CODIMENSION REDUCTION
FOR REAL SUBMANIFOLDS OF
QUATERNIONIC PROJECTIVE SPACE

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Abstract. In this paper we prove a reduction theorem of the codimension for real submanifold of quaternionic projective space as a quaternionic analogue corresponding to those in Cecil [4], Erbacher [5] and Okumura [9], and apply the theorem to quaternionic CR-submanifold of quaternionic projective space.

1. Introduction

In general, it is very hard to classify submanifolds immersed in a Riemannian manifold even though the ambient manifold is specified, and so the so-called codimension reduction problem is sometimes very important role in the theory of submanifolds.

The codimension reduction problem was investigated by Allendoerfer [1] in the case that the ambient manifold is a Euclidean space and by Erbacher [5] in the case that the ambient manifold is a real space form. On the other hand, as a complex analogue for submanifold of complex projective space, Cecil [4] proved a codimension reduction theorem for complex submanifold and Okumura [9] a theorem corresponding to those in [4] and [5] for real submanifold.

In this paper we prove a quaternionic analogue for real submanifold of quaternionic projective space which may correspond to those in [4], [5] and [9]. We mainly follow Okumura’s method in his paper [9].

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2. Preliminaries

Let $\overline{M}$ be a real $(n + p)$-dimensional quaternionic Kähler manifold. Then, by definition, there is a 3-dimensional vector bundle $V$ consisting with tensor fields of type (1,1) over $\overline{M}$ satisfying the following conditions (a), (b) and (c):

(a) In any coordinate neighborhood $\overline{U}$, there is a local basis $\{F, G, H\}$ of $V$ such that

$$
\begin{cases}
F^2 = -I, & G^2 = -I, & H^2 = -I, \\
\tilde{F}G = -GF = H, & GH = -HG = F, & HF = -FH = G.
\end{cases}
$$

(b) There is a Riemannian metric $g$ which satisfies the Hermitian property with respect to all of $F$, $G$ and $H$.

(c) For the Levi-Civita connection $\overline{\nabla}$ with respect to $g$

$$
\begin{pmatrix}
\nabla F \\
\nabla G \\
\nabla H
\end{pmatrix} = 
\begin{pmatrix}
0 & r & -q \\
-r & 0 & p \\
q & -p & 0
\end{pmatrix}
\begin{pmatrix}
F \\
G \\
H
\end{pmatrix}
$$

where $p$, $q$ and $r$ are local 1-forms defined in $\overline{U}$. Such a local basis $\{F, G, H\}$ is called a canonical local basis of the bundle $V$ in $\overline{U}$ (cf. [6,7]).

For canonical local bases $\{F, G, H\}$ and $\{F', G', H'\}$ of $V$ in coordinate neighborhoods $\overline{U}$ and $\overline{U}'$ respectively, it follows from (2.1) that in $\overline{U} \cap \overline{U}'$

$\begin{pmatrix}
F' \\
G' \\
H'
\end{pmatrix} = (s_{xy})
\begin{pmatrix}
F \\
G \\
H
\end{pmatrix} 
(x, y = 1, 2, 3)$

with differentiable functions $s_{xy}$, where the matrix $S = (s_{xy})$ is contained in $SO(3)$. As is well known, every quaternionic Kähler manifold is orientable (cf. [6,7]).

Let $M$ be an $n$-dimensional submanifold isometrically immersed in $\overline{M}$ and let $i$ the isometric immersion. Then, for any tangent vector field $X$ and normal vector field $\xi$ to $M$, we have the following decompositions in tangential and normal components (In what follows we will delete $i$ and its differential $i_*$ in our notation):

$$
(2.4) \quad FX = \phi X + u(X), \quad GX = \psi X + v(X), \quad HX = \theta X + w(X),
$$

$$
(2.5) \quad F\xi = -U_\xi + P_F\xi, \quad G\xi = -V_\xi + P_G\xi, \quad H\xi = -W_\xi + P_H\xi.
$$
Then $\phi$, $\psi$ and $\theta$ are skew-symmetric endomorphisms acting on the tangent bundle $TM$, and $P_F$, $P_G$ and $P_H$ define those on the normal bundle $TM^\perp$. Also $u$, $v$ and $w$ are normal bundle valued 1-forms on $TM$. It is easily verified that

$$
(2.6) \quad g(X, U_\xi) = g(u(X), \xi), \quad g(X, V_\xi) = g(v(X), \xi), \\
\quad \quad \quad \quad g(X, W_\xi) = g(w(X), \xi)
$$

for any $X \in TM$, $\xi \in TM^\perp$, where and in the sequel we denote the induced metric form that of $\bar{M}$ by the same letter $g$. Applying $F$ to the first equation of (2.4) and using (2.1), (2.4) and (2.5), we have

$$
\phi^2 X = -X + U_u(x), \quad u(\phi X) = -P_F u(X).
$$

Similarly we have

$$
(2.7) \quad \phi^2 X = -X + U_u(x), \quad \psi^2 X = -X + V_v(x), \quad \theta^2 X = -X + W_w(x), \\
(2.8) \quad u(\phi X) = -P_F u(X), \quad v(\psi X) = -P_G v(X), \quad w(\theta X) = -P_H w(X).
$$

Next, applying $G$ and $H$ to the first equation of (2.4), respectively and using (2.1), (2.4) and (2.5), we have

$$
\theta X + w(X) = -\psi(\phi X) - v(\phi X) + V_v(x) - P_G u(X), \\
\psi X + v(X) = \theta(\phi X) + w(\phi X) - W_w(x) + P_H u(X),
$$

and consequently

$$
(2.9) \quad \psi \phi X = -\theta X + V_u(x), \quad v(\phi X) + P_G u(X) = -w(X), \\
(2.10) \quad \theta \phi X = \psi X + W_u(x), \quad w(\phi X) + P_H u(X) = v(X).
$$

Similarly we have from the other equations of (2.4)

$$
(2.11) \quad \phi \psi X = \theta X + U_v(x), \quad u(\psi X) + P_F v(X) = w(X), \\
(2.12) \quad \theta \psi X = -\phi X + W_v(x), \quad w(\psi X) + P_H v(X) = -u(X), \\
(2.13) \quad \phi \theta X = -\psi X + U_w(x), \quad u(\theta X) + P_F w(X) = -v(X), \\
(2.14) \quad \psi \theta X = \phi X + V_w(x), \quad v(\theta X) + P_G w(X) = u(X).
$$
By the quite similar method as above, we have from (2.5) that

\begin{align}
(2.15) & \quad P_F^{-2} \xi = -\xi + u(U_\xi), \quad P_G^{-2} \xi = -\xi + v(V_\xi), \\
& \quad P_H^{-2} \xi = -\xi + w(W_\xi), \\
(2.16) & \quad \phi(U_\xi) = -U_{P_F \xi}, \quad \psi(V_\xi) = -V_{P_G \xi}, \quad \theta(W_\xi) = -W_{P_H \xi}, \\
(2.17) & \quad W_\xi = -V_{P_F \xi} - \psi U_\xi, \quad P_G P_F \xi = -P_H \xi + v(U_\xi), \\
(2.18) & \quad V_\xi = W_{P_F \xi} + \theta U_\xi, \quad P_H P_F \xi = P_G \xi + w(U_\xi), \\
(2.19) & \quad W_\xi = U_{P_G \xi} + \phi V_\xi, \quad P_F P_G \xi = P_H \xi + u(V_\xi), \\
(2.20) & \quad U_\xi = -W_{P_G \xi} - \theta V_\xi, \quad P_H P_G \xi = -P_F \xi + w(V_\xi), \\
(2.21) & \quad V_\xi = -U_{P_H \xi} - \phi W_\xi, \quad P_F P_H \xi = -P_G \xi + u(W_\xi), \\
(2.22) & \quad U_\xi = V_{P_H \xi} + \psi W_\xi, \quad P_G P_H \xi = P_F \xi + v(W_\xi). 
\end{align}

We denote by $\nabla$ the Levi-Civita connection of $M$ and by $\nabla^\perp$ the normal connection induced from $\nabla$ to $TM^\perp$. Then they are related by the Gauss and Weingarten equations (In what follows we will again delete $i$ and its differential $i^*$ in our notation):

\begin{align}
(2.23) & \quad \nabla_X Y = \nabla_X Y + h(X,Y), \\
(2.24) & \quad \nabla_X \xi = -A_\xi X + \nabla_X^\perp \xi,
\end{align}

where $h$ is the second fundamental form and $A_\xi$ the shape operator with respect to the normal vector field $\xi$.

Differentiating the first equation of (2.4) covariantly and using (2.2), (2.4), (2.5), (2.23) and (2.24), we have

\begin{align}
(2.25) & \quad (\nabla_Y \phi) X = r(Y) \psi X - q(Y) \theta X - U_{h(Y,X)} + A_u(X) Y, \\
& \quad (\ast \nabla_Y u) X = r(Y) u(X) - q(Y) w(X) - h(Y, \phi X) + P_F h(Y,X),
\end{align}

where $\ast (\nabla_Y u) X$ is defined by $(\nabla_Y u) X = \nabla_Y^\perp u(X) - u(\nabla_Y X)$.

Similarly, from the other equations of (2.4), we have

\begin{align}
(2.26) & \quad (\nabla_Y \psi) X = p(Y) \theta X - r(Y) \phi X - V_{h(Y,X)} + A_v(X) Y, \\
& \quad (\ast \nabla_Y v) X = p(Y) w(X) - r(Y) u(X) - h(Y, \psi X) + P_G h(Y,X),
\end{align}
\[(\nabla_Y \theta)X = q(Y)\phi X - p(Y)\psi X - W_{h(Y, X)} + A_w(X)Y, \quad (2.27)\]
\[*(\nabla_Y w)X = q(Y)u(X) - p(Y)v(X) - h(Y, \theta X) + P_H h(Y, X),\]

where \((\nabla_Y v)X = \nabla^{\perp}_X v(X) - v(\nabla_Y X)\) and \((\nabla_Y w)X = \nabla^{\perp}_X w(X) - w(\nabla_Y X)\).

Next, differentiating the first equation of (2.5) covariantly and making use of (2.2), (2.4), (2.5), (2.23) and (2.24), we have
\[\nabla_Y U_\xi = r(Y)V_\xi - q(Y)W_\xi + \phi A_\xi Y - A_{PF_\xi} Y + U_{\nabla^{\perp}_Y \xi}, \quad (2.28)\]
\[\nabla^{\perp}_Y P_F)\xi = r(Y)P_G Y - q(Y)P_H Y - u(A_\xi Y) + h(Y, U_\xi),\]

where \((\nabla^{\perp}_Y P_F)\xi\) is defined by \((\nabla^{\perp}_Y P_F)\xi = \nabla^{\perp}_X (P_F \xi) - P_F (\nabla^{\perp}_Y \xi)\).

Similarly, from the other equations of (2.5), we have
\[\nabla_Y V_\xi = -r(Y)U_\xi + p(Y)W_\xi + \psi A_\xi Y - A_{PG_\xi} Y + V_{\nabla^{\perp}_Y \xi}, \quad (2.29)\]
\[\nabla^{\perp}_Y P_G)\xi = -r(Y)P_F \xi + p(Y)P_H \xi - v(A_\xi Y) + h(Y, V_\xi),\]
\[\nabla_Y W_\xi = q(Y)U_\xi - p(Y)V_\xi + \theta A_\xi Y - A_{PH_\xi} Y + W_{\nabla^{\perp}_Y \xi}, \quad (2.30)\]
\[\nabla^{\perp}_Y P_H)\xi = q(Y)P_F \xi - p(Y)P_G \xi - w(A_\xi Y) + h(Y, W_\xi),\]

where \((\nabla^{\perp}_Y P_G)\xi = \nabla^{\perp}_Y (P_G \xi) - P_G (\nabla^{\perp}_Y \xi)\) and \((\nabla^{\perp}_Y P_H)\xi = \nabla^{\perp}_Y (P_H \xi) - p_H (\nabla^{\perp}_Y \xi)\).

3. Quaternionically holomorphic first normal space

Let \(N_0(x) := \{\xi \in T_x M \perp | A_\xi = 0\}\}. The first normal space \(N_1(x)\) is defined to be the orthogonal complement to \(N_0(x)\) in \(T_x M \perp\). We put
\[H_0(x) := N_0(x) \cap FN_0(x) \cap GN_0(x) \cap HN_0(x).\]
Then \(H_0(x)\) is the maximal quaternionically invariant (or briefly \(Q\)-invariant) subspace of \(N_0(x)\), that is,
\[FH_0(x) \subset H_0(x), \quad GH_0(x) \subset H_0(x), \quad HH_0(x) \subset H_0(x).\]

Since \(F, G\) and \(H\) are isomorphisms, it is clear that
\[FH_0(x) = H_0(x), \quad GH_0(x) = H_0(x), \quad HH_0(x) = H_0(x).\]
Taking account of (2.5), we can easily verify...
Lemma 3.1. For any \( \xi \in H_0(x) \), we have

\[
A_\xi = 0 \quad \text{and} \quad U_\xi = V_\xi = W_\xi = 0.
\]

Definition. The quaternionically holomorphic (or Q-holomorphic) first normal space \( H_1(x) \) is the orthogonal complement of \( H_0(x) \) in \( T_xM^\perp \).

By definition, it is clear that \( N_1(x) \subset H_1(x) \) in \( T_xM^\perp \). Moreover we have

Lemma 3.2. If \( M \) is a Q-invariant submanifold of a quaternionic Kähler manifold, then \( H_1(x) = N_1(x) \).

Proof. Since \( H_1(x) \) and \( N_1(x) \) are the orthogonal complements of \( H_0(x) \) and \( N_0(x) \), respectively, we have only to show that \( H_0(x) = N_0(x) \). Since \( T_xM^\perp \) is Q-invariant, it follows from (2.2), (2.23) and (2.24) that

\[
\overline{\nabla}_X(F\xi) = r(X)G\xi - q(X)H\xi + F(\overline{\nabla}_X^\perp \xi - A_\xi X) \\
= -A_{F\xi}X + \nabla_X^\perp (F\xi)
\]

and consequently \( A_{F\xi}X = FA_\xi X \). Similarly we have \( A_{G\xi}X = GA_\xi X \) and \( A_{H\xi}X = HA_\xi X \). Thus, if \( \xi \in N_0(x) \), then

\[
A_{F\xi} = 0 \quad \text{and} \quad \xi \in FN_0(x), \quad A_{G\xi} = 0 \quad \text{and} \quad \xi \in GN_0(x), \\
A_{H\xi} = 0 \quad \text{and} \quad \xi \in HN_0(x).
\]

This shows that \( \xi \in N_0(x) \) implies \( \xi \in H_0(x) \), which completes the proof. \( \square \)

Lemma 3.3. Let \( H(x) \) be a Q-invariant subspace of \( H_0(x) \) and \( H_2(x) \) its orthogonal complement in \( T_xM^\perp \). Then \( T_xM \oplus H_2(x) \) is a Q-invariant subspace of \( T_x\overline{M} \).
Proof. We first note that
\[ T_x \overline{M} = T_x M \oplus H_2(x) \oplus H(x). \]
Since \( F H(x) = H(x) \), for any \( \xi \in H(x) \) there exists \( \eta \in H(x) \) such that \( F \eta = \xi \). Now let \( Z \in T_x M \oplus H_2(x) \). Then for any \( \xi \in H(x) \),
\[ \langle FZ, \xi \rangle = \langle Z, \eta \rangle = 0. \]
This means that \( FZ \in T_x M \oplus H_2(x) \). By quite similar method, we can verify that \( T_x M \oplus H_2(x) \) is \( Q \)-invariant. This completes the proof. \( \square \)

Now we recall that an \((n+p+3)\)-dimensional sphere \( S^{n+p+3} \) of radius 1 in a Euclidean \((n+p+4)\)-space is a principal \( S^3 \)-bundle over \( QP^{n+4} \). Then the Hopf-fibration \( \overline{\pi} : S^{n+p+3} \rightarrow QP^{n+4} \) defines a Riemannian submersion. We construct the \( S^3 \)-bundle over the submanifold \( M \) in such a way that the diagram
\[
\begin{array}{ccc}
\pi^{-1}(M) & \xrightarrow{i} & S^{n+p+3} \\
\downarrow & & \downarrow \overline{\pi} \\
M & \xrightarrow{i} & QP^{n+4}
\end{array}
\]
is commutative (\( i, \overline{i} \) being the isometric immersions). We denote by \( X^* \) the horizontal lift of \( X \in TM \) and by \( \xi^* \) that of the normal vector field \( \xi \in TM^\perp \). We put
\[ N'_0(x') = \{ \xi' \in T_{x'} \pi^{-1}(M)^\perp \mid A_{\xi'} = 0 \}, \quad x' \in \pi^{-1}(M), \]
where \( A_{\xi'} \) denotes the shape operator with respect to the normal vector field \( \xi' \) to \( \pi^{-1}(M) \). Then, as shown in [8], we have
\[ N'_0(x') = \{ \xi^* \mid A_\xi = 0, \quad U_\xi = V_\xi = W_\xi = 0 \}. \tag{3.1} \]

Applying the reduction theorem due to Erbacher [5], we prove

Theorem 3.4. Let \( M \) be an \( n \)-dimensional real submanifold of a real \((n+p)\)-dimensional quaternionic projective space \( QP^{n+4} \) and let \( H(x) \) a \( Q \)-invariant subspace of \( H_0(x) \). If the orthogonal complement \( H_2(x) \) of \( H(x) \) in \( T_x M^\perp \) is invariant under parallel translation with respect to the normal connection and \( q \) is the constant dimension of \( H_2 \), then there exists a real \((n+q)\)-dimensional totally geodesic quaternionic projective subspace \( QP^{n+4} \) such that \( M \subset QP^{n+4} \).
Proof. Let $\xi \in H(x)$. Then $\xi \in H_0(x)$, which and Lemma 3.1 give

$$A_\xi = 0 \quad \text{and} \quad U_\xi = V_\xi = W_\xi = 0$$

and consequently $A'_{\xi*} = 0$ because of (3.1). This means that for a point $x'$ with $\pi(x') = x$

$$H(x)^* = \{\xi^* \mid \xi \in H(x)\} \subset N_0(x').$$

Hence the orthogonal complement $H_2(x)^* = \{\xi^* \mid \xi \in H_2(x)\}$ of $H(x)^*$ in $T_{x'}(\pi^{-1}(M))^\perp$ is a subspace of $T_{x'}(\pi^{-1}(M))^\perp$ such that $H'_2(x') \subset H_2(x)^*$. Since $H_2(x)$ is invariant under parallel translation with respect to the normal connection, so does $H(x)$. This shows that for any $\xi \in H(x)$, $\nabla_{\hat{\xi}}^h \xi \in H(x)$, which and

$$\nabla_{\hat{\xi}}^{h*} \xi^* = (\nabla_{\hat{\xi}}^{h*} \xi^* \in H(x)^*), \quad \nabla_{\hat{\xi}}^{h*} \xi^* = -(F \xi)^* \in H(x)^*,$$

$$\nabla_{\hat{\xi}}^{h*} \xi^* = -(G \xi)^* \in H(x)^*), \quad \nabla_{\hat{\xi}}^{h*} \xi^* = -(H \xi)^* \in H(x)^*$$

imply that $H(x)^*$ is invariant under parallel translation with respect to the normal connection $\nabla^h$ of $\pi^{-1}(M)$. From the reduction theorem ([5], p. 339), we know that there exists a totally geodesic submanifold $S^{n+q+3}$ such that $\pi^{-1}(M) \subset S^{n+q+3}$. Let $\hat{U}(x')$ be a neighborhood of a point $x'$ with $\pi(x') = x$. Then the tangent space $T_{y'}(S^{n+q+3})$ of the totally geodesic submanifold at $y' \in \hat{U}(x')$ is

$$T_{y'}(\pi^{-1}(M)) \oplus H_2(y)^* = (T_y M \oplus H_2(y))^* \oplus T_{y'}(\pi^{-1}(y)),$$

where $y = \pi(y')$. Since $S^{n+q+3}$ is totally geodesic in $S^{n+p+3}$, the maximal integral submanifold $S^3$ of the distribution $y' \mapsto T_{y'}(\pi^{-1}(y))$ is a 3-dimensional great sphere of $S^{n+p+3}$. Hence the Hopf-fibration $S^{n+q+3} \to QP^{n+q}_4$ by $S^3$ is compatible with the Hopf-fibration $\tilde{\pi} : S^{n+p+3} \to QP^{n+p}_4$ and the tangent space of $QP^{n+q}_4$ at $x$ is $T_x M \oplus H_2(x)$. Moreover, by Lemma 3.3, $QP^{n+q}_4$ is $Q$-invariant in $QP^{n+p}_4$. This completes the proof. \qed

For a $Q$-invariant submanifold, by Lemma 3.2 we see that $H_0(x) = N_0(x)$ at any $x$ in $M$. Thus we have
Corollary 3.5. Let $M$ be a real $n$-dimensional $Q$-invariant submanifold of $QP^{n+\mathbb{R}}$. Assume that a $Q$-invariant subspace of the first normal space $N_1(x)$ has constant dimension $q$ and is invariant under parallel translation with respect to the normal connection. Then there exists a totally geodesic real $(n+q)$-dimensional quaternionic projective space $QP^{n+3}$ such that $M \subset QP^{n+3}$.

4. Quaternionic CR-submanifolds

In this section, let $M$ be an $n$-dimensional real submanifold of a quaternionic Kähler manifold, if there is a $Q$-invariant distribution $\mathcal{D}: x \mapsto \mathcal{D}_x \subset T_x M$ such that its complementary orthogonal distribution $\mathcal{D}^\perp: x \mapsto \mathcal{D}_x^\perp$ in $TM$ is anti-quaternionic, that is,

$$FD_x^\perp \subset T_x M^\perp, \quad GD_x^\perp \subset T_x M^\perp, \quad HD_x^\perp \subset T_x M^\perp,$$

then $M$ is called a \textit{quaternionic CR-submanifold} ([2,3]). In particular, if $\dim \mathcal{D}_x = 0$ for any $x$ in $M$, the quaternionic CR-submanifold is called an \textit{anti-quaternionic submanifold} ([3,10]).

Let $M$ be a quaternionic CR-submanifold of a quaternionic Kähler manifold $\overline{M}$. Then, by definition, the tangent space $T_x \overline{M}$ at $x$ in $M$ is decomposed as

$$(4.1) \quad T_x \overline{M} = T_x M \oplus FD_x^\perp \oplus GD_x^\perp \oplus HD_x^\perp \oplus N_x M,$$

where $N_x M$ is the orthogonal complement of $FD_x^\perp \oplus GD_x^\perp \oplus HD_x^\perp$ in $T_x M^\perp$.

Lemma 4.1. $N_x M$ is a $Q$-invariant, that is,

$$FN_x M \subset N_x M, \quad GN_x M \subset N_x M, \quad HN_x M \subset N_x M.$$

Proof. Let $X \in T_x M \oplus FD_x^\perp \oplus GD_x^\perp \oplus HD_x^\perp$ and $\xi \in N_x M$. Since $X$ is decomposed as

$$X = X_1 + X_2 + FY_1 + GY_2 + HY_3$$
for some $X_1 \in D_x$ and $X_2, Y_1, Y_2, Y_3 \in D_x^\perp$, it is clear that
\[
\langle X, F\xi \rangle = -\langle FX, \xi \rangle \\
= -\langle FX_1, \xi \rangle - \langle FX_2, \xi \rangle + \langle Y_1, \xi \rangle + \langle Y_2, \xi \rangle + \langle Y_3, \xi \rangle \\
= 0.
\]
Similarly we have $\langle X, G\xi \rangle = \langle X, H\xi \rangle = 0$, and consequently
\[
FN_xM \subset N_xM, \quad GN_xM \subset N_xM, \quad HN_xM \subset N_xM.
\]
This completes the proof.

\textbf{Lemma 4.2.} Assume that $NM$ is invariant under parallel translation with respect to the normal connection. Then, for any $\xi \in NM$ and $\eta \in TM^\perp$, \[
A_\xi U_\eta = 0, \quad A_\xi V_\eta = 0, \quad A_\xi W_\eta = 0.
\]

\textit{Proof.} By means of Lemma 4.1 and our assumption, it follows that for any $\xi \in NM$
\begin{align*}
F\xi &= P_F\xi, \quad G\xi = P_G\xi, \quad H\xi = P_H\xi, \quad \nabla^\perp_X\xi, \\
F\nabla^\perp_X\xi &= P_F\nabla^\perp_X\xi, \quad G\nabla^\perp_X\xi = P_G\nabla^\perp_X\xi, \quad H\nabla^\perp_X\xi = P_H\nabla^\perp_X\xi
\end{align*}
are all contained in $NM$. Differentiating the first three equations of those covariantly, we have
\begin{equation}
\nabla_X(F\xi) = \nabla_X(P_F\xi) = -A_{P_F\xi}X + \nabla^\perp_X(P_F\xi),
\end{equation}
\begin{equation}
\nabla_X(G\xi) = \nabla_X(P_G\xi) = -A_{P_G\xi}X + \nabla^\perp_X(P_G\xi),
\end{equation}
\begin{equation}
\nabla_X(H\xi) = \nabla_X(P_H\xi) = -A_{P_H\xi}X + \nabla^\perp_X(P_H\xi).
\end{equation}
Also we have
\begin{equation}
\nabla_X(F\xi) = r(X)P_G\xi - q(X)P_H\xi - \phi A_\xi X - u(A_\xi X) + P_F\nabla^\perp_X\xi,
\end{equation}
\begin{equation}
\nabla_X(G\xi) = p(X)P_H\xi - r(X)P_F\xi - \psi A_\xi X - v(A_\xi X) + P_G\nabla^\perp_X\xi,
\end{equation}
\begin{equation}
\nabla_X(H\xi) = q(X)P_F\xi - p(X)P_G\xi - \theta A_\xi X - w(A_\xi X) + P_H\nabla^\perp_X\xi.
\end{equation}
We notice that \( U_\zeta = V_\zeta = W_\zeta = 0 \) for any \( \zeta \in NM \). Consequently (2.6) implies
\[
u(X), \ v(X), \ w(X) \in FD^\perp \oplus GD^\perp \oplus HD^\perp
\]
for any \( X \) in \( TM \). Comparing the normal parts of (4.2) and (4.3), we have
\[
u(A_\xi X) = 0, \quad v(A_\xi X) = 0, \quad w(A_\xi X) = 0.
\]
Thus for any \( \eta \in TM^\perp \)
\[
g(A_\xi U_\eta, X) = 0, \quad g(A_\xi V_\eta, X) = 0, \quad g(A_\xi W_\eta, X) = 0
\]
and consequently \( A_\xi U_\eta = A_\xi V_\eta = A_\xi W_\eta = 0 \). This completes the proof. \( \square \)

**Theorem 4.3.** Let \( M \) be an \( n \)-dimensional anti-quaternionic submanifold of \( QP^{n+\nu} \). If \( NM \) is invariant under parallel translation with respect to the normal connection, then there exists a real \( 4n \)-dimensional totally geodesic quaternionic projective space \( QP^n \) of \( QP^{n+\nu} \) such that \( M \) is an anti-quaternionic submanifold of \( QP^n \).

**Proof.** Since \( M \) is anti-quaternionic, the tangential parts of (4.3) vanish identically. Comparing the tangential parts of (4.2) and (4.3), we have
\[
A_{PF_\xi} \xi = 0, \quad A_{PG_\xi} \xi = 0, \quad A_{PH_\xi} \xi = 0
\]
for all \( \xi \) in \( NM \). But, in an anti-quaternionic submanifold, \( P_F, P_G \) and \( P_H \) are all isomorphisms on \( NM \) and consequently \( A_\xi = 0 \) for any \( \xi \) in \( NM \). Thus by means of Lemma 4.1 \( NM \subset H_0M \).

Conversely, let \( \xi \in H_0(x) \). Then for any \( X, Y_1, Y_2, Y_3 \in T_xM \), we have
\[
\langle \xi, X + FY_1 + GY_2 + HY_3 \rangle = 0
\]
since \( H_0(x) \) is \( Q \)-invariant. Thus \( \xi \) belongs to the orthogonal complement of \( T_xM \oplus FT_xM \oplus GT_xM \oplus HT_xM \), that is, \( \xi \in N_xM \). Hence \( NM \subset H_0M \) and consequently \( FT_xM \oplus GT_xM \oplus HT_xM \) is the \( Q \)-holomorphic first normal space. Applying Theorem 3.4, we conclude that there is a real \( 4n \)-dimensional totally geodesic quaternionic projective space \( QP^n \) of \( QP^{n+\nu} \) such that \( M \) is anti-quaternionic in \( QP^n \). \( \square \)
In [8] and [11] it was already proved that the normal connection of 
\( \pi^{-1}(M) \) in \( S^{n+p+3} \) is flat if and only if the following conditions are 
satisfied on \( M \):

\[
(a) \quad R^\perp(X,Y)\xi = -2g(\phi X,Y)P_F\xi - 2g(\psi X,Y)P_G\xi \\
\quad - 2g(\theta X,Y)P_H\xi,
\]

(4.4)

(b) The structure induced on the normal bundle is parallel 
(for the definition, see [11]).

In this sense the normal connection of \( M \) in \( QP^{n+2}_{n+2} \) is said to be lift flat 
if the conditions (a) and (b) are valid.

**Lemma 4.4.** Let \( M \) be a quaternionic \( CR \)-submanifold of \( QP^{n+2}_{n+2} \) 
with lift flat normal connection. Then \( A_\xi A_\eta = A_\eta A_\xi \) for \( \xi \in N_xM \) 
and \( \eta \in T_xM^\perp \).

**Proof.** Since the normal connection is lift flat, the equation of Ricci 
and (4.4) (a) implies

\[
0 = h(A_\xi X,Y) - h(A_\xi Y,X) + g(Y,U_\xi)u(X) - g(X,U_\xi)u(Y) \\
+ g(Y,V_\xi)v(X) - g(X,V_\xi)v(Y) + g(Y,W_\xi)w(X) - g(X,W_\xi)w(Y).
\]

In particular, for \( \xi \in N_xM, U_\xi = V_\xi = W_\xi = 0 \), and consequently

(4.5) \quad h(A_\xi X,Y) = h(A_\xi Y,X).

Hence, if \( \xi \in N_xM \) and \( \eta \in T_xM^\perp \), we have

\[
g((A_\eta A_\xi - A_\xi A_\eta)X,Y) \\
= g(A_\eta A_\xi X,Y) - g(A_\xi A_\eta X,Y) \\
= g(h(A_\xi X,Y),\eta) - g(h(A_\xi Y,X),\eta) = 0
\]

because of (4.5). This completes the proof. \(\square\)

**Theorem 4.5.** Assume that the normal connection of a quaternionic \( CR \)-submanifold \( M \) of \( QP^{n+2}_{n+2} \) is lift flat and that \( NM \) is invariant 
under parallel translation with respect to the normal connection. 
Then there is a totally geodesic quaternionic projective space \( QP^{n+2}_{n+2} \) 
such that \( M \) is a quaternionic \( CR \)-submanifold of the quaternionic projective space.
Proof. By means of Theorem 4.3, it suffices to show that \( NM = H_0 M \). We choose orthonormal normal vector fields \( \xi_1, \ldots, \xi_p \) in such a way that

\[
\xi_1, \ldots, \xi_q \in F\mathcal{D}^\perp \oplus G\mathcal{D}^\perp \oplus H\mathcal{D}^\perp, \quad \xi_{q+1}, \ldots, \xi_p \in NM
\]

\((q \text{ must be a multiple of three})\) and denote by \( A_{\alpha} \) the shape operator for \( \xi_{\alpha} \). Since \( NM \) is not only invariant under parallel translation with respect to the normal connection, but also \( Q \)-invariant, it follows that

\[
(4.6) \quad \nabla_X^\perp \xi_{\alpha} = \sum_{\lambda=q+1}^{p} s_{\alpha \lambda}(X)\xi_{\lambda}, \quad \alpha = q + 1, \ldots, p,
\]

\[
(4.7) \quad \begin{cases}
F\xi_{\alpha} = P_F\xi_{\alpha} = \sum_{\lambda=q+1}^{p} (P_F)_{\alpha \lambda} \xi_{\lambda}, \\
G\xi_{\alpha} = P_G\xi_{\alpha} = \sum_{\lambda=q+1}^{p} (P_G)_{\alpha \lambda} \xi_{\lambda}, \quad \alpha = q + 1, \ldots, p, \\
H\xi_{\alpha} = P_H\xi_{\alpha} = \sum_{\lambda=q+1}^{p} (P_H)_{\alpha \lambda} \xi_{\lambda},
\end{cases}
\]

from which we have

\[
(4.8) \quad \begin{cases}
F\nabla_X^\perp \xi_{\alpha} = \sum_{\lambda, \mu=q+1}^{p} s_{\alpha \lambda}(X)(P_F)_{\lambda \mu} \xi_{\mu}, \\
G\nabla_X^\perp \xi_{\alpha} = \sum_{\lambda, \mu=q+1}^{p} s_{\alpha \lambda}(X)(P_G)_{\lambda \mu} \xi_{\mu}, \quad \alpha = q + 1, \ldots, p. \\
H\nabla_X^\perp \xi_{\alpha} = \sum_{\lambda, \mu=q+1}^{p} s_{\alpha \lambda}(X)(P_H)_{\lambda \mu} \xi_{\mu},
\end{cases}
\]

On the other hand, comparing the tangential parts of (4.2) and (4.3), and using (4.7), we obtain

\[
(4.9) \quad \begin{cases}
\phi A_{\lambda} X = \sum_{\mu=q+1}^{p} (P_F)_{\lambda \mu} A_{\mu} X, \\
\psi A_{\lambda} X = \sum_{\mu=q+1}^{p} (P_G)_{\lambda \mu} A_{\mu} X, \quad \lambda \geq q + 1. \\
\theta A_{\lambda} X = \sum_{\mu=q+1}^{p} (P_H)_{\lambda \mu} A_{\mu} X,
\end{cases}
\]

Substituting \( A_{\lambda} X \) for \( X \) in (4.9) and summing over \( \lambda = q + 1, \ldots, p \), we have

\[
\phi \sum_{\lambda=q+1}^{p} A_{\lambda} X = \sum_{\lambda, \mu=q+1}^{p} (P_F)_{\lambda \mu} A_{\lambda} A_{\mu} X = 0
\]
because \((P_F)_{\lambda\mu}\) is skew-symmetric with respect to \(\lambda\) and \(\mu\), but \(A_{\lambda}A_{\mu} = A_{\mu}A_{\lambda}\) by Lemma 4.4. Thus we have

\[
\phi^2 \sum_{\lambda=q+1}^{p} A_{\lambda}^2 X = - \sum_{\lambda=q+1}^{p} A_{\lambda}^2 X + \sum_{\alpha=q+1}^{p} U_{u(A_{\alpha}^2 X)} = 0.
\]

However, as already shown in the proof of Lemma 4.2, \(u(A_{\alpha}^2 X) = 0\) and consequently \(\sum_{\lambda=q+1}^{n} A_{\lambda}^2 X = 0\), that is, \(A_{\lambda} = 0, \lambda \geq q+1\). Thus \(N_xM\) is a \(Q\)-invariant subspace of \(N_0M\). Since \(H_0M\) is maximal, it follows that \(N_xM \subset H_0M\). Let \(\xi \in H_0M\) and \(\eta \in F^\perp \oplus G^\perp \oplus H^\perp\). Then there exist \(Y_1, Y_2, Y_3 \in D^\perp \subset TM\) such that

\[
\eta = FY_1 + GY_2 + HY_3.
\]

Therefore it is clear that

\[
\langle \xi, \eta \rangle = -\langle F\xi, Y_1 \rangle - \langle G\xi, Y_2 \rangle - \langle H\xi, Y_3 \rangle = 0
\]

since \(H_0M\) is \(Q\)-invariant. This means that \(\xi \in NM\) and consequently \(NM = H_0M\). This completes the proof.

**Remark.** As already shown in the proof of Lemma 4.4, it suffices to assume only the condition (4.4)(a) instead of the condition "lift flatness" in order to prove Theorem 4.5.

**References**


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