N}(x_j)|},$$

and $y_N$ represents the solution of the nonlinear system (3.2.2) for the given $N$. And in table 4 the value of $\delta_{\text{max}}(N)$ and $\delta_{\text{min}}(N)$ are given for $N = 10$, 20, 40, 80 and $v = 0.9$. From table 3 and table 4, we see numerically that Theorem 4.2.1 is valid.
Convergence of finite difference approximations

Table 3. $\delta_{\text{max}}(N)$, $\delta_{\text{min}}(N)$ for $N = 10$, 20, 40, 80 and $v = 0.1$

<table>
<thead>
<tr>
<th>N</th>
<th>$\delta_{\text{max}}(N)$</th>
<th>$\delta_{\text{min}}(N)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>4.785150</td>
<td>3.386142</td>
</tr>
<tr>
<td>20</td>
<td>4.214163</td>
<td>3.845060</td>
</tr>
<tr>
<td>40</td>
<td>4.054849</td>
<td>3.961117</td>
</tr>
<tr>
<td>80</td>
<td>4.013797</td>
<td>3.990280</td>
</tr>
</tbody>
</table>

Table 4. $\delta_{\text{max}}(N)$, $\delta_{\text{min}}(N)$ for $N = 10$, 20, 40, 80 and $v = 0.9$

<table>
<thead>
<tr>
<th>N</th>
<th>$\delta_{\text{max}}(N)$</th>
<th>$\delta_{\text{min}}(N)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>3.988212</td>
<td>3.767755</td>
</tr>
<tr>
<td>20</td>
<td>3.997707</td>
<td>3.962608</td>
</tr>
<tr>
<td>40</td>
<td>3.999464</td>
<td>3.990738</td>
</tr>
<tr>
<td>80</td>
<td>3.999965</td>
<td>3.997585</td>
</tr>
</tbody>
</table>

References


H. Y. Lee, J. M. Seong  
Department of Mathematics  
Kyungsung University  
Pusan 608-736, Korea

J. Y. Shin  
Division of Mathematical Sciences  
Pukyong National University  
Pusan 608-737, Korea