CONSTRUCTIONS FOR SPARSE ROW-ORTHOGONAL MATRICES WITH A FULL ROW

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ABSTRACT. In [4], it was shown that an $n$ by $n$ orthogonal matrix which has a row of nonzeros has at least

$$(\log_2 n + 3)n - 2^{\lfloor \log_2 n \rfloor + 1}$$

nonzero entries. In this paper, the matrices achieving these bounds are constructed. The analogous sparsity problem for $m$ by $n$ row-orthogonal matrices which have a row of nonzeros is conjectured.

1. Introduction

At the 1990 SIAM Linear Algebra meeting, M. Fiedler asked:

How sparse can an $n$ by $n$ orthogonal matrix (whose rows and columns cannot be permuted to give a matrix which is a direct sum of matrices) be?

The assumption precluding direct sums is necessary, since otherwise the answer is trivially $n$. Fiedler’s question is answered in [1] (see also [5]), where it is shown that each $n$ by $n$ orthogonal matrix which is not direct summable has at least $4n - 4$ nonzero entries, and that for $n \geq 2$, there exist such orthogonal matrices with exactly $4n - 4$ nonzero entries. Recently, the $n$ by $n$ orthogonal matrices with exactly $4n - 4$ nonzero entries were constructed in [2]. The analogous sparsity problem for $m$ by $n$ row-orthogonal matrices under two natural notions of irreducibility which extends the work in [1, 5] was studied in [3].

And also, it was studied in [4], the question of how sparse an $n$ by $n$ orthogonal matrix which has a column of nonzeros can be. In

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particular, it was shown that such an $n$ by $n$ orthogonal matrix has at
least 

$$((\lceil \log_2 n \rceil + 3)n - 2^\lfloor \log_2 n \rfloor + 1)$$

(1)

nonzero entries, and matrices achieving these bounds are constructed and characterized, and are related to orthogonal matrices arising from the Haar wavelet.

Note that if $A$ is an $n$ by $n$ orthogonal matrix with a row of nonzeros then $A$ has also at least the number of nonzero entries in (1).

In this paper, we get another constructions for the $n$ by $n$ orthogonal matrices which have a full row and have exactly nonzero entries in (1), where a vector is full if each of its entries is nonzero. Furthermore, the analogous sparsity problem for $m$ by $n$ row-orthogonal matrices with a full row is conjectured.

For a matrix $A$, we denote the number of nonzero entries in $A$ by $\#(A)$.

2. Constructions for the sparsest orthogonal matrices with a full row

An $m$ by $n$ matrix is row-orthogonal provided each of its rows is nonzero, and its rows are pairwise orthogonal.

We begin by describing a way to build row-orthogonal matrices from smaller row-orthogonal matrices. Let

$$X = \begin{bmatrix} \hat{X} \\ x^T \end{bmatrix}$$

be an $s$ by $t$ row-orthogonal matrix and let

$$Y = \begin{bmatrix} y^T \\ \hat{Y} \end{bmatrix}$$

be an $k$ by $l$ row-orthogonal matrix, where $\hat{X}$ is $(s - 1)$ by $t$ matrix and $\hat{Y}$ is $(k - 1)$ by $l$ matrix. Define $X \diamond Y$ to be the $(s + k - 1)$ by $(t + l)$ matrix

$$X \diamond Y = \begin{bmatrix} \hat{X} & O \\ x^T & y^T \\ O & \hat{Y} \end{bmatrix}.$$
Certainly, $X \diamond Y$ is a row-orthogonal matrix. We can extend this construction to use any number of row-orthogonal matrices by defining $X \diamond Y \diamond Z$ as $(X \diamond Y) \diamond Z$. This construction can be used in a recursive manner to construct $m$ by $n$ row-orthogonal matrices.

Now, we describe a way of constructing an $n$ by $n$ orthogonal matrices having a full row and exactly $(([\log_2 n] + 3)n - 2^{[\log_2 n]} + 1$ nonzero entries. This is a different manner from the one used in [4].

**Lemma 2.1.** Let

$$X = \begin{bmatrix} \hat{X} \\ x^T \end{bmatrix}, \quad \text{and} \quad Y = \begin{bmatrix} y^T \\ \hat{Y} \end{bmatrix}$$

be an $r$ by $r$ orthogonal matrix and a $s$ by $s$ orthogonal matrix respectively where $\hat{X}$ is $(r - 1)$ by $r$ matrix and $\hat{Y}$ is $(s - 1)$ by $s$ matrix. Then

\begin{equation}
A = \begin{bmatrix} X & Y \\ x^T & -y^T \end{bmatrix}
\end{equation}

is an $n$ by $n$ row-orthogonal matrix where $r + s = n$. Thus the matrix, $\hat{A}$, obtained from $A$ by normalizing the row $r$ and the row $n$ of $A$ is an $n$ by $n$ orthogonal matrix with the same zero pattern as $A$.

**Proof.** Since $X \diamond Y$ is an $(n - 1)$ by $n$ row-orthogonal matrix, it is sufficient to show that the row $r$ and the row $n$ of $A$ are orthogonal each other. Indeed,

$$[x^T \ y^T][x^T \ -y^T]^T = ||x^T||^2 - ||y^T||^2 = 1 - 1 = 0.$$ 

Thus the proof is completed.

Note that if both $x^T$ and $y^T$ in Lemma 2.1 are full rows then $\hat{A}$ is an $n$ by $n$ orthogonal matrix with a full row.

Throughout in this paper, we define $\rho(n)$ by

$$\rho(n) = ([\log_2 n] + 3)n - 2^{[\log_2 n]} + 1.$$
Theorem 2.2. Let
\[ X = \begin{bmatrix} \hat{X} \\ x^T \end{bmatrix} \]
be an \( r \) by \( r \) orthogonal matrix with the full row \( x^T \) which has \( \rho(r) \) nonzero entries, and let
\[ Y = \begin{bmatrix} y^T \\ \hat{Y} \end{bmatrix} \]
be a \( s \) by \( s \) orthogonal matrix with the full row \( y^T \) which has \( \rho(s) \) nonzero entries, where \( r + s = n \). If
\[ 2^{\lfloor \log_2 n \rfloor - 1} \leq r, s \leq 2^{\lfloor \log_2 n \rfloor} \]
then
\[ A = \begin{bmatrix} X & Y \\ x^T & -y^T \end{bmatrix} \]
is an \( n \) by \( n \) row-orthogonal matrix with a full row which has \( \rho(n) \) nonzero entries. Thus the matrix, \( \hat{A} \), obtained from \( A \) by normalizing the row \( r \) and the row \( n \) of \( A \) is an \( n \) by \( n \) orthogonal matrix with the same zero pattern as \( A \).

Proof. There exist \( r \) and \( s \) satisfying (3) and \( r + s = n \), since we may take \( r = \lfloor \frac{n}{2} \rfloor \) and \( s = \lfloor \frac{n+1}{2} \rfloor \). From Lemma 2.1, \( A \) is an \( n \) by \( n \) row-orthogonal matrix with a full row. It is easy to show that
\[ \begin{cases} \lfloor \log_2 r \rfloor = \lfloor \log_2 s \rfloor - 1 = \lfloor \log_2 n \rfloor - 1 & \text{if } n = 2^k - 1, \\ \lfloor \log_2 r \rfloor = \lfloor \log_2 s \rfloor = \lfloor \log_2 n \rfloor - 1 & \text{otherwise.} \end{cases} \]
Thus if \( n \neq 2^k - 1 \) then
\[ \#(A) = \#(X) + \#(Y) + \#(x^T - y^T) \]
\[ = (\lfloor \log_2 r \rfloor + 3)r - 2^{\lfloor \log_2 r \rfloor + 1} + (\lfloor \log_2 s \rfloor + 3)s - 2^{\lfloor \log_2 s \rfloor + 1} + n \]
\[ = (\lfloor \log_2 n \rfloor + 2)(r + s) - 2^{\lfloor \log_2 n \rfloor + 1} + n \]
\[ = (\lfloor \log_2 n \rfloor + 3)n - 2^{\lfloor \log_2 n \rfloor + 1}. \]
Let \( n = 2^k - 1 \). Then we take \( r = \lfloor \frac{n}{2} \rfloor \) and \( s = \lfloor \frac{n+1}{2} \rfloor \). Since \( \lfloor \log_2 n \rfloor = \lfloor \log_2 (2^k - 1) \rfloor = k - 1 \), we have
\[
s = \left\lfloor \frac{n+1}{2} \right\rfloor = 2^{k-1} = 2^{\lfloor \log_2 n \rfloor}.
\]

Thus
\[
\#(A) = \#(X) + \#(Y) + \#([x^T - y^T])
  = ([\log_2 r] + 3) r - 2^{\lfloor \log_2 r \rfloor + 1} + ([\log_2 s] + 3) s - 2^{\lfloor \log_2 s \rfloor + 1} + n
  = ([\log_2 n] + 2)(r + s) - 3 \cdot 2^{\lfloor \log_2 n \rfloor} + s + n
  = ([\log_2 n] + 3)n - 2^{\lfloor \log_2 n \rfloor + 1},
\]
which completes the proof. \( \square \)

Since \( \rho(n) = 4n - 4 \) for \( n = 2, 3, 4 \), from the result in [1], for each \( n = 2, 3, 4 \) we know zero patterns, \( B_n \), of \( n \) by \( n \) orthogonal matrices with a full row which have \( \rho(n) \) nonzero entries. That is,

\[
B_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.
\]

For \( n = 5 \), since
\[
B_3 \diamond B_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix},
\]
by lemma 2.1
\[
(4) \quad \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}
\]
is a zero pattern of 5 by 5 sparse orthogonal matrix with a full row which has $\rho(5) = 17$ nonzero entries.

Furthermore, from the result in [2], since, for each $n = 2, 3, 4$, we can get $n$ by $n$ orthogonal matrices with the same zero patterns as $B_2$, $B_3$, and $B_4$ respectively, we get a 5 by 5 orthogonal matrix with the same zero pattern as (4).

For example, let $n = 9$. From (3), since $4 \leq r, s \leq 8$, we take $r = 4$ and $s = 5$. Let $X$ be a 4 by 4 orthogonal matrix with the full row which has $\rho(4) = 12$, and let $Y$ be a 5 by 5 orthogonal matrix with the full row which has $\rho(5) = 17$. Take

\[
X = \begin{bmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2}
\end{bmatrix},
\]

\[
Y = \begin{bmatrix}
\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{bmatrix}.
\]

Then

\[
A = \begin{bmatrix}
X & \diamond Y \\
x^T & -y^T
\end{bmatrix},
\]
and

\[
\hat{A} = \begin{bmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\]

is a 9 by 9 sparse orthogonal matrix with the full row which has \(\rho(9) = 38\) nonzero entries.

By these recursive manners, we can construct sparse \(n\) by \(n\) orthogonal matrices with a full row which have \(\rho(n)\) nonzero entries.

3. Conjecture for sparse row-orthogonal matrices with a full row

We consider the case that \(A\) is an \(m\) by \(n\) row-orthogonal matrix with a full row.

Let

\[
X = \begin{bmatrix} \hat{X} \\ x^T \end{bmatrix}, \text{ and } Y = \begin{bmatrix} \hat{Y} \\ y^T \end{bmatrix}
\]

be an \(r\) by \(r\) matrix and a \(s\) by \(s\) matrix, respectively. Then both

\[
X \odot Y = \begin{bmatrix} \hat{X} & O \\ x^T & y^T \end{bmatrix}
\]
and
\[ A = \begin{bmatrix}
\hat{X} & O \\
O & \hat{Y} \\
x^T & y^T
\end{bmatrix} \]
are \((r + s - 1)\) by \((r + s)\) matrices and have the same nonzero entries. It is clear that \(A\) is an row-orthogonal matrix with the full row if and only if both \(X\) and \(Y\) are square orthogonal matrix with the full row \(x^T\) and with the full row \(y^T\), respectively.

We define an \(m\) by \(n\) matrix \(A\) with \(m \leq n\) to be indecomposable provided \(A\) does not contain a zero submatrix whose dimensions sum to \(n\). It is not difficult to verify that if both \(\hat{X}\) and \(\hat{Y}\) are non-square indecomposable row-orthogonal matrices, then so is their direct sum \(\hat{X} \oplus \hat{Y}\).

For each \(i = 1, 2, \ldots, n - m + 1\), let
\[ X_{p_i} = \begin{bmatrix}
\hat{X}_{p_i} \\
x_{p_i}^T
\end{bmatrix} \]
be a \(p_i\) by \(p_i\) orthogonal matrix with the full row \(x_{p_i}^T\) which has \(\rho(p_i)\) nonzero entries where
\[ \rho(p_i) = ([\log_2 p_i] + 3)p_i - 2^{\lfloor \log_2 p_i \rfloor + 1}. \]

Define
\[
A = \begin{bmatrix}
\hat{X}_{p_1} & O & O & O \\
O & \hat{X}_{p_2} & O & O \\
O & O & \ddots & O \\
x_{p_1}^T & x_{p_2}^T & \ldots & x_{p_{n-m+1}}^T
\end{bmatrix}
\]
(5)
\[
2^{\left\lfloor \log_2 \left(\frac{n}{n-m+1}\right) \right\rfloor} \leq p_i \leq 2^{\left\lfloor \log_2 \left(\frac{n}{n-m+1}\right) \right\rfloor + 1}
\]
(6)
and

\[ p_1 + p_2 + \cdots + p_{n-m+1} = n. \]

Certainly, \( A \) is an \( m \) by \( n \) indecomposable, row-orthogonal matrix with the full row.

There exists \( p_i \)'s satisfying (6) and (7), since we may assume \( p_1 \leq p_2 \leq \cdots \leq p_{n-m+1} \) and we may take

\[ p_1 = \left\lfloor \frac{n}{n-m+1} \right\rfloor, \quad p_2 = \left\lfloor \frac{n+1}{n-m+1} \right\rfloor, \quad \cdots, \]

\[ p_{n-m+1} = \left\lfloor \frac{n+(n-m)}{n-m+1} \right\rfloor. \]

For example, let \( A \) be a 17 by 19 row-orthogonal matrix with the form in (5). From (6) since \( 4 \leq p_i \leq 8 \), \((p_1,p_2,p_3)\)'s satisfying \( p_1 + p_2 + p_3 = 19 \) are \((4,7,8)\), \((5,7,7)\), \((6,6,7)\), and \( A \) has the following forms respectively:

\[
\begin{bmatrix}
\hat{X}_4 & O & O \\
O & \hat{X}_7 & O \\
O & O & \hat{X}_8 \\
x_4^T & x_7^T & x_8^T
\end{bmatrix}
\quad \text{or} \quad
\begin{bmatrix}
\hat{X}_5 & O & O \\
O & \hat{X}_7 & O \\
O & O & \hat{X}_7 \\
x_5^T & x_7^T & x_7^T
\end{bmatrix}
\quad \text{or} \quad
\begin{bmatrix}
\hat{X}_6 & O & O \\
O & \hat{X}_6 & O \\
O & O & \hat{X}_7 \\
x_6^T & x_6^T & x_7^T
\end{bmatrix}
\]

where for each \( i = 1, 2, 3 \),

\[
\begin{bmatrix}
\hat{X}_{p_i} \\
x_{p_i}^T
\end{bmatrix}
\]

is a \( p_i \) by \( p_i \) orthogonal matrix with the full row \( x_{p_i}^T \) which has \( \rho(p_i) \) nonzero entries. These matrices are determined from Theorem 2.2. It is easy to compute that \#(\( A \)) = 71 for the matrices in (8). But note that if

\[
A =
\begin{bmatrix}
\hat{X}_3 & O & O \\
O & \hat{X}_8 & O \\
x_3^T & x_8^T & x_8^T
\end{bmatrix}
\]

then \#(\( A \)) = 72. This means that the condition (6) for \( p_i \)'s is necessary to get sparse row-orthogonal matrices with a full row.
Now, we determine the number of nonzero entries of $A$ in (5). We claim

$$\#(A) = (k + 3)n - (n - m + 1)2^{k+1}$$

where

$$k = \left\lfloor \log_2 \left( \frac{n}{n - m + 1} \right) \right\rfloor.$$ 

Since $2^k \leq p_i \leq 2^{k+1}$ for each $i = 1, 2, \ldots, n - m + 1$,

$$[\log_2 p_i] = \begin{cases} 
  k & \text{if } 2^k \leq p_i < 2^{k+1} \\
  k + 1 & \text{if } p_i = 2^{k+1}. 
\end{cases}$$

Thus if $2^k \leq p_i < 2^{k+1}$ for each $i = 1, 2, \ldots, n - m + 1$, then

$$\#(A) = \#(X_{p_1}) + \#(X_{p_2}) + \cdots + \#(X_{p_{n-m+1}})$$

$$= (k + 3)(p_1 + p_2 + \cdots + p_{n-m+1}) - (n - m + 1)2^{k+1}$$

$$= (k + 3)n - (n - m + 1)2^{k+1}.$$ 

Let $p_i = 2^{k+1}$ for $i = j, j + 1, \ldots, n - m + 1$. Since $p_j + p_{j+1} + \cdots + p_{n-m+1} = (n - m - j + 2)2^{k+1}$,

$$\#(A) = \#(X_{p_1}) + \#(X_{p_2}) + \cdots + \#(X_{p_{n-m+1}})$$

$$= (k + 3)(p_1 + p_2 + \cdots + p_{j-1}) - (j - 1)2^{k+1}$$

$$+ (k + 4)(p_j + p_{j+1} + \cdots + p_{n-m+1}) - (n - m - j + 2)2^{k+2}$$

$$= (k + 3)n - (n - m + 1)2^{k+1}.$$ 

In the above example, i.e., if $A$ is a 17 by 19 row-orthogonal matrix with the full row in (8) then $k = 2$, and thus $\#(A) = 5 \cdot 19 - 3 \cdot 2^3 = 71$.

Thus, for positive integers $m$ and $n$ with $m \leq n$, if $f(m, n)$ denote the least number of nonzero entries in an $m$ by $n$ indecomposable, row-orthogonal matrix with a full row then we conclude that

$$f(m, n) \leq (k + 3)n - (n - m + 1)2^{k+1}$$

where

$$k = \left\lfloor \log_2 \left( \frac{n}{n - m + 1} \right) \right\rfloor.$$ 

And we have the following conjecture.
Conjecture. For positive integers \( m \) and \( n \) with \( m \leq n \), let \( f(m, n) \) denote the least number of nonzero entries in an \( m \) by \( n \) indecomposable, row-orthogonal matrix with a full row, then the equality holds in (9). Furthermore, the equality holds in (9) if and only if, up to row and column permutations, the matrix is \( A \) in (5).

Note that if \( A \) is an \( n \) by \( n \) indecomposable, orthogonal matrix with a full row, from [4], since

\[
\#(A) \geq (\lfloor \log_2 n \rfloor + 3)n - 2^{\lfloor \log_2 n \rfloor + 1},
\]

this conjecture holds for \( m = n \). Thus this conjecture is a generalization of the result in [4].

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