DISCRETE SOBOLEV ORTHOGONAL POLYNOMIALS
AND SECOND ORDER DIFFERENCE EQUATIONS

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ABSTRACT. Let \( \{R_n(x)\}_{n=0}^\infty \) be a discrete Sobolev orthogonal polynomials (DSOPS) relative to a symmetric bilinear form

\[
\phi(p, q) = \int_R p d\mu_0 + \int_R \Delta p \Delta q d\mu_1,
\]

where \( d\mu_0 \) and \( d\mu_1 \) are signed Borel measures on \( \mathbb{R} \). We find necessary and sufficient conditions for \( \{R_n(x)\}_{n=0}^\infty \) to satisfy a second order difference equation

\[
\ell_2(x) \Delta \nabla y(x) + \ell_1(x) \Delta y(x) = \lambda_n y(x)
\]

and classify all such \( \{R_n(x)\}_{n=0}^\infty \). Here, \( \Delta \) and \( \nabla \) are forward and backward difference operators defined by \( \Delta f(x) = f(x + 1) - f(x) \) and \( \nabla f(x) = f(x) - f(x - 1) \).

1. Introduction

Let \( \mathcal{P} \) be the space of real polynomials in a single variable and \( \deg(\pi) \) the degree of any \( \pi(x) \in \mathcal{P} \) with the convention that \( \deg(0) = -1 \). By a polynomial system (PS), we mean a sequence of polynomials \( \{\phi_n(x)\}_{n=0}^\infty \) with \( \deg(\phi_n) = n \), \( n \geq 0 \).

Any bilinear form \( \phi(\cdot, \cdot) \) defined on \( \mathcal{P} \times \mathcal{P} \) is called quasi-definite (respectively, positive-definite) if the double sequence (called the moments of \( \phi(\cdot, \cdot) \))

\[
\phi_{mn} := \phi(x^m, x^n) \quad (m \text{ and } n \geq 0)
\]

satisfy the Hamburger condition

\[
\Delta_n(\phi) := \det(\phi_{i,j})_{i,j=1}^n \neq 0 \quad \text{(respectively, } \Delta_n(\phi) > 0 \text{)}
\]

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for each \( n \geq 0 \). It is well known ([7]) that if the bilinear form \( \phi(\cdot, \cdot) \) is quasi-definite (respectively, positive-definite), there is a PS \( \{R_n(x)\}_{n=0}^\infty \) such that

\[
(1.1) \quad \phi(R_m, R_n) = K_n \delta_{mn}, \quad m \text{ and } n \geq 0,
\]

where \( K_n \) is a non-zero (respectively, a positive) constant and vice versa. In this case, each \( R_n(x) \) is uniquely determined up to a non-zero constant multiple and we call \( \{R_n(x)\}_{n=0}^\infty \) a (generalized) orthogonal polynomial system (OPS) relative to \( \phi(\cdot, \cdot) \).

We now let \( \Delta \) and \( \nabla \) be the forward and backward difference operators defined by

\[
\Delta f(x) = f(x + 1) - f(x) \quad \text{and} \quad \nabla f(x) = f(x) - f(x - 1)
\]

and consider a symmetric bilinear form on \( \mathcal{P} \times \mathcal{P} \) given by

\[
(1.2) \quad \phi(p, q) := \int_\mathbb{R} p(x)q(x) \, d\mu_0(x) + \int_\mathbb{R} \Delta p(x)\Delta q(x) \, d\mu_1(x),
\]

where \( d\mu_i(x) \) \((i = 1, 2)\) is a signed Borel measure on the real line \( \mathbb{R} \). When \( \phi(\cdot, \cdot) \) in (1.2) is quasi-definite, we call any PS \( \{R_n(x)\}_{n=0}^\infty \) satisfying (1.1) a discrete Sobolev orthogonal polynomial system (DSOPS).

When \( d\mu_1(x) \equiv 0 \), \( \{R_n(x)\}_{n=0}^\infty \) is just an ordinary OPS relative to \( d\mu_0(x) \).

In this work, we first find necessary and sufficient conditions for a DSOPS relative to \( \phi(\cdot, \cdot) \) in (1.2) to satisfy the second order difference equation

\[
(1.3) \quad L[y](x) = \ell_2(x)\Delta y + \ell_1(x)\Delta y = \lambda_n y,
\]

where \( \ell_2(x) = \ell_{22}x^2 + \ell_{21}x + \ell_{20} \) and \( \ell_1(x) = \ell_{11}x + \ell_{10} \) are polynomials and \( \lambda_n \) is the eigenvalue parameter given by

\[
\lambda_n = \ell_{22} n(n - 1) + \ell_{11} n, \quad n \geq 0.
\]

We then classify all such DSOPS's which generalize the discrete classical orthogonal polynomial systems, that is, OPS's relative to \( d\mu_0(x) \), which are eigenfunctions of the difference equation (1.3). The general theory of discrete classical OPS's is rather well developed, for which we refer to [3, 5, 6, 8]. Similar problem, where the difference operator is replaced by the differential operator, is handled in [7].
2. Polynomials satisfying difference equations

Due to Boas' ([1]) or Duran's ([2]) theorem on the moment problem, any linear functional \( \sigma \) on \( \mathcal{P} \), which we call a moment functional, can be represented as an the integral of the form

\[
\langle \sigma, \pi \rangle = \int_{\mathbb{R}} \pi(x) \, d\mu(x), \quad (\pi \in \mathcal{P})
\]

or

\[
\langle \sigma, \pi \rangle = \int_{\mathbb{R}} \pi(x) w(x) \, dx, \quad (\pi \in \mathcal{P}),
\]

where \( \mu(x) \) is a function of bounded variation on \( \mathbb{R} \) and \( w(x) \) is a \( C^\infty \)-function in the Schwartz space of rapidly decaying functions. Hence, in studying DSOPS's the symmetric bilinear form in (1.2) can be replaced by

\[
\phi(p, q) = \langle \sigma, pq \rangle + \langle \tau, \Delta p \Delta q \rangle,
\]

where \( \sigma \) and \( \tau \) are moment functionals. As we shall see later, it is much more convenient to use moment functionals instead of their integral representations as in (1.2). We call a DSOPS \( \{P_n(x)\}_{n=0}^\infty \) relative to \( \phi(\cdot, \cdot) \) in (2.1) with \( \tau = 0 \) to be an OPS relative to \( \sigma \).

For a PS \( \{P_n(x)\}_{n=0}^\infty \), we call any moment functional \( \sigma \) satisfying

\[
\langle \sigma, P_0 \rangle \neq 0 \quad \text{and} \quad \langle \sigma, P_n \rangle = 0, \quad n \geq 1
\]

a canonical moment functional for \( \{P_n(x)\}_{n=0}^\infty \), which is unique up to a non-zero constant multiple. A PS \( \{P_n(x)\}_{n=0}^\infty \) is called a weak orthogonal polynomial system (WOPS) if there is a non-zero moment functional \( \sigma \) such that \( \langle \sigma, P_m P_n \rangle = 0 \) for \( m \neq n \). Note that if \( \{R_n(x)\}_{n=0}^\infty \) is a WOPS relative to \( \sigma \) or a DSOPS relative to \( \phi(\cdot, \cdot) \) in (2.1), then \( \sigma \) must be a canonical moment functional for \( \{R_n(x)\}_{n=0}^\infty \).

For a moment functional \( \sigma \), a polynomial \( \pi(x) \), and a real constant \( a, \Delta \sigma, \nabla \sigma, \pi \sigma \) and \( \tau_a \sigma \) are moment functionals defined by

\[
\langle \Delta \sigma, \psi \rangle = -\langle \sigma, \nabla \psi \rangle, \quad \langle \nabla \sigma, \psi \rangle = -\langle \sigma, \Delta \psi \rangle
\]

\[
\langle \pi \sigma, \psi \rangle = \langle \sigma, \pi \psi \rangle
\]

\[
\langle \tau_a \sigma, \psi \rangle = \langle \sigma, \tau_{-a} \psi \rangle = \langle \sigma, \psi(x + a) \rangle, \quad (\psi \in \mathcal{P}).
\]

For convenience we denote \( \tau_a \sigma \) by \( \sigma(x-a) \). Then, the followings are easy consequences of definitions.
Lemma 2.1. Let $\pi(x)$ be a polynomial and $\sigma$ a moment functional. Then

(i) for any $a \in \mathbb{R}$, $\sigma(x - a) = 0$ if and only if $\sigma = 0$;
(ii) $\Delta \sigma = 0$ (or $\nabla \sigma = 0$) if and only if $\sigma = 0$;
(iii) $\Delta (\pi \sigma) = (\Delta \pi) \sigma + \pi(x + 1) \Delta \sigma = \pi \Delta \sigma + \sigma(x + 1) \Delta \pi$;
(iv) when $\sigma$ is quasi-definite, $\pi \sigma = 0$ if and only if $\pi(x) \equiv 0$.

By a direct calculation, it is easy to see that the difference equation (1.3) has a unique monic polynomial solution of degree $n$ for each $n \geq 0$ except possibly for a finite number of values of $n$ and for those exceptional values of $n$, there is either no polynomial solution of degree $n$ or infinitely many monic polynomial solutions of degree $n$.

Definition 2.1. ([4]) The difference operator $L[\cdot]$ in (1.3) (or the equation (1.3) itself) is called admissible if $\lambda_m \neq \lambda_n$ for $m \neq n$.

Lemma 2.2. For the difference equation (1.3), the followings are equivalent:

(i) $L[\cdot]$ in (1.3) is admissible;
(ii) $s_n := \ell_{22} n + \ell_{11} \neq 0$ for $n \geq 0$;
(iii) For each $n \geq 0$, the difference equation (1.3) has a unique monic polynomial solution of degree $n$.

Proof. See Lemma 2.4 in [6].

To discuss the orthogonality of polynomials satisfying the difference equation (1.3), we need the following which involves discrete moment equations.

Lemma 2.3. If the difference equation (1.3) has a PS $\{P_n(x)\}_{n=0}^{\infty}$ of solutions, then any canonical moment functional $\sigma$ of $\{P_n(x)\}_{n=0}^{\infty}$ satisfies

\[
\Delta (\ell_2 \sigma) - \ell_1 \sigma = 0
\]

or equivalently

\[
s_n \sigma_{n+1} + (n(2n - 1) \ell_{22} + n \ell_{21} + n \ell_{11} + \ell_{10}) \sigma_n + n \ell_2(n - 1) \sigma_{n-1},
\]

\[= 0 \quad n \geq 1,
\]

where

\[
\sigma_n := \langle \sigma, x^n \rangle, \quad n \geq 0, \quad \sigma_n = 0 \quad \text{for} \quad n < 0.
\]
and $x^{[n]}$ are factorial polynomials:

$$x^{[0]} = 1, \quad x^{[n]} := x(x - 1) \cdots (x - n + 1).$$

Moreover if the difference equation (1.3) is admissible, then the equation (2.2) or equivalently (2.3) has a unique linearly independent solution.

Proof. See Lemma 2.5 and Lemma 2.6 in [6] (see also [3]).

We call (2.2) the discrete moment equation for the difference equation (1.3). By Favard's theorem, any monic PS $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS if and only if $\{P_n(x)\}_{n=0}^{\infty}$ satisfy a three term recurrence relation

$$P_{n+1}(x) = (x - b_n)P_n(x) - c_nP_{n-1}(x), \quad n \geq 0 \quad (P_{-1}(x) = 0),$$

where $c_n \neq 0$, $n \geq 1$. In particular, $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS relative to a positive-definite moment functional if and only if $c_n > 0$, $n \geq 1$. In this case, if we let $\sigma$ be a canonical moment functional of $\{P_n(x)\}_{n=0}^{\infty}$ with $\langle \sigma, 1 \rangle = 1$ and $c_0 = 1$, then $\langle \sigma, P_n^2 \rangle = c_nc_{n-1}\cdots c_0$, $n \geq 0$.

The next theorem gives a necessary and sufficient condition for the difference equation (1.3) to have an OPS as solutions.

**Theorem 2.4.** The difference equation (1.3) has a monic OPS $\{P_n(x)\}_{n=0}^{\infty}$ as solutions if and only if

(i) $s_n \neq 0$, $n \geq 0$;

(ii) $\ell_2(-\frac{t_n}{s_n}) \neq 0$, where $t_n = t_{2n}n^2 + (\ell_{21} + \ell_{11})n + \ell_{10}$, $n \geq 0$.

In this case, the coefficients of the three term recurrence relation are

$$b_n = \frac{nt_{n-1}}{s_{2n-2}} - \frac{(n + 1)t_n}{s_{2n}} + n, \quad n \geq 0,$$

and

$$c_n = -\frac{n(s_{n-2})}{s_{2n-3}s_{2n-1}}\ell_2\left(-\frac{t_{n-1}}{s_{2n-2}}\right), \quad s_{-1} = 1, \quad n \geq 1.$$  

Proof. See Theorem 3.5 in [6].

Note that for any moment functional $\sigma$ satisfying (2.2) and any polynomial $p(x)$, we have

$$L[p](x)\sigma = \Delta[(\nabla p)\ell_2\sigma].$$

Now assume that the equation (1.3) is admissible, that is, $s_n \neq 0$, $n \geq 0$ and let $\sigma$ be a canonical moment functional of the unique monic PS
\( \{P_n(x)\}_{n=0}^\infty \) of solutions of the equation (1.3). Then
\[
\lambda_n(\sigma, P_m P_n) = \langle L[P_n]\sigma, P_m \rangle = \langle \Delta[(\nabla P_n)\ell_2\sigma], P_m \rangle \\
= -\langle \ell_2\sigma, \nabla P_n \nabla P_m \rangle = \lambda_m(\sigma, P_m P_n), \quad m \text{ and } n \geq 0
\]
so that \( \langle \sigma, P_m P_n \rangle = 0 \) for \( m \neq n \), that is, \( \{P_n(x)\}_{n=0}^\infty \) is a WOPS. But, \( \{P_n(x)\}_{n=0}^\infty \) need not be an OPS in general, unless the condition (ii) in Theorem 2.4 is satisfied.

The condition (i) in Theorem 2.4 is just the admissibility of \( L[\cdot] \) (cf. Lemma 2.2). Hence, if the equation (1.3) has an OPS \( \{P_n(x)\}_{n=0}^\infty \) as solutions, then \( \{P_n(x)\}_{n=0}^\infty \) must be orthogonal relative to any non-zero solution \( \sigma \) of the discrete moment equation (2.2).

Now, we are ready to classify all OPS’s satisfying the difference equation (1.3) (cf. [6]). There are four cases, up to a real linear change of variable, to be considered according to the root system of \( \ell_2(x) \):
\[
\ell_2(x) = x(A - x), \quad x^2 + \zeta^2, \quad x, \quad 1,
\]
where \( A \) and \( \zeta (> 0) \) are real numbers. In case \( \ell_2(x) \equiv 0 \), it is easy to see that the equation (1.3) cannot have an OPS as solutions (see Remark 3.1).

In the following classification, we always let \( b_n \) and \( c_n \) be the coefficients of the three term recurrence relation satisfied by a monic PS of solutions of the corresponding difference equation.

**Case 1:** \( \ell_2(x) = x(A - x) \). In this case, set
\[
\alpha + \gamma = A, \quad \alpha + \beta + 2 = -\ell_{11}, \quad \text{and} \quad (\beta + 1)(\gamma - 1) = \ell_{10}
\]
so that the equation (1.3) becomes
\[
(2.4) \quad x(\gamma + \alpha - x)\Delta \nabla y + [(\beta + 1)(\gamma - 1) - (\alpha + \beta + 2)x]\Delta y \\
= -n(n + \alpha + \beta + 1)y.
\]
By Theorem 2.4, (2.4) has a monic OPS \( \{h_n^{(\alpha, \beta)}(x, \gamma)\}_{n=0}^\infty \) as solutions if and only if
\[
\alpha, \beta, -\gamma, \alpha + \beta + 1, \alpha + \beta + \gamma \notin \mathbb{Z}^- := \{-1, -2, \cdots \}.
\]
In this case,
\[
b_n = \frac{n(n - \gamma)(n + \beta)}{2n + \alpha + \beta} - \frac{(n + 1)(n - \gamma + 1)(n + \beta + 1)}{2n + \alpha + \beta + 2} + n, \quad n \geq 0,
\]
and
\[ c_n = \frac{n(n + \alpha + \beta)(n + \alpha)(n + \beta)(\gamma - n)(n + \alpha + \beta + \gamma)}{(2n + \alpha + \beta - 1)(2n + \alpha + \beta)^2(2n + \alpha + \beta + 1)}, \quad n \geq 1. \]

Note that \( \{h_n^{(\alpha,\beta)}(x, \gamma)\}_{n=0}^{\infty} \) cannot be a positive-definite OPS since \( c_n < 0 \) for \( n \) large.

**REMARK 2.1.** It is worth to note that the constants \( \alpha, \beta, \) and \( \gamma \) need not be real but \( \{h_n^{(\alpha,\beta)}(x, \gamma)\}_{n=0}^{\infty} \) is always a real PS. For example, the following difference equation
\[ x(1 - x)\Delta \nabla y - (1 + x)\Delta y = -n^2 y \]
has a real OPS \( \{h_n^{(\alpha,\beta)}(x, \gamma)\}_{n=0}^{\infty} \) as solutions by Theorem 2.4, where \( \alpha = -i, \beta = -1 + i, \) and \( \gamma = 1 + i. \)

**REMARK 2.2.** If \( \alpha + \beta + 1 \notin \mathbb{Z}^- \), then the difference equation (2.4) is admissible and so has a unique monic PS \( \{h_n^{(\alpha,\beta)}(x, \gamma)\}_{n=0}^{\infty} \) as solutions, which is a WOPS. In particular, if either \( \alpha > -1, \beta > -1, \) and \( \gamma = N \) or \( \alpha < 1 - N, \beta < 1 - N, \) and \( \gamma = N \) for some positive integer \( N, \) then \( \{h_n^{(\alpha,\beta)}(x, N)\}_{n=0}^{\infty} \) has the finite orthogonality:
\[ \langle \sigma, [h_n^{(\alpha,\beta)}(x, N)]^2 \rangle = \begin{cases} \text{positive,} & 0 \leq n \leq N - 1 \\ 0, & n \geq N, \end{cases} \]

where \( \sigma \) is a canonical moment functional of \( \{h_n^{(\alpha,\beta)}(x, N)\}_{n=0}^{\infty}. \) In the literature([8]), \( \{h_n^{(\alpha,\beta)}(x, N)\}_{n=0}^{N-1} \) are known as Hahn polynomials of type 1 if \( \alpha > -1 \) and \( \beta > -1 \) and Hahn polynomials of type 2 if \( \alpha < 1 - N \) and \( \beta < 1 - N \).

**Case 2:** \( \ell_2(x) = x^2 + \zeta^2 \) (\( \zeta > 0 \)). In this case, the equation (1.3) becomes
\[ (x^2 + \zeta^2)\Delta \nabla y + (ax + b)\Delta y = (n + a - 1)ny. \]

By Theorem 2.4, (2.5) has a monic OPS \( \{h_n^{(a,b)}(x, \zeta)\}_{n=0}^{\infty} \) as solutions if and only if \( a - 1 \notin \mathbb{Z}^- \). In this case,
\[ b_n = \frac{n[(n - 1)^2 + a(n - 1) + b]}{2n + a - 2} - \frac{(n + 1)[n^2 + an + b]}{2n + a} + n, \quad n \geq 0. \]
and

\[ c_n = \frac{-n(n + a - 2)}{(2n + a - 3)(2n + a - 1)} \left( \frac{(n - 1)^2 + a(n - 1) + b^2}{(2n + a - 2)^2} + \zeta^2 \right), \quad n \geq 1. \]

Hence, \( \{ \tilde{h}_n^{(a,b)} (x, \zeta) \}_{n=0}^{\infty} \) cannot be a positive definite OPS.

**Case 3:** \( \ell_2 (x) = x \). In this case, the equation (1.3) can be parameterized as

\[ x \Delta \nabla y + [a - (1 - \mu)x] \Delta y = (\mu - 1)ny. \tag{2.6} \]

By Theorem 2.4, (2.6) has a monic OPS as solutions if and only if \( \mu \neq 1 \) and \( \mu n + a \neq 0, \quad n = 0, 1, \ldots \).

If \( \mu = 0 \) and \( a \neq 0 \), then the difference equation (2.6) becomes

\[ x \Delta \nabla y + (a - x) \Delta y = -ny, \]

which has a monic OPS \( \{ c_n^{(a)} (x) \}_{n=0}^{\infty} \), known as Charlier polynomials ([8]), as solutions. In this case,

\[ b_n = n + a, \quad n \geq 0, \quad \text{and} \quad c_n = an, \quad n \geq 1. \]

Note that \( \{ c_n^{(a)} (x) \}_{n=0}^{\infty} \) is a positive-definite OPS only for \( a > 0 \).

If \( \mu \neq 0 \), then by setting \( a = \gamma \mu \), the difference equation (2.6) becomes

\[ x \Delta \nabla y + [\gamma \mu - (1 - \mu)x] \Delta y = (\mu - 1)ny, \]

which has a monic OPS \( \{ m_n^{(\gamma, \mu)} (x) \}_{n=0}^{\infty} \), known as Meixner polynomials ([8]), as solutions if and only if \( \mu \neq 0, 1 \) and \( \gamma - 1 \notin \mathbb{Z}^- \). In this case,

\[ b_n = \frac{1}{1 - \mu} ((1 + \mu)n + \gamma \mu), \quad n \geq 0 \]

and

\[ c_n = \frac{\mu n}{(1 - \mu)^2} (n + \gamma - 1), \quad n \geq 1. \]

Note that \( \{ m_n^{(\gamma, \mu)} (x) \}_{n=0}^{\infty} \) is a positive-definite OPS only for \( \mu > 0, \mu \neq 1, \) and \( \gamma > 0 \). In the literature ([8]), Meixner polynomials \( \{ m_n^{(\gamma, \mu)} (x) \}_{n=0}^{\infty} \) is usually introduced with \( 0 < \mu < 1 \) and \( \gamma > 0 \). However, \( \{ m_n^{(\gamma, \mu)} (x) \}_{n=0}^{\infty} \)
is also a positive-definite OPS for \(\mu > 1\) and \(\gamma > 0\), which is orthogonal with respect to a discrete weight function

\[
w(x) = \sum_{k=0}^{\infty} \frac{\Gamma(\gamma + k)\mu^k}{k!\Gamma(\gamma)} \delta(-\gamma - k).
\]

**Remark 2.3.** When \(\mu \neq 1\) and \(a = -\mu N\) for some positive integer \(N\), we set \(p = \frac{\mu}{\mu - 1}\) so that the equation (2.6) becomes

\[(2.7) \quad x\Delta \nabla y(x) + \frac{Np - x}{q} \Delta y(x) = -\frac{n}{q} y(x) \quad (q \neq 0, \ p + q = 1).
\]

The equation (2.7) is admissible so that has a unique monic WOPS \(\{k_n^{(p)}(x, N)\}_{n=0}^{\infty}\) as solutions, which has the finite orthogonality:

\[
\langle \sigma, [k_n^{(p)}(x, N)]^2 \rangle = \begin{cases} 
\text{positive,} & 0 \leq n \leq N \\
0, & n \geq N + 1.
\end{cases}
\]

In the literature ([8]), \(\{k_n^{(p)}(x, N)\}_{n=0}^{N}\) are known as Krawchuk polynomials.

**Case 4:** \(l_2(x) = 1\). In this case, the equation (1.3) becomes

\[(2.8) \quad \Delta \nabla y + (l_{11}x + l_{10}) \Delta y = l_{11}ny.
\]

By a real linear change of variable: \(x \mapsto -x - \frac{1+\delta_0}{l_{11}}\) and using \(\Delta - \nabla = \Delta \nabla\), the difference equation (2.8) can be transformed into

\[x\Delta \nabla y + (a - x) \Delta y = -ny \quad (a = -1/l_{11}),
\]

which has Charlier polynomials \(\{c_n^{(a)}(x)\}_{n=0}^{\infty}\) as solutions (cf. Case 3).

Lastly in this section, we examine more closely the orthogonality of polynomial solutions of the difference equation (1.3). For any monic PS \(\{P_n(x)\}_{n=0}^{\infty}\), we can write \(P_{n+1}(x)\) as

\[(2.9) \quad P_{n+1}(x) = (x - \xi_n^k)P_n(x) - \xi_n^{n-1}P_{n-1}(x) - \sum_{k=0}^{n-2} \xi_n^k P_k(x) \quad (n \geq 1),
\]

where \(\xi_1^0 = \xi_1^{-1} = 0\) and \(P_1(x) \equiv 0\).

**Lemma 2.5.** Assume that the difference equation (1.3) has a monic PS \(\{P_n(x)\}_{n=0}^{\infty}\) as solutions. Let \(N \geq 0\) be the largest integer such that \(\lambda_N = 0\). Then
(i) $\xi_n^0 = 0$ if $n \geq 2$ and $n + 1 \neq N$

and

(ii) for any moment functional solution $\sigma$ of the discrete moment equation (2.2),

$$\langle \sigma, P_n^2 \rangle = \xi_n^{n-1} \cdot \xi_{n-1}^{n-2} \cdots \xi_{N+1}^{N}(\sigma, P_N^2), \quad n \geq N + 1,$$

where $\xi_n^0$ and $\xi_n^{n-1}$ are the constants given in (2.9).

**Proof.** First note that $\lambda_m = \lambda_n$ for $m \neq n$ if and only if $m + n = N$. Hence we have for $m \neq n$ and $m + n \neq N$ (cf. Proposition 3.2),

$$\langle \sigma, P_m P_n \rangle = 0$$

for any solution $\sigma$ of the discrete moment equation (2.2).

(i) Let $\sigma$ be a canonical moment functional of $\{P_n(x)\}_{n=0}^{\infty}$. Then $\sigma$ satisfies the discrete moment equation (2.2) by Lemma 2.3. If we apply $\sigma$ to the equation (2.9), we obtain for $n \geq 2$

$$0 = \langle \sigma, P_{n+1} \rangle = \langle \sigma, x P_n \rangle - \xi_n^n \langle \sigma, P_n \rangle - \xi_{n-1}^{n-1} \langle \sigma, P_{n-1} \rangle - \sum_{k=0}^{n-2} \xi_n^k \langle \sigma, P_k \rangle$$

$$= \langle \sigma, x P_n \rangle - \xi_n^0 \langle \sigma, P_0 \rangle$$

since $\langle \sigma, P_n \rangle = 0$ for $n \geq 1$. Hence we have (i) since $\langle \sigma, P_0 \rangle \neq 0$ and $\langle \sigma, x P_n \rangle = \langle \sigma, P_1 P_n \rangle = 0$ for $n \geq 2$ and $n + 1 \neq N$.

(ii) Now let $\sigma$ be any solution of (2.2). If we multiply the equation (2.9) by $P_{n-1}(x)$ and apply $\sigma$, then we obtain from (2.11)

$$0 = \langle \sigma, P_{n+1} P_{n-1} \rangle = \langle \sigma, P_n^2 \rangle - \xi_{n-1}^{n-1} \langle \sigma, P_{n-1}^2 \rangle - \xi_n^0 \langle \sigma, P_0 P_{n-1} \rangle$$

for $n \geq N + 1$. If $n > N + 1$, then $\langle \sigma, P_0 P_{n-1} \rangle = 0$ by (2.11). If $n = N + 1$ and $N \geq 1$, then $\xi_{N+1}^0 = 0$ by (i). Finally if $N = 0$, then $\xi_1^0 = 0$. Therefore, we have

$$\langle \sigma, P_n^2 \rangle = \xi_n^{n-1} \langle \sigma, P_{n-1}^2 \rangle, \quad n \geq N + 1,$$

from which (ii) follows immediately. \qed

We are now ready to prove the following result which we need later in section three and is interesting in its own right.

**Theorem 2.6.** Assume that the difference equation (1.3) has a monic PS $\{P_n(x)\}_{n=0}^{\infty}$ as solutions. If $\{P_n(x)\}_{n=0}^{\infty}$ is not an OPS, then for any
solution $\sigma$ of the discrete moment equation \((2.2)\), there is an integer $m \geq 0$ such that

$$\langle \sigma, P_n^2 \rangle = 0 \quad \text{for all} \quad n \geq m + 1.$$ 

Proof. If $\lambda_n \neq 0$, $n \geq 1$, then $L[-\cdot]$ is admissible and $\{P_n(x)\}_{n=0}^\infty$ is a WOPS. Consequently, $\xi_k^{n-1} = 0$ for some $k \geq 1$ by Lemma 2.5 since $\{P_n(x)\}_{n=0}^\infty$ is not an OPS. Hence $\langle \sigma, P_n^2 \rangle = 0, n \geq k$ by \((2.10)\). Now, we assume that $\lambda_N = 0$ for some $N \geq 1$. For each $n \geq 0$, let $P_n(x) = \sum_{k=0}^n \eta_n^k x^k$ ($\eta_n^n = 1$) be a monic polynomial of degree $n$. Then $P_n(x)$ satisfies the difference equation \((1.3)\) if and only if

\begin{equation}
(\lambda_n - \lambda_k)\eta_k^n = (k + 1)[\ell_{22}k^2 + \ell_{21}k + \ell_{11}k + \ell_{10}]\eta_{k+1}^n 
+ \sum_{j=k}^n \eta_{j+2}^n (j+2)(j+1)(-1)^{j-k}P(j, j-k), \quad 0 \leq k \leq n,
\end{equation}

where $\eta_{n+1}^n = \eta_{n+2}^n = 0$ and $P(n, m) = n(n-1)\cdots(n-m+1)$. Since $\lambda_{N+1} - \lambda_k \neq 0$ for $0 \leq k \leq N$ and $\lambda_{N+2} - \lambda_k \neq 0$ for $0 \leq k \leq N + 1$, the equation \((2.12)\) is uniquely solvable for $\eta_{N+1}^{N+1}, \eta_{N+2}^{N+1}, \eta_{N+2}^{N+2}$, and $\eta_{N+1}^{N+2}$. Actually, we have

$$\eta_{n+1}^n = \frac{(n+1)t_n}{s_{2n}}, \quad (n = N, N+1)$$

and

$$\eta_{n-1}^{n+1} = \frac{n(n+1)(t_{n+1}t_n + ns_{2n})}{2s_{2n-1}s_{2n}}, \quad (n = N, N+1).$$

Note that since $\lambda_N = 0$ or equivalently $s_{N-1} = 0$, $s_{2N} \neq 0$, $s_{2N} \neq 0$ and $s_{2N+1} \neq 0$. Hence, we have

$$\xi_{N+1}^N = \frac{-(N+1)s_{N-1}}{s_{2N-1}s_{2N+1}} \ell_2 \left( \frac{-t_N}{s_{2N}} \right) = 0$$

and the conclusion follows by \((2.10)\). \hfill \Box

3. Classification of discrete Sobolev orthogonal polynomials

From here on, we shall consider DSOPS's relative to a symmetric bilinear form \((2.1)\). We first obtain necessary and sufficient conditions for
a DSOPS relative to $\phi(\cdot, \cdot)$ to satisfy a second order difference equation (1.3).

**Theorem 3.1.** For a bilinear form $\phi(\cdot, \cdot)$ in (2.1), the followings are all equivalent.

(i) The difference operator $L[\cdot]$ in (1.3) is symmetric on polynomials relative to $\phi(\cdot, \cdot)$, that is,

$$\phi(L[p], q) = \phi(p, L[q]), \quad (p, q \in \mathcal{P}).$$

(ii) The moment functionals $\sigma$ and $\tau$ satisfy the functional equations

$$\Delta(\ell_2 \sigma) - \ell_1 \sigma = 0$$

and

$$\Delta(\ell_2 \tau) - [\Delta(\ell_2 + \ell_1) + \ell_1] \tau = 0.$$ 

(iii) The moments of $\sigma$ and $\tau$ given by $\sigma_n := \langle \sigma, x^{[n]} \rangle$ and $\tau_n := \langle \tau, x^{[n]} \rangle$ satisfy for $n \geq 0$,

$$((\ell_{22}n + \ell_{11})\sigma_{n+1} + [\ell_{22}n(2n-1) + (\ell_{21} + \ell_{11})n + \ell_{10}]\sigma_n + n[\ell_{22}(n-1)^2 + \ell_{21}(n-1) + \ell_{20}]\sigma_{n-1} = 0$$

and

$$(\ell_{22}n + 2\ell_{22} + \ell_{11})\tau_{n+1} + [\ell_{22}n(2n-1) + (\ell_{21} + 2\ell_{22} + \ell_{11})n + \ell_{21} + \ell_{11} + \ell_{10}]\tau_n + n[\ell_2(n) + \ell_1(n)]\tau_{n-1} = 0,$$

where $\sigma_n = 0$ and $\tau_n = 0$ if $n < 0$.

Furthermore if $\phi(\cdot, \cdot)$ is quasi-definite and $\{R_n(x)\}_{n=0}^{\infty}$ is a DSOPS relative to $\phi(\cdot, \cdot)$, then the statements (i), (ii), and (iii) are also equivalent to

(iv) $\{R_n(x)\}_{n=0}^{\infty}$ satisfies the difference equation (1.3).

**Proof.** (i)⇔(ii): It is easy to check the following identities: for $p, q \in \mathcal{P}$,

$$\phi(L[p], q) = \langle L^+[q\sigma], p \rangle - \langle L^+[\nabla(\Delta q)\tau], p \rangle,$$

$$\phi(p, L[q]) = \langle L[q]\sigma, p \rangle - \langle \nabla(\Delta L[q])\tau, p \rangle,$$

where $L^+[\cdot]$ is an adjoint of $L[\cdot]$ defined by

$$L^+[y] = \Delta \nabla(\ell_2 y) - \nabla(\ell_1 y).$$
Hence, the equation (3.1) is equivalent to
\[ L^+[q\sigma] - L^+[\nabla[(\Delta q)\tau]] - L[q]\sigma + \nabla[(\Delta L[q])\tau] = 0, \quad (q \in \mathcal{P}), \]
which can be written out as
\[
\begin{aligned}
&\Delta(\ell_2 \tau) - (\Delta(\ell_2 + \ell_1) + \ell_1)\tau)\Delta^2\nabla q + [(\ell_2 + \ell_1)\nabla \tau - \ell_1\tau] \Delta\nabla^2 q \\
&+ \nabla[(\ell_2 + \ell_1)\nabla \tau - \ell_1\tau] \nabla^2 q + 2\Delta[(\ell_2 + \ell_1)\nabla \tau - \ell_1\tau] \Delta\nabla q \\
&+ [\Delta\nabla[(\ell_2 + \ell_1)\nabla \tau - \ell_1\tau] + (\nabla(\ell_2\sigma) - \ell_1\sigma + \nabla(\ell_1\sigma)]\nabla q \\
&+ [\Delta(\ell_2\sigma) - \ell_1\sigma] \Delta q + \nabla[\Delta(\ell_2\sigma) - \ell_1\sigma]q = 0.
\end{aligned}
\tag{3.6}
\]
We can see that the condition (3.6) is equivalent to the fact that \( \sigma \) and \( \tau \) satisfy (3.2) and (3.3) since \( \Delta(\ell_2 \tau) - (\ell_1 + \Delta(\ell_1 + \ell_2))\tau = 0 \) if and only if \( (\ell_2 + \ell_1)\nabla \tau - \ell_1\tau = 0 \) and \( \nabla(\ell_2\sigma) - \ell_1\sigma + \nabla(\ell_1\sigma) = 0 \) if and only if \( \Delta(\ell_2\sigma) = \ell_1\sigma \) by Lemma 2.1.

(ii)\( \Leftrightarrow \) (iii): The equivalence of (3.2) and (3.4) is proved in Lemma 2.3 and the equivalence of (3.3) and (3.5) can be proved similarly.

We now assume that \( \{R_n(x)\}_{n=0}^\infty \) is a DSOPS relative to \( \phi(\cdot, \cdot) \).

(i)\( \Rightarrow \) (iv): Since \( L[R_n](x) \) is a polynomial of degree \( \leq n \), we may write
\[ L[R_n](x) = \sum_{k=0}^n C_{nj} R_j(x) \]
for some real constant \( C_{nj} \), \( j = 0, 1, \cdots, n \). Then for \( 0 \leq k \leq n - 1 \),
\[ C_{nk}\phi(R_k, R_k) = \sum_{k=0}^n C_{nj}\phi(R_j, R_k) = \phi(L[R_n], R_k) = \phi(R_n, L[R_k]) = 0; \]
since \( \deg(L[R_k]) \leq k \). Hence, \( C_{nk} = 0 \) for \( k = 0, 1, \cdots, n - 1 \) so that
\[ L[R_n](x) = C_{nn} R_n(x) = \lambda_n R_n(x) \]
by comparing the coefficients of \( x^n \) on both sides.

(iv)\( \Rightarrow \) (i): (3.1) follows immediately from
\[ \phi(L[R_n], R_m) = \lambda_n\phi(R_n, R_m) = \lambda_m\phi(R_n, R_m) = \phi(R_n, L[R_m]) \]
since \( \{R_n\}_{n=0}^\infty \) is a basis of \( \mathcal{P} \).

When \( \tau = 0 \), the equivalences of (i), (ii), (iii), and (iv) in Theorem 3.1 are proved in [6] (see also [3]).
Remark 3.1. When $\ell_2(x) \equiv 0$, the difference equation (1.3) reduces to the first order difference equation

\[(3.7) \quad \ell_1(x) \Delta y = \ell_{11} n y,\]

which can have a PS as solutions only when $\ell_{11} \neq 0$. In this case, the general solutions of the moment equations (3.2) and (3.3) are

$$\sigma = c_1 \delta(x + \ell_{10}/\ell_{11}) \quad \text{and} \quad \tau = c_2 \delta(x + \ell_{10}/\ell_{11}),$$

where $c_1$ and $c_2$ are arbitrary constants. Then the corresponding symmetric bilinear form $\phi(\cdot, \cdot)$ cannot be quasi-definite. Hence, the equation (3.7) cannot have a DSOPS as solutions. Similarly, we can also show that if $\ell_2(x) + \ell_1(x) \equiv 0$, then the equation (1.3) can not have a DSOPS as solutions.

Remark 3.2. If we act $\Delta$ on both sides of (1.3), then $z(x) = \Delta y(x)$ satisfies

\[(3.8) \quad \ell_2 \Delta \nabla z + [\ell_1 + \Delta(\ell_1 + \ell_2)] \Delta z = (\lambda_n - \Delta \ell_1) z.\]

Note that (3.3) is the discrete moment equation for the difference equation (3.8).

Proposition 3.2. If $L[p] = \lambda p$ and $L[q] = \mu q$ for some $p, q \in \mathcal{P}$ and $\lambda \neq \mu$, then

$$\phi(p, q) = 0$$

for any solutions $\sigma$ and $\tau$ of the discrete moment equations (3.2) and (3.3) respectively.

Proof. It immediately follows from the fact (cf. Theorem 3.1)

$$(\lambda - \mu) \phi(p, q) = \phi(L[p], q) - \phi(p, L[q]).$$

We are now ready to classify all DSOPS’s relative to the bilinear form $\phi(\cdot, \cdot)$ in (2.1) satisfying the difference equation (1.3). In the following, we shall assume $\ell_2(x) \neq 0$ and $\ell_2(x) + \ell_1(x) \neq 0$ (see Remark 3.1). Concerning the symmetric bilinear form $\phi(\cdot, \cdot)$ in (2.1), there arise the following three cases:

Type A: $\sigma$ is quasi-definite;

Type B: Both $\sigma$ and $\tau$ are not quasi-definite;

Type C: $\sigma$ is not quasi-definite but $\tau$ is quasi-definite.
We now consider three cases individually.

**Type A:** \( \sigma \) is quasi-definite.

It is well known (cf. Proposition 2.5 in [5]) that if \( \{P_n(x)\}_{n=0}^{\infty} \) is a discrete classical OPS relative to \( \sigma \) satisfying the equation (1.3), then \( \{\nabla P_n(x)\}_{n=0}^{\infty} \) is also an OPS relative to \( \ell_2(x)\sigma \). In this case, \( \{\Delta P_n(x)\}_{n=0}^{\infty} \) is also an OPS relative to \( \ell_2(x+1)\sigma(x+1) = (\ell_2(x) + \ell_1(x))\sigma \) since \( \Delta f(x) = \nabla f(x+1) \) and \( (\ell_2\sigma)' = \ell_1\sigma \).

**Theorem 3.3.** If the difference equation (1.3) has a DSOPS \( \{R_n(x)\}_{n=0}^{\infty} \) relative to \( \phi(\cdot, \cdot) \) as solutions and \( \sigma \) is quasi-definite, then \( \{R_n(x)\}_{n=0}^{\infty} \) must be a discrete classical OPS relative to \( \sigma \) and either \( \tau = 0 \) or \( \tau \) is also quasi-definite.

**Proof.** Since \( \sigma \) is a canonical moment functional of \( \{R_n(x)\}_{n=0}^{\infty} \), \( \{R_n(x)\}_{n=0}^{\infty} \) must be an OPS so that a discrete classical OPS relative to \( \sigma \) when \( \sigma \) is quasi-definite. Then \( \{\Delta R_n(x)\}_{n=0}^{\infty} \) is also an OPS relative to \( \tilde{\sigma} = (\ell_2(x) + \ell_1(x))\sigma \) and satisfy the equation (3.8). Hence, \( \tau \) and \( \tilde{\tau} \) satisfy the moment equation (3.3), which is uniquely solvable (up to a constant factor) by Lemma 2.3 so that \( \tau = a\tilde{\tau} \) for some constant \( a \). Thus, \( \tau = 0 \) if \( a = 0 \) or \( \tau \) is quasi-definite if \( a \neq 0 \).

**Type B:** Both \( \sigma \) and \( \tau \) are not quasi-definite.

Here, we will show that any DSOPS relative to a bilinear form \( \phi(\cdot, \cdot) \) in (2.1) cannot satisfy a second order difference equation of the form (1.3).

**Theorem 3.4.** Let \( \{R_n(x)\}_{n=0}^{\infty} \) be a DSOPS relative to a quasi-definite bilinear form \( \phi(\cdot, \cdot) \) in (2.1). If both \( \sigma \) and \( \tau \) are not quasi-definite, then \( \{R_n(x)\}_{n=0}^{\infty} \) cannot satisfy a difference equation of the form (1.3).

**Proof.** Assume that \( \{R_n(x)\}_{n=0}^{\infty} \) satisfies the difference equation (1.3). Then by Theorem 3.1, \( \sigma \) and \( \tau \) must be non-trivial solutions of the moment equations (3.2) and (3.3) respectively. Hence, \( \{R_n(x)\}_{n=0}^{\infty} \) cannot be an OPS since \( \sigma \) is not quasi-definite. On the other hand, \( \{\Delta R_n(x)\}_{n=1}^{\infty} \) is a PS satisfying the difference equation (3.8) and \( \tau \) satisfies the corresponding moment equation (3.3). Hence, \( \{\Delta R_n(x)\}_{n=1}^{\infty} \) cannot be an OPS since \( \tau \) is not quasi-definite. Then, by Theorem 2.6, we have

\[
\langle \sigma, R_n^2 \rangle = \langle \tau, (\Delta R_n)^2 \rangle = 0
\]
for all $n$ large enough and so
\[ \phi(R_n, R_n) = \langle \sigma, R_n^2 \rangle + \langle \tau, (\Delta R_n)^2 \rangle = 0 \]

for all $n$ large enough, which contradicts the fact that $\{R_n(x)\}_{n=0}^{\infty}$ is a DSOPS relative to $\phi(\cdot, \cdot)$. \hfill \Box

**Type C: $\sigma$ is not quasi-definite but $\tau$ is quasi-definite.**

**Theorem 3.5.** Assume that the difference equation (1.3) has a PS $\{R_n(x)\}_{n=0}^{\infty}$ of solutions, which is a DSOPS relative to the bilinear form $\phi(\cdot, \cdot)$ in (2.1). If $\tau$ is quasi-definite, then

(i) $\{\Delta R_n(x)\}_{n=1}^{\infty}$ is a discrete classical OPS relative to $\tau$ and satisfies the equation (3.8);

(ii) $\{R_n(x)\}_{n=0}^{\infty}$ is a WOPS relative to $\sigma$;

(iii) $(\ell_2 + \ell_1)\sigma = a\tau$ for some constant $a$ so that either $(\ell_2 + \ell_1)\sigma = 0$ or $(\ell_2 + \ell_1)\sigma$ is quasi-definite.

**Proof.** (i) Let $\{Q_n(x)\}_{n=0}^{\infty}$ be the monic OPS relative to $\tau$. Since $\tau$ satisfies the moment equation (3.3), $\{Q_n(x)\}_{n=0}^{\infty}$ is a discrete classical OPS satisfying the difference equation (3.8) (cf. Theorem 3.1). Hence, the equation (3.8) must be admissible. On the other hand, the PS $\{\Delta R_{n+1}(x)\}_{n=0}^{\infty}$ also satisfies the difference equation (3.8). Hence, $\Delta R_{n+1}(x) = C_n Q_n(x)$, $n \geq 0$ by Lemma 2.2, where $C_n$ is the coefficient of $x^n$ in $\Delta R_{n+1}(x)$.

(ii) It follows from the orthogonalities of $\{R_n(x)\}_{n=0}^{\infty}$ relative to $\phi(\cdot, \cdot)$ and $\{\Delta R_n(x)\}_{n=1}^{\infty}$ relative to $\tau$.

(iii) Since the equation (3.8) is admissible, the moment equation (3.3) has only one linearly independent solution. Since $\tau$ and $(\ell_2 + \ell_1)\sigma$ satisfy the moment equation (3.3), we have $(\ell_2 + \ell_1)\sigma = a\tau$ for some constant $a$ so that either $(\ell_2 + \ell_1)\sigma = 0$ if $a = 0$ or $(\ell_2 + \ell_1)\sigma$ is quasi-definite if $a \neq 0$. \hfill \Box

As in section two, we may assume that
\[ \ell_2(x) = x(A - x), \quad x^2 + \zeta^2, \quad x, \quad 1 \quad (A \text{ real and } \zeta > 0). \]

In each case, we look for conditions that the equation (3.2) has no quasi-definite moment functional solution and the equation (3.3) has a quasi-definite moment functional solution.
Case C.1. \( \ell_2(x) = x(A - x) \). In this case, the difference equation (1.3) can be written as

\[
\begin{align*}
\Delta (x + \alpha - x) \Delta y + [(\beta + 1)(\gamma - 1) - (\alpha + \beta + 2)x] \Delta y &= -n(n + \alpha + \beta + 1)y.
\end{align*}
\]

Then the corresponding discrete moment equations are

\[
\begin{align*}
\Delta [x(\gamma + \alpha - x)\sigma] &= [(\beta + 1)(\gamma - 1) - (\alpha + \beta + 2)x] \sigma \\
\Delta [x(\gamma + \alpha - x)\tau] &= [(\beta + 2)(\gamma - 2) - (\alpha + \beta + 4)x] \tau.
\end{align*}
\]

The equation (3.10) has no quasi-definite moment functional solution if and only if \( \alpha \notin \mathbb{Z}^- \) or \( \beta \notin \mathbb{Z}^- \) or \( \alpha + \beta + 1 \notin \mathbb{Z}^- \) or \( \gamma \in \mathbb{Z}^+ \) or \( \alpha + \beta + \gamma \in \mathbb{Z}^- \). The equation (3.11) has a quasi-definite moment functional solution if and only if \( \alpha + 1 \notin \mathbb{Z}^- \), \( \beta + 1 \notin \mathbb{Z}^- \), \( \alpha + \beta + 2 \notin \mathbb{Z}^- \), \( \gamma - 1 \notin \mathbb{Z}^+ \) and \( \alpha + \beta + \gamma + 1 \notin \mathbb{Z}^- \). Hence, there are five cases:

(i) \( \alpha = -1, \beta + 1, 1 - \gamma, \) and \( \beta + \gamma \notin \mathbb{Z}^- \);
(ii) \( \beta = -1, \alpha, 1 - \gamma, \) and \( \alpha + \gamma \notin \mathbb{Z}^- \);
(iii) \( \gamma = 1, \alpha, \beta, \) and \( \alpha + \beta + 2 \notin \mathbb{Z}^- \);
(iv) \( \alpha + \beta = -2, \alpha, \beta \notin \mathbb{Z}^+ \), and \( \gamma \notin \mathbb{Z}^+ \cup \{0\} \cup \mathbb{Z}^- \);
(v) \( \alpha + \beta + \gamma = -1, \alpha, \beta, \alpha + \beta + 1 \notin \mathbb{Z}^- \), and \( \gamma \notin \mathbb{Z}^+ := \{1, 2, \cdots\} \).

We now have:

**Theorem 3.6.**

(i) If \( \alpha = -1, \beta + 1, 1 - \gamma, \) and \( \beta + \gamma \notin \mathbb{Z}^- \), then the difference equation (3.9) always has a DSOPS \( \{h_n^{(1,1)}(x,\gamma)\}_{n=0}^{\infty} \) as solutions, which are orthogonal relative to

(a) \( \phi_A(p, q) = Ap(\gamma - 1)q(\gamma - 1) + \langle W^{(0,\beta+1)}(x,\gamma - 1), \Delta p \Delta q \rangle \) if \( \beta \neq -1 \), where \( A \) is any non-zero constant;

(b) \( \phi_{AB}(p, q) = Ap(0)q(0) + Bp(\gamma - 1)q(\gamma - 1) + \langle W^{(0,0)}(x,\gamma - 1), \Delta p \Delta q \rangle \) if \( \beta = -1, \gamma \neq 1, \) and \( h_1^{(1,-1)}(x,\gamma) = x + a, \) where \( a, A, \) and \( B \) are arbitrary constants with \( A + B \neq 0, \) \( Aa^2 + B(\gamma - 1 + a)^2 = 0, \) \( Aa + B(\gamma - 1 + a) = 0; \)

(c) \( \phi_{AB}(p, q) = Ap(0)q(0) - B[p(0)q(0) + p'(0)q(0)] + \langle W^{(0,0)}(x,0), \Delta p \Delta q \rangle \) if \( -\beta = \gamma = 1 \) and \( h_1^{(1,-1)}(x,1) = x + a, \) where \( a, A, \) and \( B \) are arbitrary constants with \( A \neq 0, \) \( Ab^2 - 2Bb + 1 \neq 0, \) and \( Aa - B = 0. \)

(ii) If \( \beta = -1, \alpha, 1 - \gamma, \) and \( \alpha + \gamma \notin \mathbb{Z}^- \), then the difference equation (3.9) always has a unique DSOPS \( \{h_n^{(a,1)}(x,\gamma)\}_{n=0}^{\infty} \) as solutions,
which are orthogonal relative to

$$
\phi_A(p, q) = Ap(0)q(0) + \langle W^{(\alpha+1,0)}(x, \gamma - 1), \Delta p\Delta q \rangle,
$$

where $A$ is an arbitrary nonzero constant.

(iii) If $\gamma = 1$, $\alpha, \beta$, and $\alpha + \beta + 2 \not\in \mathbb{Z}^-$, then the difference equation (3.9) always has a DSOPS $\{h_n^{(\alpha,\beta)}(x, 1)\}_{n=0}^{\infty}$ as solutions, which are orthogonal relative to

(a) $\phi_A(p, q) = Ap(0)q(0) + \langle W^{(\alpha+1,\beta+1)}(x, 0), \Delta p\Delta q \rangle$ if $\alpha + \beta \neq -2$, where $A$ is an arbitrary non-zero constant;

(b) $\phi_{A,B}(p, q) = Ap(0)q(0) + Bp(\alpha + 1)q(\alpha + 1) + \langle W^{(\alpha+1,\beta+1)}(x, 0), \Delta p\Delta q \rangle$ if $\alpha + \beta = -2$, and $h_1^{(\alpha,\beta)}(x, 1) = x + a$, where $a, A$, and $B$ are arbitrary constants with $A+B \neq 0$, $AB(\alpha+1)^2 + A+B \neq 0$, and $a(A+B) + (\alpha + 1)B = 0$.

(iv) If $\alpha + \beta = -2$, $\alpha, \beta \not\in \mathbb{Z}^-$, and $\gamma \not\in \mathbb{Z}^+ \cup \{0\} \cup \mathbb{Z}^-$, then the difference equation (3.9) has no DSOPS as solutions.

(v) If $\alpha + \beta + \gamma = -1$, $\alpha, \beta, -\gamma$, and $\alpha + \beta + 1 \not\in \mathbb{Z}^-$, then the difference equation (3.9) always has a DSOPS $\{h_n^{(\alpha,\beta)}(x, \gamma)\}_{n=0}^{\infty}$ as solutions, which are orthogonal relative to

$$
\phi_A(p, q) = Ap(\alpha + \gamma)q(\alpha + \gamma) + \langle W^{(\alpha+1,\beta+1)}(x, \gamma - 1), \Delta p\Delta q \rangle,
$$

where $A$ is an arbitrary nonzero constant.

Here, $W^{(\alpha,\beta)}(x, \gamma)$ is the canonical moment functional of the monic discrete classical OPS $\{h_n^{(\alpha,\beta)}(x, \gamma)\}_{n=0}^{\infty}$ with $\langle W^{(\alpha,\beta)}(x, \gamma), 1 \rangle = 1$.

Before giving the proof, we note that in cases (a) of (i), (ii), (a) of (iii), and (v) above, the equation (3.9) is admissible so that each $h_n^{(\alpha,\beta)}(x, \gamma)$, $n \geq 0$, is unique. However, in other cases except (iv), the equation (3.9) is not admissible but $h_n^{(\alpha,\beta)}(x, \gamma)$, $n \neq 1$ is unique and for $n = 1$, $h_1^{(\alpha,\beta)}(x, \gamma) = x + a$ satisfies the equation (3.9) for any constant $a$.

Proof of Theorem 3.6. We prove only (b) of (i) and (iv) since all the other cases can be proved similarly. When $\alpha = -1, \beta = -1$ and $\gamma \neq 1$, the equation (3.9) becomes

$$
(3.12) \quad x(\gamma - 1 - x)\Delta \nabla y = -n(n - 1)y,
$$

which is not admissible. However, it is easy to see that the equation (3.12) has a unique monic polynomial solution $h_n^{(-1,-1)}(x, \gamma)$ for $n \neq 1$
and for \( n = 1 \), any polynomial \( h_{1}^{(-1,-1)}(x) = x + a \) with arbitrary constant \( a \) is a solution of (3.12). Then we have from (2.4) and (3.12)

\[
\Delta h_{n}^{(-1,-1)}(x, \gamma) = nh_{n-1}^{(0,0)}(x, \gamma - 1), \quad n \geq 1.
\]

On the other hand, the general solutions of the moment equations (3.10) and (3.11) are

\[
\sigma = d_{1} \delta(x) + d_{2} \delta(x - \gamma + 1) \quad \text{and} \quad \tau = d_{3} W^{(0,\beta+1)}(x, \gamma - 1)
\]

where \( d_{i} \) \((i = 1, 2, 3)\) are arbitrary constants. Hence, by Proposition 3.2, (3.13)

\[
\phi_{A,B}(h_{m}^{(-1,-1)}(x, \gamma), h_{n}^{(-1,-1)}(x, \gamma)) = \\
\begin{cases} 
A + B & \text{if } m = n = 0 \\
Aa^2 + B(\gamma - 1 + a)^2 + 1 & \text{if } m = n = 1 \\
n^2(W^{(0,\beta+1)}(x, \gamma - 1), h_{n-1}^{(0,0)}(x, \gamma - 1)^2) \neq 0 & \text{if } m = n \geq 2 \\
Aa + B(\gamma - 1 + a) & \text{if } m = 0, n = 1 \\
0 & \text{if } m \neq n, m \geq 2
\end{cases}
\]

since \( h_{n}^{(-1,-1)}(x, \gamma) = 0, n \geq 2 \) at \( x = 0 \) and \( \gamma - 1 \) and \( \{h_{n}^{(0,0)}(x, \gamma - 1)\}_{n=0}^{\infty} \) is a discrete classical OPS relative to \( W^{(0,\beta+1)}(x, \gamma - 1) \) (see Case 1 in Section 2). Hence (b) of (i) follows immediately from (3.13).

Now, let \( \alpha + \beta = -2 \), \( \alpha, \beta \notin \mathbb{Z}^- \), and \( \gamma \notin \mathbb{Z}^+ \cup \{0\} \cup \mathbb{Z}^- \). Then the equation (3.9) become

\[
x(\gamma + \alpha - x)\Delta \nabla y + (\beta + 1)(\gamma - 1)\Delta y = -n(n - 1)y.
\]

In this case, the equation (3.14) has no polynomial solution of degree 1.

\( \square \)

**Case C.2.** \( \ell_2(x) = x^2 + \zeta^2 \) (\( \zeta > 0 \)). In this case, the difference equation (1.3) can be written as

\[
(x^2 + \zeta^2)\Delta \nabla y + (ax + b)\Delta y = n(n + a - 1)y.
\]

Then the corresponding discrete moment equations are

\[
\Delta[(x^2 + \zeta^2)\sigma] = (ax + b)\sigma
\]

(3.16)

\[
\Delta[(x^2 + \zeta^2)\tau] = [(a + 2)x + a + b + 1]\tau.
\]

(3.17)

The equation (3.16) has no quasi-definite moment functional solution if and only if \( a - 1 \in \mathbb{Z}^- \). The equation (3.17) has a quasi-definite moment functional solution if and only if \( a + 1 \notin \mathbb{Z}^- \). Hence, there are two cases:
If $a = 0$, $b \neq 0$ or $a = -1$, then the difference equation (3.15) has no polynomial solution of degree 1 or 2, respectively. When $a = b = 0$, we have:

**Theorem 3.7.** If $a = b = 0$, then difference equation (3.15) always has a DSOPS $\{\hat{h}_n^{(0,0)}(x, \zeta)\}_{n=0}^\infty$ as solutions, which are orthogonal relative to

$$
\phi_{AB}(p, q) = A\langle \sigma^{(1)}, pq \rangle + B\langle \sigma^{(2)}, pq \rangle + \langle W^{(2,1)}(x, \zeta), \Delta p \Delta q \rangle,
$$

if $\hat{h}_1^{(0,0)}(x, \zeta) = x + a$, where $a$, $A$, and $B$ are arbitrary constants with $A \neq 0$, $A - 2A \zeta^2 - 2B^2 \zeta^2 \neq 0$, and $Aa - B\zeta = 0$. Here, $\sigma^{(1)}$ and $\sigma^{(2)}$ are moment functionals defined by

$$
\langle \sigma^{(1)}, x^n \rangle = (\zeta i)^n + (-\zeta i)^n \quad \text{and} \quad \langle \sigma^{(2)}, x^n \rangle = [(\zeta i)^n - (-\zeta i)^n]i,
$$

$n \geq 0$ \(i = \sqrt{-1}\)

and $W^{(2,1)}(x, \zeta)$ is the canonical moment functional of the monic discrete classical OPS $\{\hat{h}_n^{(2,1)}(x, \zeta)\}_{n=0}^\infty$ with $\langle W^{(2,1)}(x, \zeta), 1 \rangle = 1$.

**Case C.3.** $\ell_2(x) = x$. In this case, the difference equation (1.3) can be written as

$$
(3.18) \quad x \Delta \nabla y + [a - (1 - \mu)x] \Delta y = -n(1 - \mu)y.
$$

Then the corresponding discrete moment equations are

$$
(3.19) \quad \Delta(x \sigma) = [a - (1 - \mu)x] \sigma
$$

$$
(3.20) \quad \Delta(x \tau) = [a + \mu - (1 - \mu)x] \tau.
$$

The equation (3.19) has no quasi-definite moment functional solution if and only if $\mu = 1$ or $a = -\mu n$ for some $n \in \{0, 1, \cdots \}$. The equation (3.20) has a quasi-definite moment functional solution if and only if $\mu \neq 1$ and $a \neq -\mu(n + 1)$ for all $n \in \{0, 1, \cdots \}$. Hence, there is only one case: $a = 0$ and $\mu \neq 0, 1$.

**Theorem 3.8.** If $a = 0$ and $\mu \neq 0, 1$, then the difference equation (3.18) always has a unique DSOPS $\{m_n^{(0,\mu)}(x)\}_{n=0}^\infty$ as solutions, which are orthogonal relative to

$$
\phi_A(p, q) = A p(0) q(0) + \langle W^{(1,\mu)}(x), \Delta p \Delta q \rangle,
$$

where $A$ is any non-zero constant and $W^{(1,\mu)}(x)$ is the canonical moment functional of the monic discrete classical OPS $\{m_n^{(1,\mu)}(x)\}_{n=0}^\infty$ with $\langle W^{(1,\mu)}(x), 1 \rangle = 1$. 
Proofs of Theorem 3.7 and Theorem 3.8 are essentially the same as that of Theorem 3.6.

Case C.4. $\ell_2(x) = 1$. In this case, the difference equation (1.3) can be written as

$$\Delta \nabla y + (ax + b)\Delta y = -ny.$$  \hfill (3.21)

Then the corresponding discrete moment equations are

$$\Delta \sigma = (ax + b)\sigma$$ \hfill (3.22)

$$\Delta \tau = (ax + a + b)\tau.$$ \hfill (3.23)

The equation (3.22) has no quasi-definite moment functional solution if and only if $a - 1 \in \mathbb{Z}^-$. The equation (3.23) has a quasi-definite moment functional solution if and only if $a \notin \mathbb{Z}^-$. Hence, there is only one case: $a = 0$, for which the difference equation (3.21) has no polynomial solution of degree 1.

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