PARALLEL BLOCK ILU PRECONDITIONERS FOR A BLOCK-TRIDIAGONAL M-MATRIX

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ABSTRACT. We propose new parallel block ILU (Incomplete LU) factorization preconditioners for a nonsymmetric block-tridiagonal M-matrix. Theoretical properties of these block preconditioners are studied to see the convergence rate of the preconditioned iterative methods. Lastly, numerical results of the right preconditioned GMRES and BiCGSTAB methods using the block ILU preconditioners are compared with those of these two iterative methods using a standard ILU preconditioner to see the effectiveness of the block ILU preconditioners.

1. Introduction

The discretization of partial differential equations in 2D or 3D, by finite difference or finite element approximation, leads often to large sparse block-tridiagonal linear systems. In this paper, we consider the linear system of equations

\[ Ax = b, \quad x, b \in \mathbb{R}^n \]

where \( A \in \mathbb{R}^{n \times n} \) is a large sparse nonsymmetric block-tridiagonal M-matrix blocked in the form

\[
A = \begin{pmatrix}
B_1 & -C_1 & 0 & \cdots & 0 \\
-E_1 & B_2 & -C_2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & -E_{m-2} & B_{m-1} & -C_{m-1} \\
0 & \cdots & 0 & -E_{m-1} & B_m
\end{pmatrix}
\]
It is assumed that the diagonal blocks $B_i$ of $A$ are square matrices, and $C_i$'s and $E_i$'s are nonnegative matrices. Since $A$ is a large sparse matrix, direct solvers become prohibitively expensive because of the large amount of work and storage required. As an alternative, we usually consider nonstationary iterative methods such as the GMRES [15], BCG [9], CGS [16], and BiCGSTAB [17]. Given an initial guess $x_0$, these algorithms compute iteratively new approximations $x_k$ to the true solution $x^* = A^{-1}b$. The iterate $x_k$ is accepted as a solution if the residual $r_k = b - Ax_k$ satisfies $\|r_k\|/\|b\| \leq \text{(Tolerance)}$. In general, the convergence is not guaranteed or may be extremely slow. Hence, the original problem (1) must be transformed into a more tractable form. To do so, we consider an easily invertible matrix $K$ called the preconditioning matrix or preconditioner and apply the iterative solvers either to the left preconditioned linear system $K^{-1}Ax = K^{-1}b$ or to the right preconditioned linear system $AK^{-1}y = b$, where $y = Kx$. The preconditioner $K$ should be chosen so that $K^{-1}A$ or $AK^{-1}$ is a good approximation to the identity matrix.

Since the ultimate goal of the preconditioned iterative methods is to reduce the total execution time, the computation of preconditioner $K$ should be done in parallel. One of the powerful preconditioning methods in terms of reducing the number of iterations is the incomplete LU (called ILU) factorization method studied by Meijerink and van der Vorst [12]. A detailed review for the ILU factorization method can be found in [3, 5, 10, 14]. However, it is very difficult to parallelize the ILU factorization algorithm because of the recursive nature of the computation. On the other hand, polynomial preconditioners defined by $K^{-1} = p(A)$, where $p$ is a polynomial, are easy to parallelize since they only involve the computation of matrix-vector operations, but they are not as powerful as the ILU factorization preconditioners. In order to make the ILU factorization method more suitable for vector computers and parallel architectures, incomplete block LU factorizations using matrix blocks as basic entities were proposed [1, 2, 6, 7, 10, 13, 14].

The block incomplete Cholesky factorization preconditioners for a symmetric block-tridiagonal M-matrix have been studied recently by Yun [18]. The purpose of this paper is to propose new parallel block

ILU factorization preconditioners for a nonsymmetric block-tridiagonal M-matrix which extend the ideas for symmetric problems introduced in [18]. The block ILU factorization preconditioners to be proposed in
this paper are quite different from the existing incomplete block LU factorization preconditioners which require the approximate inverses of pivot blocks (see [1, 2, 4, 7, 8, 13] for more details). In other words, the block ILU factorization preconditioners to be proposed in this paper are obtained by performing the standard ILU factorization on each matrix block independently, so that they have no block recurrence which requires sparse approximate inverses for pivot blocks and thus they can be computed in parallel based on matrix blocks.

In section 2, we review some properties of the ILU factorization on M-matrices. In section 3, we propose new block ILU factorization preconditioners for a nonsymmetric block-tridiagonal M-matrix and their theoretical properties are studied to see the convergence rate of the preconditioned nonstationary iterative methods. In section 4, we describe how to construct the effective block preconditioners for a special type of matrix which arises from five-point discretization of the second-order partial differential equation. In section 5, we present numerical results of the right preconditioned GMRES(ℓ) and BiCGSTAB methods with the block ILU factorization preconditioners developed in this paper, and their results are compared with those of the right preconditioned GMRES(ℓ) and BiCGSTAB with a standard ILU factorization preconditioner. Lastly, some conclusions are drawn.

2. ILU (Incomplete LU) factorizations

A general algorithm for building ILU factorizations for M-matrices can be derived by performing Gaussian elimination and dropping some elements in predetermined nondiagonal positions. To better understand the ILU factorization process for an M-matrix, we review some important results in this section. Let \( P_n \) denote the set of all pairs of indices of off-diagonal matrix entries, that is,

\[
P_n = \{(i, j) \mid i \neq j, \ 1 \leq i \leq n, \ 1 \leq j \leq n\}.
\]

A representation \( A = K - N \) is called a splitting of \( A \) when \( K \) is nonsingular. The spectral radius \( \rho(A) \) of a matrix \( A \) is \( \rho(A) = \max\{ |\lambda| \mid \lambda \in \sigma(A) \} \), where \( \sigma(A) \) denotes the spectrum of \( A \), that is, the set of eigenvalues of \( A \). For two matrices \( A = (a_{ij}) \) and \( B = (b_{ij}) \), \( A \preceq B \) denotes \( a_{ij} \leq b_{ij} \) for all \( i \) and \( j \), and \( A \succeq B \) denotes \( a_{ij} \geq b_{ij} \) for all \( i \) and \( j \). It can be easily shown that \( A \preceq B \) and \( C \succeq 0 \) implies \( AC \succeq BC \) and
$CA \geq CB$. A matrix $A = (a_{ij})$ is called an $M$-matrix if $a_{ij} \leq 0$ for $i \neq j$, $A$ is nonsingular, and $A^{-1} \geq 0$. A splitting $A = K - N$ is called a regular splitting of $A$ if $K$ is nonsingular, $K^{-1} \geq 0$, and $N \geq 0$. A splitting $A = K - N$ is a convergent splitting of $A$ if $K$ is nonsingular and $\rho(K^{-1}N) < 1$. It is well-known that if $A = K - N$ is a convergent splitting, then any stationary iterative method of the form

$$Kx_{k+1} = Nx_k + b, \quad k \geq 0$$

converges to the exact solution of $Ax = b$ for every choice of $x_0$. The following theorem shows the existence of the (standard) ILU factorization for an $M$-matrix.

**Theorem 2.1 ([12]).** Let $A$ be an $M$-matrix. Then, for every zero pattern set $P \subset P_n$, there exist a unit lower triangular matrix $L = (l_{ij})$, an upper triangular matrix $U = (u_{ij})$, and a matrix $R = (r_{ij})$, with $l_{ij} = u_{ij} = 0$ if $(i, j) \in P$ and $r_{ij} = 0$ if $(i, j) \notin P$, such that $A = LU - R$ is a regular splitting of $A$. The factor $L$ and $U$ are unique. Moreover, $L$ and $U$ are also $M$-matrices.

In Theorem 2.1, $A = LU - R$ is called an ILU factorization of $A$ corresponding to a zero pattern set $P \subset P_n$. In particular, if $P$ is an empty set, then $R = 0$ and thus a complete LU factorization of $A$ such that $A = LU$ is obtained. Lemma 2.2 in the following will be used to prove Lemma 3.1.

**Lemma 2.2.** Let $A$ and $B$ be $M$-matrices. If $A \leq B$, then $B^{-1} \leq A^{-1}$.

**Proof.** The matrix $A$ can be splitted into

$$A = A - 0 = B - (B - A),$$

where 0 denotes the zero matrix. Since $A$ and $B$ are $M$-matrices and $B - A \geq 0$, from Theorem 6.22 in [3] $B^{-1} \leq A^{-1}$. \hfill $\square$

The following theorem describes a relation between ILU factorizations of two different $M$-matrices.

**Theorem 2.3 ([18]).** Let $A$ and $B$ be $n \times n$ $M$-matrices, and let $A = L_1U_1 - R_1$ and $B = L_2U_2 - R_2$ be ILU factorizations corresponding to the same zero pattern set $P \subset P_n$. If $A \leq B$, then $L_2^{-1} \leq L_1^{-1}$ and $U_2^{-1} \leq U_1^{-1}$.

A comparison theorem for regular splittings which will be used for the proof of main results in section 3 is presented below.
THEOREM 2.4 ([3]). Let \( A = K_1 - N_1 = K_2 - N_2 \) be regular splittings of \( A \). If \( K_2^{-1} \leq K_1^{-1} \), then
\[
\rho(K_1^{-1}N_1) \leq \rho(K_2^{-1}N_2).
\]

3. Block ILU factorization preconditioners

We first consider block ILU factorization preconditioners for a non-symmetric block-tridiagonal \( M \)-matrix of the simplest form
\[
A = \begin{pmatrix}
B_1 & -C_1 \\
-E_1 & B_2
\end{pmatrix}.
\]
Since \( A \) is an \( M \)-matrix, \( B_1 \) and \( B_2 \) are \( M \)-matrices. From the ILU factorization process, we can find a unit lower triangular matrix \( L_i \), an upper triangular matrix \( U_i \), and a matrix \( R_i \) such that \( B_i = L_i U_i - R_i \) is a regular splitting of \( B_i \) for each \( i = 1, 2 \), see Theorem 2.1. If \( A = K - N \) is a splitting of \( A \) and \( K \) is a matrix which is easily invertible, then \( K \) can be used as a preconditioner for nonstationary iterative methods. The effectiveness of the preconditioner \( K \) depends on how well \( K \) approximates \( A \).

LEMMA 3.1. Let \( B_i = L_i U_i - R_i \) be a regular splitting of \( B_i \) which can be obtained by the ILU factorization process for each \( i = 1, 2 \), and let \( D_i = diag(U_i) \) be the diagonal matrix which is obtained by taking the diagonal part of \( U_i \) for each \( i = 1, 2 \). Then each of the following holds:

(a) \( L_i^{-1} \geq 0, \ U_i^{-1} \geq 0 \) for \( i = 1, 2 \).
(b) \( U_i^{-1} \geq D_i^{-1} \geq 0 \) for \( i = 1, 2 \).
(c) \( I - L_i \geq 0, \ I - D_i^{-1} U_i \geq 0 \) for \( i = 1, 2 \).
(d) \( L_i^{-1} - I \geq 0, \ (D_i^{-1} U_i)^{-1} - I \geq 0 \) for \( i = 1, 2 \).

Proof. For the proof of part (a), from Theorem 2.1, \( L_i \) and \( U_i \) are \( M \)-matrices. Hence, part (a) holds. For the proof of part (b), since \( U_i \) is an \( M \)-matrix, \( D_i = diag(U_i) \) is a diagonal matrix which has positive diagonal elements. Hence, \( D_i^{-1} \geq 0 \) for \( i = 1, 2 \). Also, since \( U_i \) and \( D_i \) are \( M \)-matrices and \( U_i \leq D_i, \ U_i^{-1} \geq D_i^{-1} \) from Lemma 2.2. Hence, part (b) follows. For the proof of part (c), since \( L_i \) is a unit lower triangular \( M \)-matrix, \( I - L_i \geq 0 \). Also, since \( D_i^{-1} U_i \) is a unit upper triangular matrix whose off-diagonal elements are nonpositive, \( I - D_i^{-1} U_i \geq 0 \) for
\( i = 1, 2 \). For the proof of part (d), since \( L_i \) is a unit lower triangular \( M \)-matrix, \( L_i^{-1} \) is a unit lower triangular matrix whose off-diagonal elements are nonnegative. Thus, \( L_i^{-1} - I \geq 0 \). Since \( U_i^{-1} \geq 0 \), \( (D_i^{-1}U_i)^{-1} \) is a unit upper triangular matrix whose off-diagonal elements are nonnegative. Hence, \( (D_i^{-1}U_i)^{-1} - I \geq 0 \) for \( i = 1, 2 \). \( \square \)

**Theorem 3.2.** Let \( A \) be a nonsymmetric \( M \)-matrix of the form (3), and let \( B_i = L_iU_i - R_i \) be a regular splitting of \( B_i \) which can be obtained by the ILU factorization process for each \( i = 1, 2 \). Let \( D_i = \text{diag}(U_i) \) be the diagonal matrix which is obtained by taking the diagonal part of \( U_i \) for each \( i = 1, 2 \). Let

\[
L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}, \quad U = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix},
\]

\[
L_\alpha = \begin{pmatrix} L_1 & 0 \\ -E_1D_1^{-1} & L_2 \end{pmatrix}, \quad U_\alpha = \begin{pmatrix} U_1 & -C_1 \\ 0 & U_2 \end{pmatrix},
\]

\[
L_\beta = \begin{pmatrix} L_1 & 0 \\ -E_1U_1^{-1} & L_2 \end{pmatrix}, \quad U_\beta = \begin{pmatrix} U_1 & -L_1^{-1}C_1 \\ 0 & U_2 \end{pmatrix}.
\]

If we let \( M = LU \), \( M_\alpha = L_\alpha U_\alpha \), and \( M_\beta = L_\beta U_\beta \), then each of the following holds:

(a) \( R = M - A \geq 0 \), \( R_\alpha = M_\alpha - A \geq 0 \), and \( R_\beta = M_\beta - A \geq 0 \).

(b) \( 0 \leq L^{-1} \leq L_\alpha^{-1} \leq L_\beta^{-1} \).

(c) \( 0 \leq U^{-1} \leq U_\alpha^{-1} \leq U_\beta^{-1} \).

(d) \( 0 \leq M^{-1} \leq M_\alpha^{-1} \leq M_\beta^{-1} \).

(e) \( A = M - R = M_\alpha - R_\alpha = M_\beta - R_\beta \) are regular splittings of \( A \).

(f) \( \rho(M_\beta^{-1}R_\beta^\beta) \leq \rho(M_\alpha^{-1}R_\alpha^\alpha) \leq \rho(M^{-1}R) < 1 \).

**Proof.** Since \( R = M - A \), \( R_\alpha = M_\alpha - A \), and \( R_\beta = M_\beta - A \), one obtains

\[
R = \begin{pmatrix} R_1 & C_1 \\ E_1 & R_2 \end{pmatrix},
\]

\[
R_\alpha = \begin{pmatrix} R_1 \\ E_1(I - D_1^{-1}U_1) \\ E_1D_1^{-1}C_1 + R_2 \end{pmatrix},
\]

\[
R_\beta = \begin{pmatrix} R_1 \\ 0 \\ E_1U_1^{-1}L_1^{-1}C_1 + R_2 \end{pmatrix}.
\]
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Since \( A \) is an \( M \)-matrix and \( R_i \geq 0 \) for \( i = 1, 2, R \geq 0 \). From Lemma 3.1, it follows that \( R^\alpha \geq 0 \) and \( R^\beta \geq 0 \). Hence, part (a) is proved. For the proof of part (b), if we compute inverse matrices of \( L \), \( L_\alpha \), and \( L_\beta \), then

\[
L^{-1} = \begin{pmatrix} L^{-1}_1 & 0 \\ 0 & L^{-1}_2 \end{pmatrix},
\]

\[
L_\alpha^{-1} = \begin{pmatrix} L^{-1}_1 & 0 \\ L^{-1}_2 E_1 D^{-1}_1 L^{-1}_1 & L^{-1}_2 \end{pmatrix},
\]

\[
L_\beta^{-1} = \begin{pmatrix} L^{-1}_1 & 0 \\ L^{-1}_2 E_1 L^{-1}_1 & L^{-1}_2 \end{pmatrix}.
\]

Using parts (a) and (b) of Lemma 3.1, part (b) is obtained. For the proof of part (c), if we compute inverse matrices of \( U \), \( U_\alpha \), and \( U_\beta \), then

\[
U^{-1} = \begin{pmatrix} U^{-1}_1 & 0 \\ 0 & U^{-1}_2 \end{pmatrix},
\]

\[
U_\alpha^{-1} = \begin{pmatrix} U^{-1}_1 & U^{-1}_1 C U_2^{-1} \\ 0 & U^{-1}_2 \end{pmatrix},
\]

\[
U_\beta^{-1} = \begin{pmatrix} U^{-1}_1 & U^{-1}_1 L^{-1}_1 C U_2^{-1} \\ 0 & U^{-1}_2 \end{pmatrix}.
\]

Using parts (a) and (d) of Lemma 3.1, part (c) is proved. Since \( M^{-1} = U^{-1} L^{-1} \), \( M_\alpha^{-1} = U_\alpha^{-1} L_\alpha^{-1} \), and \( M_\beta^{-1} = U_\beta^{-1} L_\beta^{-1} \), parts (b) and (c) imply part (d). From parts (a) and (d), part (e) is immediately obtained. Since \( A \) is an \( M \)-matrix and \( A = M - R \) is a regular splitting of \( A \), it is easy to show that \( \rho(M^{-1} R) < 1 \). Hence, from Theorem 2.4, part (f) is proved.

Many other types of the block ILU factorization preconditioners except those introduced in the above theorem have been discussed in [11].

Next, we consider block ILU factorization preconditioners for a nonsymmetric block-tridiagonal \( M \)-matrix of the general form (2). Generalization of Theorem 3.2 to an \( M \)-matrix of the form (2) is complicated but easy, so that the following theorem is described without proof:

**Theorem 3.3.** Let \( A \) be a nonsymmetric block-tridiagonal \( M \)-matrix of the form (2) and let \( B_i = L_i U_i - R_i \) be a regular splitting of \( B_i \) which can be obtained by the ILU factorization process for each \( i = 1, 2, \ldots, m \). Let \( D_i \) denote a diagonal matrix consisting of diagonal elements of \( U_i \)
for each \( i = 1, 2, \ldots, m \). Let

\[
L = \begin{pmatrix} L_1 & 0 & \cdots & 0 \\
0 & L_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & L_m 
\end{pmatrix},
U = \begin{pmatrix} U_1 & 0 & \cdots & 0 \\
0 & U_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & U_m 
\end{pmatrix},
\]

\[
L_\alpha = \begin{pmatrix} L_1 & 0 & \cdots & 0 \\
-E_1D_1^{-1} & L_2 & \cdots & 0 \\
0 & -E_2D_2^{-1} & L_3 & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -E_{m-1}D_{m-1}^{-1} & L_m 
\end{pmatrix},
\]

\[
U_\alpha = \begin{pmatrix} U_1 & -C_1 & 0 & \cdots & 0 \\
0 & U_2 & -C_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & U_{m-1} & -C_{m-1} \\
0 & 0 & 0 & 0 & U_m 
\end{pmatrix},
\]

\[
L_\beta = \begin{pmatrix} L_1 & 0 & \cdots & 0 \\
-E_1U_1^{-1} & L_2 & \cdots & 0 \\
0 & -E_2U_2^{-1} & L_3 & \cdots \\
\vdots & \vdots & \ddots & \ddots \\
0 & 0 & \cdots & -E_{m-1}U_{m-1}^{-1} & L_m 
\end{pmatrix},
\]

\[
U_\beta = \begin{pmatrix} U_1 & -L_1^{-1}C_1 & 0 & \cdots & 0 \\
0 & U_2 & -L_2^{-1}C_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & U_{m-1} & -L_{m-1}^{-1}C_{m-1} \\
0 & 0 & 0 & 0 & U_m 
\end{pmatrix},
\]

\[M = LU, \ M_\alpha = L_\alpha U_\alpha, \text{ and } M_\beta = L_\beta U_\beta.\] Then, each of the following holds:

(a) \( R = M - A \geq 0, \ \text{and} \ \ R_\alpha = M_\alpha - A \geq 0, \ \text{and} \ \ R_\beta = M_\beta - A \geq 0. \)

(b) \( 0 \leq L^{-1} \leq L_\alpha^{-1} \leq L_\beta^{-1}. \)

(c) \( 0 \leq U^{-1} \leq U_\alpha^{-1} \leq U_\beta^{-1}. \)

(d) \( 0 \leq M^{-1} \leq M_\alpha^{-1} \leq M_\beta^{-1}. \)

(e) \( A = M - R = M_\alpha - R_\alpha = M_\beta - R_\beta \) are regular splittings of \( A. \)

(f) \( \rho(M_\beta^{-1}R_\beta^{-1}) \leq \rho(M_\alpha^{-1}R_\alpha^{-1}) \leq \rho(M^{-1}R) < 1. \)
If \( B_i = L_i U_i - R_i \) is an ILU factorization of \( B_i \), then \( L_i U_i \) can be viewed as a complete LU factorization of \( B_i + R_i \) and the following equation holds

\[
\begin{pmatrix}
B_i + R_i & -C_i \\
-E_i & 0
\end{pmatrix} =
\begin{pmatrix}
L_i & 0 \\
-E_i U_i^{-1} & I
\end{pmatrix}
\begin{pmatrix}
U_i & -L_i^{-1} C_i \\
0 & -E_i (L_i U_i)^{-1} C_i
\end{pmatrix}.
\]

This equation shows that \( L_i^{-1} C_i \) and \( E_i U_i^{-1} \) required for the construction of \( L_\beta \) and \( U_\beta \) in Theorem 3.3 can be computed efficiently at the time when the ILU factorization of \( B_i \) is executed. In other words, since \( L_i^{-1} \) is a product of elementary lower triangular matrices which are generated during the ILU factorization process of \( B_i \), \( L_i^{-1} C_i \) is not computed explicitly using matrix-solve operations, but computed implicitly using elementary lower triangular matrices. Also, \( E_i U_i^{-1} \) can be computed implicitly without using matrix-solve operations because \( L_i \\
-E_i U_i^{-1}
\) can be computed during the Gaussian elimination process which transforms

\[
\begin{pmatrix}
B_i + R_i \\
-E_i
\end{pmatrix}
\]
to

\[
\begin{pmatrix}
U_i \\
0
\end{pmatrix}.
\]

Since \( L_i \)'s and \( U_i \)'s can be computed independently of one another, three types of the block ILU factorization preconditioners \( M \), \( M_a^\beta \), and \( M_\beta^\alpha \) presented in Theorem 3.3 can be computed in parallel. This inherent parallelism is a big advantage of the block ILU factorization preconditioners. In this paper, the right preconditioned iterative methods are used to test the effectiveness of the block preconditioners in Theorem 3.3.

**Lemma 3.4.** Let \( A = K - N \) be a splitting of \( A \). Then

(a) \( \sigma(K^{-1}N) = \sigma(N K^{-1}) \) and \( \rho(K^{-1}N) = \rho(N K^{-1}) \),

(b) \( \sigma(K^{-1}A) = \sigma(A K^{-1}) \) and \( \rho(K^{-1}A) = \rho(A K^{-1}) \).

**Proof.** Since there is a nonsingular matrix \( K \) such that \( NK^{-1} = K(K^{-1}N)K^{-1} \), \( K^{-1}N \) is similar to \( NK^{-1} \). Hence, \( \sigma(K^{-1}N) = \sigma(N K^{-1}) \) and thus \( \rho(K^{-1}N) = \rho(N K^{-1}) \). Since \( K^{-1}A = I - K^{-1}N \) and \( AK^{-1} = I - NK^{-1} \), part (b) is obtained from part (a). \( \square \)

Let \( A = K - N \) be a splitting of \( A \). Then, convergence rate of the right preconditioned iterative methods with the preconditioner \( K \) for solving \( Ax = b \) largely depends upon how clustered the eigenvalues of \( AK^{-1} \) are about 1. Note that \( \sigma(AK^{-1}) = \{ 1 - \lambda | \lambda \in \sigma(NK^{-1}) \} \). It follows that the smaller the value of \( \rho(NK^{-1}) \) is, the more clustered the
eigenvalues of $AK^{-1}$ are about 1. In Lemma 3.4, we have shown that $\rho(NK^{-1}) = \rho(K^{-1}N)$. Hence, we want to make $\rho(K^{-1}N)$ as small as possible to ensure fast convergence of the right preconditioned iterative methods. From this point of view, the right preconditioned iterative methods with the block ILU factorization preconditioner of type $M^g_\beta$ will converge to the exact solution faster than those with any other type of the block ILU factorization preconditioner presented in Theorem 3.3.

4. Applications of block ILU factorization preconditioners

The construction of three types of the block ILU factorization preconditioners presented in Theorem 3.3 will be considered in this section for a special type of matrix $A$ described below. The matrix $A$ arises from five-point discretization of the following second-order partial differential equation:

$$(4) \quad -(au_x)_x - (bu_y)_y + (cu)_x + (du)_y + f u = g$$

with $a(x,y)$, $b(x,y)$, $c(x,y)$, $d(x,y)$, and $f(x,y)$ defined on a square region $\Omega$, and with suitable boundary conditions on $\partial\Omega$ which denotes the boundary of $\Omega$. The structure of the resulting matrix $A$ is of the form (2) with $B_i$’s tridiagonal matrices and $C_i$’s and $E_i$’s diagonal matrices. Suppose that $a(x,y)$, $b(x,y)$, $c(x,y)$, $d(x,y)$, and $f(x,y)$ are chosen so that the resulting matrix $A$ becomes a nonsymmetric $M$-matrix.

Since $B_i$ is a tridiagonal matrix, the complete LU factorization of $B_i$ has no fill-in elements. More specifically, if $B_i = L_iU_i$ is the complete LU factorization of $B_i$, then $L_i$ is a unit lower bidiagonal matrix and $U_i$ is an upper bidiagonal matrix. Hence, three types of block ILU factorization preconditioners are constructed using the complete LU factorizations of $B_i$’s rather than using the ILU factorizations of $B_i$’s. The block preconditioners defined in Theorems 3.3 which are constructed based on the complete LU factorizations of $1 \times 1$ block matrices $B_i$ are from now on called 1-block ILU factorization preconditioners. Suppose that $B_i = L_iU_i$ is the complete LU factorization of $B_i$ and $D_i$ is a diagonal matrix consisting of diagonal elements of $U_i$.

For the purpose of getting more effective block preconditioners than 1-block ILU factorization preconditioners mentioned above, we now consider 2-block ILU factorization preconditioners which are constructed based on the ILU factorizations of $2 \times 2$ block matrices rather than $1 \times 1$
block matrices $B_i$. For simplicity of exposition, suppose that $A$ is a $4 \times 4$ nonsymmetric block-tridiagonal M-matrix of the form (2), i.e., $m = 4$ is assumed in the form (2). From now on, let $d$ denote the order of matrices $B_i, C_i,$ and $E_i$. First, $A$ is partitioned into

$$
A = \begin{pmatrix}
B_1 & -C_1 \\
-\mathcal{E}_1 & B_2
\end{pmatrix}
$$

where $B_1 = \begin{pmatrix} B_1 & -C_1 \\ -E_1 & B_2 \end{pmatrix}$, $B_2 = \begin{pmatrix} B_3 & -C_3 \\ -E_3 & B_4 \end{pmatrix}$, $C_1 = \begin{pmatrix} 0 & 0 \\ C_2 & 0 \end{pmatrix}$, and $\mathcal{E}_1 = \begin{pmatrix} 0 & E_2 \\ 0 & 0 \end{pmatrix}$ are $2 \times 2$ block submatrices of $A$. Since $A$ is assumed to be an M-matrix, $B_i$ ($i = 1, 2$) is also an M-matrix. It follows from Theorem 2.1 that the ILU factorization of $B_i$ exists. If $B_i = \mathcal{L}_i \mathcal{U}_i$ is the complete LU factorization of $B_i$, then the nonzero structures of $B_i, \mathcal{L}_i,$ and $\mathcal{U}_i$ for $d = 7$ are illustrated in Fig. 1.

As can be seen in Fig. 1, the complete LU factorization of $B_i$ now has a lot of fill-in elements, so that the ILU factorization of $B_i$ with some fill-ins needs to be considered for the construction of 2-block ILU factorization preconditioners. Note that the complete LU factorization of $B_i$ is used for the construction of 1-block ILU factorization preconditioners since no fill-ins occur during the LU factorization of $B_i$. For each fixed $i$, let $B_i = \mathcal{L}_{ij} \mathcal{U}_{ij} - \mathcal{R}_{ij}$ be the ILU factorization of $B_i$, where $0 \leq j \leq d - 1$, and the nonzero structures of $\mathcal{L}_{ij}$'s and $\mathcal{U}_{ij}$'s for $d = 7$ are illustrated in Fig. 2 and Fig. 3 respectively.
Notice that if $B_i = \mathcal{L}_i \mathcal{U}_i$ is the complete LU factorization of $B_i$, then $\mathcal{L}_{i,d-1} = \mathcal{L}_i$, $\mathcal{U}_{i,d-1} = \mathcal{U}_i$, and $\mathcal{R}_{i,d-1} = 0$. Let $\mathcal{D}_{ij}$ denote the diagonal matrix which is obtained by taking diagonal part of $\mathcal{U}_{ij}$. If we let for each $0 \leq j \leq d - 1$

$$\mathcal{L}^2_j = \begin{pmatrix} \mathcal{L}_{1j} & 0 \\ 0 & \mathcal{L}_{2j} \end{pmatrix}, \quad \mathcal{U}^2_j = \begin{pmatrix} \mathcal{U}_{1j} & 0 \\ 0 & \mathcal{U}_{2j} \end{pmatrix},$$

$$(\mathcal{L}_\alpha)^2_j = \begin{pmatrix} \mathcal{L}_{1j} & 0 \\ -\mathcal{E}_i \mathcal{D}_{1j}^{-1} & \mathcal{L}_{2j} \end{pmatrix}, \quad (\mathcal{U}_\alpha)^2_j = \begin{pmatrix} \mathcal{U}_{1j} & -\mathcal{C}_1 \\ 0 & \mathcal{U}_{2j} \end{pmatrix},$$

$$(\mathcal{L}_\beta)^2_j = \begin{pmatrix} \mathcal{L}_{1j} & 0 \\ -\mathcal{E}_i \mathcal{U}_{1j}^{-1} & \mathcal{L}_{2j} \end{pmatrix}, \quad (\mathcal{U}_\beta)^2_j = \begin{pmatrix} \mathcal{U}_{1j} & -\mathcal{C}_1 \mathcal{L}_{1j}^{-1} \\ 0 & \mathcal{U}_{2j} \end{pmatrix},$$

then $M^2_j = \mathcal{L}^2_j \mathcal{U}^2_j$, $(M^\alpha)^2_j = (\mathcal{L}_\alpha)^2_j (\mathcal{U}_\alpha)^2_j$, and $(M^\beta)^2_j = (\mathcal{L}_\beta)^2_j (\mathcal{U}_\beta)^2_j$ are 2-block ILU factorization preconditioners of type $M$, $M^\alpha$, and $M^\beta$ respectively, where the superscript 2 is used to represent 2-block preconditioners. The nonzero structures of $\mathcal{E}_i \mathcal{D}_{1j}^{-1}$, $\mathcal{E}_i \mathcal{U}_{1j}^{-1}$, $\mathcal{C}_1$, and $\mathcal{L}_{1j}^{-1} \mathcal{C}_1$ for $d = 7$ are illustrated in Fig. 4.
From Fig. 4, it can be seen that the block preconditioner of type $M^\beta_\beta$ has much more fill-ins than that of type $M^\alpha_\alpha$.

In the similar way as was done for 2-block preconditioners, $k$-block ILU factorization preconditioners $M^k_j$, $(M^\alpha_\alpha)^k_j$, and $(M^\beta_\beta)^k_j$ which are based on the ILU factorizations of $k \times k$ block matrices can be easily constructed. That is, the $m \times m$ block matrix $A$ of the form (2) is first partitioned so that each submatrix of $A$ is a $k \times k$ block matrix (it is assumed that $m$ is divisible by $k$), and then the ILU factorizations of $k \times k$ block matrices are carried out to construct $k$-block preconditioners. Since the complete LU factorizations of tridiagonal matrices $B_i$ which have no fill-in elements are used for construction of 1-block preconditioners, for each $0 \leq j \leq d - 1$ $M^1_j = M$, $(M^\alpha_\alpha)^1_j = M^\alpha_\alpha$, and $(M^\beta_\beta)^1_j = M^\beta_\beta$. In other words, the superscript 1 representing 1-block preconditioners can be omitted and 1-block preconditioners of types $M$, $M^\alpha_\alpha$, and $M^\beta_\beta$ are independent of $j$. Notice that the construction of $(k + 1)$-block preconditioners requires a little more storage and arithmetic than that of $k$-block preconditioners.

5. Numerical results

In this section, we provide numerical results of the right preconditioned GMRES(20) (called PGMRES(20)) and the right preconditioned BiCGSTAB (called PBSTAB) methods using two different types of the $k$-block ILU factorization preconditioners $M^k_j$ and $(M^\alpha_\alpha)^k_j$ for solving $Ax = b$ with the special type of matrix $A$ described in section 4. However, numerical experiments for the $k$-block ILU factorization preconditioner $(M^\beta_\beta)^k_j$ are not provided here since they require a lot of fill-in elements causing too much storage and arithmetic. For each type of block preconditioner, numerical experiments are carried out for $0 \leq j \leq 3$ and...
1 ≤ k ≤ 4. To evaluate the effectiveness of the k-block ILU factorization preconditioners, we also provide numerical results of PGMRES(20) and PBSTAB methods using the standard ILU factorization preconditioner with 0 extra diagonals which is called ILU(0) preconditioner. In all cases, the preconditioned iterative methods were started with x₀ = 0, and they were stopped when ∥rᵢ∥₂/∥b∥₂ < 10⁻⁸, where ∥·∥₂ refers to L₂-norm. All numerical experiments have been carried out in double precision floating point arithmetic. All data presented in Tables 1 to 8 represent the number of iterations satisfying the stopping criterion mentioned above, and the data under the column I of Tables 1 to 8 represent the number of iterations of unpreconditioned iterative methods (which can be viewed as preconditioned iterative methods with identity matrix I as a preconditioner). NC in Tables 1 to 8 indicates that iterative methods do not converge within 1000 iterations. For all test problems, the unit square region Ω = (0, 1) × (0, 1) and the Dirichlet boundary condition u(x, y) = 0 on ∂Ω are used. Only the discretized matrix A is of importance, so the right-hand side vector b is created artificially. Therefore, the right-hand side function g(x, y) in equation (4) is not relevant.

Example 5.1. We consider equation (4) with a(x, y) = b(x, y) = 1, c(x, y) = 10(x + y), d(x, y) = 10(x - y), and f(x, y) = 0. We have used two uniform meshes of Δx = Δy = 1/49 and Δx = Δy = 1/73, which lead to two matrices of order n = 48 × 48 and n = 72 × 72, where Δx and Δy refer to the mesh sizes in the x-direction and y-direction, respectively. Once the matrix A is constructed from five-point finite difference discretization of the PDE, the right-hand side vector b is chosen so that b = A(1, 1, ..., 1)ᵀ. Numerical results for this problem are listed in Tables 1 and 2.

Example 5.2. This example is the same as Example 5.1 except for a(x, y) = b(x, y) which is defined as

\[ a(x, y) = \begin{cases} 
10^3 & \text{if } 1/4 < x, y < 3/4 \\
1 & \text{otherwise}
\end{cases} \]

We have used the same uniform meshes as Example 5.1, and the right-hand side vector b is chosen so that b = A(1, 1, ..., 1)ᵀ. Numerical results for this problem are listed in Tables 3 and 4.
TABLE 1: Number of iterations of PGMRES(20) for Example 5.1

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EXAMPLE 5.3. We consider equation (4) with \( a(x, y) = 2e^{x+y} \), \( b(x, y) = 3e^{x+y} \), \( c(x, y) = \sin(x+y) \), \( d(x, y) = \cos(x-y) \), and \( f(x, y) = 10/(1+x+y) \). We have used the same uniform meshes as Example 5.1, and the right-hand side vector \( b \) is chosen so that the exact solution is the discretization of \( xe^{xy} \sin(\pi x) \sin(\pi y) \). Numerical results for this problem are listed in Tables 5 and 6.

EXAMPLE 5.4. We consider equation (4) with \( c(x, y) = \sin(x+y) \), \( d(x, y) = \cos(x-y) \), \( f(x, y) = 2/(1+x+y) \), and \( a(x,y) = b(x,y) \) defined as

\[
a(x,y) = \begin{cases} 
3e^{x+y} & \text{if } 1/4 < x, y < 3/4 \\
6e^{x+y} & \text{otherwise}
\end{cases}
\]

We have used the same uniform meshes as Example 5.1, and the right-hand side vector \( b \) is chosen so that the exact solution is the discretization of \( 10xy(1-x)(1-y)e^{x-y} \). Numerical results for this problem are listed in Tables 7 and 8.

As can be seen in Tables 1 to 8, the numerical results presented are in good agreement with the theoretical results presented in Theorem 3.3. That is, the block preconditioner of type \( M_{i}^{5} \) is more effective than the block preconditioner of type \( M \). It can be also seen that PGMRES(20) and PBSTAB with \((k+1)\)-block preconditioners converge faster than those with \( k \)-block preconditioners. As compared with the standard ILU factorization preconditioner, the block preconditioner \((M_{i}^{5})_{j}\) is relatively effective when \( j \geq 1 \) and \( k \geq 2 \).
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### Table 3: Number of iterations of PGMRES(20) for Example 5.2

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Parallel block ILU preconditioners

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### Table 7: Number of iterations of PGMRES(20) for Example 5.4

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### Table 8: Number of iterations of PBSTAB for Example 5.4

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### 6. Conclusions

We presented in this paper three types of block ILU factorization preconditioners which can be computed in parallel. The block ILU factorization preconditioner of type $M_3^1$ may not be used in practical situations since it requires a lot of fill-in elements causing too much storage and arithmetic. Block ILU factorization preconditioner of type $M$ has rich parallelism since both the computation of preconditioner and preconditioner solver step can be done in parallel, but its effectiveness is
much worse than the block preconditioner of type $M_\alpha$ (see Tables 1 to 8).

The block ILU factorization preconditioner of type $M_\alpha$ is relatively effective as compare with the standard ILU factorization preconditioner. Thus, $(M_\alpha^j)_j^k$ with $j \geq 1$ and $k \geq 2$ is recommended as a preconditioner of the nonstationary iterative methods. Notice that the number of arithmetic operations for constructing $(M_\alpha^j)_j^k$ grows as $j$ becomes large. From our experiments, it is not recommended to use large value of $j$ and the optimal value of $j$ usually ranges from 1 to 3.

References


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