THE DIMENSION OF THE CONVOLUTION OF
BIPARTITE ORDERED SETS

DEOK RAK BAE

ABSTRACT. In this paper, for any two bipartite ordered sets $P$ and $Q$, we define the convolution $P * Q$ of $P$ and $Q$. For $\dim(P) = s$ and $\dim(Q) = t$, we prove that $s + t - (U + V) - 2 \leq \dim(P * Q) \leq s + t -(U + V) + 2$, where $U + V$ is the max-min integer of the certain realizers. In particular, we also prove that $\dim(P_{n,k}) = n + k - \lfloor \frac{n+k}{2} \rfloor$ for $2 \leq k \leq n < 2k$ and $\dim(P_{n,k}) = n$ for $n \geq 2k$, where $P_{n,k} = S_n \ast S_k$ is the convolution of two standard ordered sets $S_n$ and $S_k$.

1. Introduction

Let $X$ be a set. An order $R$ on $X$ is a reflexive, antisymmetric and transitive binary relation on $X$. Then $P = (X, R)$ is called an ordered set. In this paper, we assume that $X$ is finite. An order $R$ on a set is called an extension of another order $S$ on the same set if $S \subseteq R$. For $a, b \in X$, we usually write $a \leq b$ for $(a, b) \in R$ and also $a < b$ when $a \leq b$ and $a \neq b$. For elements $b > a$ in an ordered set $P$, we write $b > a$ or $a < b$ ($b$ covers $a$ or $a$ is covered by $b$) if $b \geq c > a$ implies $b = c$ for every element $c$ of $P$. A linear extension of an ordered set $P$ is a linear order $E : x_1 < x_2 < \cdots < x_n$ containing the order of $P$. Szpilrajn [3] shows that any order has a linear extension. Dushnik and Miller [2] later defined the dimension of an ordered set $P$, denoted by $\dim(P)$, to be the minimal cardinality of a family of its linear extensions whose intersection is its order itself. An incomparable pair $(a, b)$ in an ordered set $P$ is called a critical pair if $x < a$ implies $x < b$ and $x > b$ implies $x > a$ and $\text{crit}(P)$ denotes the set of all critical pairs. A bipartite ordered set is a triple $P = (X, Y, I_P)$ where $X$ and $Y$ are disjoint sets and $I_P$ is an order on $X \cup Y$.

Received January 12, 1999.
1991 Mathematics Subject Classification: 06A07.
Key words and phrases: bipartite ordered set, dimension, Ferrers relation, Ferrers dimension, realizer, $R$-irreducible, max-min integer.
with \( \{(x, y) \in I_P \mid x \neq y\} \subseteq X \times Y \). In [5], Trotter defined the interval dimension of \( P \), denoted by \( \text{dim}_I(P) \), as the least positive integer \( t \) for which there exists a family \( \mathcal{R} = \{L_1, L_2, \ldots, L_t\} \) of linear extensions of \( P \) reversing all critical pairs in \( \text{crit}(P) \cap (X \times Y) \). And Trotter [4] defined the ordered set \( P_n \), for integer \( n \geq 2 \), as follows: \( \min(P_n) = \{x_1, x_2, \ldots, x_n\} \), \( \mid \text{mid}(P_n) = \{z_{ij} \mid 1 \leq i, j \leq n\} \) and \( \max(P_n) = \{b_1, b_2, \ldots, b_n\} \). For all \( i, j, u, v = 1, 2, \ldots, n \), \( x_i < z_{uj} \) in \( P_n \) for \( i \neq u \), \( z_{iv} < b_j \) in \( P_n \) for \( v \neq j \), and \( x_i \mid z_{ij} \mid b_j \) in \( P_n \). It is known [4] that \( \text{dim}(P_n) = \lceil \frac{4n}{3} \rceil \).

In this paper, we define the convolution \( P * Q \) of any two bipartite ordered sets \( P \) and \( Q \). And we prove that \( \text{dim}(P) + \text{dim}(Q) - (U + V) - 2 \leq \text{dim}(P * Q) \leq \text{dim}(P) + \text{dim}(Q) - (U + V) + 2 \), where \( U + V \) is the max-min integer of \( P \) and \( Q \). In particular, we also prove that \( \text{dim}(P_{n,k}) = n + k - \lceil \frac{n+k}{3} \rceil \) for \( 2 \leq k \leq n < 2k \) and \( \text{dim}(P_{n,k}) = n \) for \( n \geq 2k \), where \( P_{n,k} = S_n \ast S_k \) is the convolution of two standard ordered sets \( S_n \) and \( S_k \). Furthermore, we see that \( P_n = S_n \ast S_n \) and \( \text{dim}(P_n) = \lceil \frac{4n}{3} \rceil = 2n - \lceil \frac{2n}{3} \rceil \).

2. Definitions and examples

Let \( G \) and \( M \) be the sets and let \( I \) be a binary relation between \( G \) and \( M \). We define a context as a triple \((G, M, I)\) (see Wille [7]). A relation \( F \subseteq G \times M \) is called a Ferrers relation if

\[ g_1 F m_1 \text{ and } g_2 F m_2 \text{ implies } g_1 F m_2 \text{ or } g_2 F m_1 \]

for all \( g_1, g_2 \in G \) and \( m_1, m_2 \in M \). The Ferrers dimension of a context \((G, M, I)\), denoted by \( \text{fdim}(G, M, I) \), is defined to be the smallest number of Ferrers relations \( F_1, F_2, \ldots, F_n \) with \( I = \bigcap F_i \). Observe that the complement of a Ferrers relation \( F \) is again a Ferrers relation in \( G \times M - I \). Therefore, one can alternatively define \( \text{fdim}(G, M, I) \) as the minimum number of Ferrers relations \( F_1, F_2, \ldots, F_n \) with \( F_i \subseteq G \times M - I \) such that \( \bigcup F_i = G \times M - I \). Throughout this paper, for any context \((G, M, I)\), we assume that \( F \) is a Ferrers relation in \( G \times M \) is the same meaning as \( F \) is a Ferrers relation in \( G \times M - I \).

Let \( P = (X, \leq) \) be an ordered set and let \( S \subseteq X \). Let \( J(P) = \{x \in X \mid x \in \bigvee S \Rightarrow x \in S\} \) and \( M(P) = \{x \in X \mid x \in \bigwedge S \Rightarrow x \in S\} \). Then it is known [7] that \((X, X, \leq)\) and \((J(P), M(P), \leq_{J(P) \times M(P)})\) are contexts and that

\[ \text{dim}(P) = \text{fdim}(X, X, \leq) = \text{fdim}(J(P), M(P), \leq_{J(P) \times M(P)}) \].
The dimension of the convolution of bipartite ordered sets

For a bipartite ordered set $P = (X, Y, I_P)$, we can easily show that

$$\dim_l(P) = \text{fdim}(X, Y, I_P).$$

Trotter obtained the following result:

**Theorem 2.1** [5]. Let $P = (X, Y, I_P)$ be a bipartite ordered set. Then

$$\dim(P) - 1 \leq \dim_l(P) \leq \dim(P).$$

Let $(G, M, I)$ be a context and let $F$ be any Ferrers relation in $G \times M$. We define the two subsets $C(F)$ and $R(F)$ of $G$ and $M$, respectively, as follows:

$$C(F) = \bigcup \{ a \mid (a, b) \in F \} \quad \text{and} \quad R(F) = \bigcup \{ b \mid (a, b) \in F \}.$$  

In fact, we know that $C(F)$ is a set of the first coordinate of the certain longest column in $F$ and $R(F)$ is a set of the second coordinate of the certain longest row in $F$.

Let $P = (X, Y, I_P)$ be a bipartite ordered set and let $\mathcal{F}$ be a family of Ferrers relations in $X \times Y$. We say that $\mathcal{F}$ is a *(optimal)* realization of $X \times Y$ if $|\mathcal{F}| = \text{fdim}(X, Y, I_P)$ and $\bigcup_{F \in \mathcal{F}} F = X \times Y - I_P$.

The following Theorem done by Bae and Lee was a corollary in [1].

**Theorem 2.2** [1]. Let $P = (X, Y, I_P)$ be a bipartite ordered set and let $\mathcal{F}$ be a realization of $X \times Y$ with $|\mathcal{F}| \geq 2$. If there are elements $F_i, F_j \in \mathcal{F}$ such that $R(F_i) \cap R(F_j) = \emptyset$ and $C(F_i) \cap C(F_j) = \emptyset$, then

$$\dim(P) = \dim_l(P).$$
Then we define a convolution $P \ast Q$ of $P$ and $Q$ induced by $P$ and $Q$ is an ordered set on $X \cup Z \cup B$ with the following order relations:

$$
\begin{align*}
    x_i < z_{uv} & \text{ in } P \ast Q \quad \text{if } x_i < y_u \text{ in } P, \\
    z_{uv} < b_j & \text{ in } P \ast Q \quad \text{if } a_v < b_j \text{ in } Q.
\end{align*}
$$

![Figure 1. $S_3 \ast S_3$](image)

Let $P = (X, Y, I_P)$ be a bipartite ordered set and let $\mathcal{F}$ be a realizer of $X \times Y$. A subfamily $\mathcal{F}_i$ of $\mathcal{F}$ is said to be $R$-irreducible of $\mathcal{F}$ if

(i) $\bigcap_{F \in \mathcal{F}_i} R(F) = \emptyset$,

(ii) There do not exist nonempty families $\mathcal{E}_1$ and $\mathcal{E}_2$ of Ferrers relations in $\bigcup_{F \in \mathcal{F}_i} F$ such that $\bigcap_{E \in \mathcal{E}_1} R(E) = \emptyset$ or $\bigcap_{E \in \mathcal{E}_2} R(E) = \emptyset$ with $|\mathcal{E}_1| + |\mathcal{E}_2| = |\mathcal{F}_i|$.

Similarly, a subfamily $\mathcal{G}_j$ of $\mathcal{F}$ is said to be $C$-irreducible of $\mathcal{F}$ if

(i)' $\bigcap_{G \in \mathcal{G}_j} C(G) = \emptyset$,

(ii)' There do not exist nonempty families $\mathcal{D}_1$ and $\mathcal{D}_2$ of Ferrers relations in $\bigcup_{G \in \mathcal{G}_j} G$ such that $\bigcap_{D \in \mathcal{D}_1} C(D) = \emptyset$ or $\bigcap_{D \in \mathcal{D}_2} C(D) = \emptyset$ with $|\mathcal{D}_1| + |\mathcal{D}_2| = |\mathcal{G}_j|$.

**Remark.** Let $P = (X, Y, I_P)$ be a bipartite ordered set and let $\mathcal{F}$ be a realizer of $X \times Y$. If $\mathcal{F}_i$ and $\mathcal{G}_j$ are $R$-irreducible and $C$-irreducible, respectively, of $\mathcal{F}$, then we have

1. $|\mathcal{F}_i| \geq 2$ and $|\mathcal{G}_j| \geq 2$,
2. $\bigcap_{F \in \mathcal{F}_i \setminus \{F_0\}} R(F) \neq \emptyset$ for all $F_0 \in \mathcal{F}_i$,
3. $\bigcap_{G \in \mathcal{G}_j \setminus \{G_0\}} C(G) \neq \emptyset$ for all $G_0 \in \mathcal{G}_j$. 

Let \( P = (X, Y, I_P) \) be a bipartite ordered set and let \( \mathcal{F} \) be a realizer of \( X \times Y \). A collection \( \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_w\} \) of subfamilies of \( \mathcal{F} \) is said to be an \emph{R-irreducible family} of \( \mathcal{F} \) if

(i) each \( \mathcal{F}_i (i = 1, 2, \ldots, w) \) is R-irreducible of \( \mathcal{F} \),

(ii) \( \cap \{R(F) \mid F \in \mathcal{F} - \bigcup_{i=1}^{w} \mathcal{F}_i\} \neq \emptyset \) or \( \mathcal{F} = \bigcup_{i=1}^{w} \mathcal{F}_i \).

In this case, we say that \( \mathcal{F} \) has an R-irreducible family. Similarly, a collection \( \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_w\} \) of subfamilies of \( \mathcal{F} \) is said to be a \emph{C-irreducible family} of \( \mathcal{F} \) if

(i)' each \( \mathcal{G}_j (j = 1, 2, \ldots, w') \) is C-irreducible of \( \mathcal{F} \),

(ii)' \( \cap \{C(F) \mid F \in \mathcal{F} - \bigcup_{j=1}^{w'} \mathcal{G}_j\} \neq \emptyset \) or \( \mathcal{F} = \bigcup_{j=1}^{w'} \mathcal{G}_j \).

In this case, we say that \( \mathcal{F} \) has a C-irreducible family.

\( \mathcal{F} \) is said to be \emph{max-min R-irreducible} of \( X \times Y \) if

(i) There is a collection \( \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_w\} \) of subfamilies of \( \mathcal{F} \) such that

\[ \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_w\} \] is an R-irreducible family of \( \mathcal{F} \),

(ii) If \( E_R = \{E_1, E_2, \ldots, E_u\} \) is an arbitrary R-irreducible family of any realizer \( \mathcal{E} \) of \( X \times Y \), then

\[ u \leq w \quad \text{and} \quad \sum_{i=1}^{k} |\mathcal{F}_i| \leq \min_{E' \in E_R} \left\{ \sum_{E' \in E_R} |E'| : E' \in \mathcal{E}' \right\} \]

for all \( k \) with \( 1 \leq k \leq u \).

Similarly, \( \mathcal{F} \) is said to be \emph{max-min C-irreducible} of \( X \times Y \) if

(i)' There is a collection \( \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_w\} \) of subfamilies of \( \mathcal{F} \) such that

\[ \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_w\} \] is a C-irreducible family of \( \mathcal{F} \),

(ii)' If \( D_C = \{D_1, D_2, \ldots, D_v\} \) is an arbitrary C-irreducible family of any realizer \( D \) of \( X \times Y \), then

\[ v \leq w' \quad \text{and} \quad \sum_{j=1}^{k} |\mathcal{G}_j| \leq \min_{D' \in D_C} \left\{ \sum_{D' \in D_C} |D'| : D' \in \mathcal{D}' \right\} \]

for all \( k \) with \( 1 \leq k \leq v \).

\textbf{Remark.} Let \( P = (X, Y, I_P) \) be a bipartite ordered set. If \( \mathcal{F} \) is a max-min R-irreducible realizer of \( X \times Y \), then there is a max-min R-irreducible family \( \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_w\} \) of \( \mathcal{F} \) such that \( |\mathcal{F}_i| \leq |\mathcal{F}_{i+1}| \) for all \( i = 1, 2, \ldots, w \). Similarly, if \( \mathcal{F} \) is a max-min C-irreducible realizer of \( X \times Y \), then there is a max-min C-irreducible family \( \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_w\} \) of \( \mathcal{F} \) such that \( |\mathcal{G}_j| \leq |\mathcal{G}_{j+1}| \) for all \( j = 1, 2, \ldots, w' \).
Let \( P = (X, Y, I_P) \) and \( Q = (A, B, I_Q) \) be bipartite ordered sets. Let \( \mathcal{F} \) and \( \mathcal{G} \) be max-min \( R \)- and \( C \)-irreducible realizers of \( X \times Z \) and \( Z \times B \), respectively, where \( X = \min(P \ast Q), Z = \text{mid}(P \ast Q) \) and \( B = \max(P \ast Q) \). An integer \( u + v \) is said to be an \( R \& C \)-irreducible integer of \( \mathcal{F} \) and \( \mathcal{G} \) if there are max-min \( R \)-irreducible family \( \{ \mathcal{F}_1, \mathcal{F}_2, \ldots , \mathcal{F}_w \} \) of \( \mathcal{F} \) and max-min \( C \)-irreducible family \( \{ \mathcal{G}_1, \mathcal{G}_2, \ldots , \mathcal{G}_w \} \) of \( \mathcal{G} \) such that \( v + \sum_{i=1}^{w} |\mathcal{F}_i| \leq \dim(P) \) and \( u + \sum_{j=1}^{w} |\mathcal{G}_j| \leq \dim(Q) \) for some \( u \in \{1, 2, \ldots , w \} \) and \( v \in \{1, 2, \ldots , w' \} \). The integer \( U + V \) is called the max-min integer of \( \mathcal{F} \) and \( \mathcal{G} \), which is defined by

\[
U + V = \max_{\mathcal{F}, \mathcal{G}} \{ u + v \mid u + v \text{ is an } R \& C \text{-irreducible integer of } \mathcal{F} \text{ and } \mathcal{G} \}.
\]

**Remark.** Let \( P = (X, Y, I_P) \) and \( Q = (A, B, I_Q) \) be bipartite ordered sets. Let \( \mathcal{F} \) and \( \mathcal{G} \) be max-min \( R \)- and \( C \)-irreducible realizers of \( X \times Z \) and \( Z \times B \), respectively, with the max-min integer \( U + V \) of \( \mathcal{F} \) and \( \mathcal{G} \), where \( X = \min(P \ast Q), Z = \text{mid}(P \ast Q) \) and \( B = \max(P \ast Q) \). Note that \( |\mathcal{F}_i| \geq 2 \) and \( |\mathcal{G}_j| \geq 2 \) for all \( \mathcal{F}_i \in \mathcal{F} \) and \( \mathcal{G}_j \in \mathcal{G} \). Hence we have \( U + V \leq \left\lfloor \frac{\dim(P) + \dim(Q)}{3} \right\rfloor \).

**Example 1.** For \( 2k > n \geq k \geq 2 \), let \( S_n = (X, Y, \leq) \) and \( S_k = (A, B, \leq) \) be standard ordered sets with \( X = \{x_1, x_2, \ldots , x_n\}, Y = \{y_1, y_2, \ldots , y_n\} \), \( A = \{a_1, a_2, \ldots , a_k\} \) and \( B = \{b_1, b_2, \ldots , b_k\} \). Then \( \mathcal{F} = \{(x_i, z_j) \mid 1 \leq i \leq n \} \) and \( \mathcal{G} = \{(z_j, b_j) \mid 1 \leq j \leq k \} \) are max-min \( R \)- and \( C \)-irreducible realizers of \( X \times Z \) and \( Z \times B \), respectively, where \( X = \min(S_n \ast S_k), Z = \text{mid}(S_n \ast S_k) \) and \( B = \max(S_n \ast S_k) \). For all \( i = 1, 2, \ldots , \left\lfloor \frac{n}{2} \right\rfloor \) and \( j = 1, 2, \ldots , \left\lfloor \frac{k}{2} \right\rfloor \), let

\[
\mathcal{F}_i = \{(x_i, z_{i1}), (x_i, z_{i2}), \ldots , (x_i, z_{ik})\} \cup \{(x_p, z_{p1}), (x_p, z_{p2}), \ldots , (x_p, z_{pk})\},
\]

\[
\mathcal{G}_j = \{(z_{1j}, b_j), (z_{2j}, b_j), \ldots , (z_{nj}, b_j)\} \cup \{(z_{1q}, b_q), (z_{2q}, b_q), \ldots , (z_{nq}, b_q)\}
\]

where \( p = \left\lfloor \frac{n}{2} \right\rfloor + i \) and \( q = \left\lfloor \frac{k}{2} \right\rfloor + j \). Then we see that \( \{\mathcal{F}_1, \mathcal{F}_2, \ldots , \mathcal{F}_{\left\lfloor \frac{n}{2} \right\rfloor} \} \) and \( \{\mathcal{G}_1, \mathcal{G}_2, \ldots , \mathcal{G}_{\left\lfloor \frac{k}{2} \right\rfloor} \} \) are max-min \( R \)- and \( C \)-irreducible realizers of \( X \times Z \) and \( Z \times B \), respectively, and that \( |\mathcal{F}_i| = |\mathcal{G}_j| = 2 \) for all \( i = 1, 2, \ldots , \left\lfloor \frac{n}{2} \right\rfloor \) and \( j = 1, 2, \ldots , \left\lfloor \frac{k}{2} \right\rfloor \). Then there exist \( R \& C \) irreducible integer \( u + v \) of \( \mathcal{F} \) and \( \mathcal{G} \) such that \( 1 \leq u \leq \left\lfloor \frac{n}{2} \right\rfloor \) and \( 1 \leq v \leq w \), where \( w = \min\{n - 2u, \left\lfloor \frac{k-u}{2} \right\rfloor \} \). Since \( 2k > n \geq k \geq 2 \), it follows from the above Remark that \( U + V = \max\{u + v \mid 1 \leq u \leq \left\lfloor \frac{n}{2} \right\rfloor, 1 \leq v \leq w \} = \left\lfloor \frac{n+k}{3} \right\rfloor \).
EXAMPLE 2. Consider the relations in the below Table 1 defined as follows: for any pair \((p, q) \in (S_5 \ast S_3) \times (S_5 \ast S_3)\),

\[
p \lessdot q \iff (p, q) \in H_i,
\]

\[
p \lessdot^j q \iff (p, q) \in H_i \text{ and } (p, q) \in H_j,
\]

\[
p \lessdot^\circ q \iff p < q \in S_5 \ast S_3
\]

for some Ferrers relations \(H_i\) and \(H_j\) in \((S_5 \ast S_3) \times (S_5 \ast S_3)\).

| \(x_1\) | \(x_2\) | \(x_3\) | \(x_4\) | \(x_5\) | \(x_{11}\) | \(x_{12}\) | \(x_{13}\) | \(x_{14}\) | \(x_{15}\) | \(x_{21}\) | \(x_{22}\) | \(x_{23}\) | \(x_{24}\) | \(x_{25}\) | \(x_{31}\) | \(x_{32}\) | \(x_{33}\) | \(x_{34}\) | \(x_{35}\) | \(x_{41}\) | \(x_{42}\) | \(x_{43}\) | \(x_{44}\) | \(x_{45}\) | \(x_{51}\) | \(x_{52}\) | \(x_{53}\) | \(b_1\) | \(b_2\) | \(b_3\) |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| \(1\) | \(1\) | \(1\) | \(1\) | \(1\) | \(1\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) |
| \(2\) | \(2\) | \(2\) | \(2\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) |
| \(3\) | \(2\) | \(2\) | \(2\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(5\) | \(5\) | \(5\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) |
| \(4\) | \(2\) | \(1\) | \(0\) | \(2\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) |
| \(5\) | \(2\) | \(1\) | \(1\) | \(1\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) |
| \(6\) | \(2\) | \(1\) | \(1\) | \(2\) | \(O\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) |
| \(7\) | \(2\) | \(1\) | \(1\) | \(2\) | \(O\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) |
| \(8\) | \(2\) | \(1\) | \(1\) | \(2\) | \(O\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) |
| \(9\) | \(2\) | \(1\) | \(1\) | \(2\) | \(O\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) | \(3\) | \(4\) |

Table 1. Ferrers relations in \((S_5 \ast S_3) \times (S_5 \ast S_3)\)
Then we see that there are distinct 6-Ferrers relations $H_1, H_2, H_3, H_4, H_5$ and $H_6$ such that $\bigcup_{i=1}^6 H_i = (S_5 \ast S_3) \times (S_5 \ast S_3) - I$ and hence $\dim(S_5 \ast S_3) \leq 6$. From Example 1 we obtain $\dim(S_5 \ast S_3) \geq 6$. Hence we conclude that $\dim(S_5 \ast S_3) = 6$. Similarly, we have $\dim(S_4 \ast S_3) = 5$ and $\dim(S_3 \ast S_3) = 4$.

3. Main results

Let $(G, M, I)$ be a context. For arbitrary subset $S \subseteq G \times M$, we denoted by

$$\text{fdim}(S) = \min \{n \mid \bigcup_{i=1}^n F_i = S - I\},$$

where $F_i$ is a Ferrers relation in $G \times M - I$.

**Lemma 3.1.** Let $P = (X, Y, I_P)$ and $Q = (A, B, I_Q)$ be bipartite ordered sets and let $\mathcal{F}$ and $\mathcal{G}$ be realizers of $X \times Z$ and $Z \times B$, respectively, where $X = \min(P \ast Q), Z = \text{mid}(P \ast Q)$ and $B = \max(P \ast Q)$. If $\dim(P) = \dim_1(P)$ and $\dim(Q) = \dim_1(Q)$, then we have the following properties:

1. If $\mathcal{F}_i$ is an $R$-irreducible of $\mathcal{F}$, then there is at most one $G_0 \in \mathcal{G}$ such that $\text{fdim}(\bigcup_{F \in \mathcal{F}_i} F \cup G_0 \cup M) = |\mathcal{F}_i|$ for some $M \subset Z \times X - I$.

2. If $\mathcal{G}_j$ is a $C$-irreducible of $\mathcal{G}$, then there is at most one $F_0 \in \mathcal{F}$ such that $\text{fdim}(\bigcup_{G \in \mathcal{G}_j} G \cup F_0 \cup M') = |\mathcal{G}_j|$ for some $M' \subset Z \times Z - I$.

**Proof.** Let $P = (X, Y, I_P)$ and $Q = (A, B, I_Q)$ be bipartite ordered sets. Let $\mathcal{F}$ and $\mathcal{G}$ be realizers of $X \times Z$ and $Z \times B$, respectively, where $X = \{x_1, x_2, \ldots, x_n\}, Z = \{z_{uv} \mid 1 \leq u, v \leq l \}$ and $B = \{b_1, b_2, \ldots, b_k\}$.

Consider the projection maps $q_1 : X \times Z \rightarrow X \times Y$ and $q_2 : Z \times B \rightarrow A \times B$, which are defined by

$$q_1(x_i, z_{uv}) = (x_i, y_u) \text{ and } q_2(z_{uv}, b_j) = (a_u, b_j).$$

Now, we have the following observations:

(i) For $F \in \mathcal{F}$, $q_1(F)$ is a Ferrers relation in $X \times Y$ and $\{q_1(F) \mid F \in \mathcal{F}\}$ is also a realizer of $X \times Y$,

(ii) For $G \in \mathcal{G}$, $q_2(G)$ is also a Ferrers relation in $A \times B$ and $\{q_2(G) \mid G \in \mathcal{G}\}$ is also a realizer of $A \times B$,
(iii) For \( F \in \mathcal{F} \), there is \((x_i, y_u) \in X \times Y = I_P\) such that
\[
\{(x_i, z_{u1}), (x_i, z_{u2}), \ldots, (x_i, z_{um})\} \subseteq F,
\]
(iv) For \( G \in \mathcal{G} \), there is \((a_v, b_j) \in A \times B = I_Q\) such that
\[
\{(z_{v1}, b_j), (z_{v2}, b_j), \ldots, (z_{vI}, b_j)\} \subseteq G.
\]

(1) Suppose that \( \mathcal{F} \) is an \( R\)-irreducible of \( \mathcal{F} \) and that there are distinct \( G_1, G_2 \in \mathcal{G} \) such that \( \text{fdim}(\bigcup_{F \in \mathcal{F}} F \cup G_1 \cup G_2 \cup M) = |\mathcal{F}| = w\) for some \( M \subseteq Z \times Z - I \). Let \( \mathcal{E} \) be an arbitrary realizer of \( \bigcup_{F \in \mathcal{F}} F \cup G_1 \cup G_2 \cup M \) with \( |\mathcal{E}| = |\mathcal{F}| \). Since \( q_2(G_1) \) and \( q_2(G_2) \) are distinct Ferrers relations in \( A \times B = I_Q \), it follows that there are \((a_1, b_1) \in q_2(G_1) \) and \((a_2, b_2) \in q_2(G_2) \) such that \((a_1, b_2) \notin q_2(G_1 \cup G_2) \) and \((a_2, b_1) \notin q_2(G_1 \cup G_2) \). Therefore \( \{E \in \mathcal{E} \mid E \cap G_1 \neq \emptyset\} \cap \{E \in \mathcal{E} \mid E \cap G_2 \neq \emptyset\} = \emptyset \). Furthermore, since \( \mathcal{E} \) is a realizer of \( \bigcup_{F \in \mathcal{F}} F \cup G_1 \cup G_2 \cup M \), it follows that \( G_1 \subseteq \bigcup\{E \in \mathcal{E} \mid E \cap G_1 \neq \emptyset\} \) and \( G_2 \subseteq \bigcup\{E \in \mathcal{E} \mid E \cap G_2 \neq \emptyset\} \). Hence we have \( \bigcap\{R(E) \mid E \in \mathcal{E} \text{ with } E \cap G_1 \neq \emptyset\} = \emptyset \) and \( \bigcap\{R(E) \mid E \in \mathcal{E} \text{ with } E \cap G_2 \neq \emptyset\} = \emptyset \), which is a contradiction as \( \mathcal{F} \) is an \( R\)-irreducible of \( \mathcal{F} \).

(2) Similarly, we can prove the result. \(\square\)

**Theorem 3.2.** Let \( P = (X, Y, I_P) \) and \( Q = (A, B, I_Q) \) be bipartite ordered sets with \( \text{dim}(P) = s \) and \( \text{dim}(Q) = t \) and let \( \mathcal{F} \) and \( \mathcal{G} \) be max-min \( R \)- and \( C \)-irreducible realizers of \( X \times Z \) and \( Z \times B \), respectively, where \( X = \text{min}(P \ast Q) \), \( Z = \text{mid}(P \ast Q) \) and \( B = \text{max}(P \ast Q) \). If \( \text{dim}(P) = \text{dim}_1(P) \) and \( \text{dim}(Q) = \text{dim}_1(Q) \), then we have
\[
s + t - (U + V) \leq \text{dim}(P \ast Q) \leq s + t - (U + V) + 2,
\]
where \( U + V \) is the max-min integer of \( \mathcal{F} \) and \( \mathcal{G} \).

**Proof.** Suppose that \( \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_w\} \) is a max-min \( R \)-irreducible family of a realizer \( \mathcal{F} \) of \( X \times Z \) and \( \{G_1, G_2, \ldots, G_w\} \) is a max-min \( C \)-irreducible family of a realizer \( \mathcal{G} \) of \( Z \times Y \), where \( X = \text{min}(P \ast Q) \), \( Z = \text{mid}(P \ast Q) \) and \( B = \text{max}(P \ast Q) \).

Consider the max-min integer \( U + V \) of \( \mathcal{F} \) and \( \mathcal{G} \) with \( 1 \leq U \leq w \) and \( 1 \leq V \leq w' \). Then there are subfamilies \( \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_U\} \) and \( \{G_1, G_2, \ldots, G_V\} \) of \( \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_w\} \) and \( \{G_1, G_2, \ldots, G_w\} \), respectively, such that \( V + \sum_{i=1}^{U} |\mathcal{F}_i| \leq \text{dim}(P) \) and \( U + \sum_{j=1}^{V} |G_j| \leq \text{dim}(Q) \).

By Lemma 3.1, for each \( i = 1, 2, \ldots, U \), there is exactly one Ferrers relation \( G_i \in \mathcal{G} - \bigcup_{j=1}^{V} G_j \) such that \( \text{fdim}(\bigcup_{F \in \mathcal{F}_i} F \cup M_i \cup G_i) = |\mathcal{F}_i| \) for
some $M_i \subset Z \times Z - I$. Similarly, for each $j = 1, 2, \cdots, V$, there is exactly one Ferrers relation $F_j \in \mathcal{F} - \bigcup_{i=1}^{U} \mathcal{F}_i$ such that $\dimm(\bigcup_{G \in \mathcal{G}_j} G \cup M'_j \cup F_j) = |G_j|$ for some $M'_j \subset Z \times Z - I$. From the definition of $U + V$, we know that $\mathcal{F} - \bigcup_{i=1}^{U} \mathcal{F}_i - \{F_i \mid 1 \leq i \leq V\}$ does not have an $R$-irreducible subfamily and $\mathcal{G} - \bigcup_{j=1}^{V} \mathcal{G}_j - \{G_j \mid 1 \leq j \leq U\}$ does not have a $C$-irreducible subfamily. Since $\dimm(P) = \dimm_1(P)$ and $\dimm(Q) = \dimm_1(Q)$, it follows that $\dimm((X \times Z) \times (Z \times B)) = \dimm(P) + \dimm(Q) - (U + V)$. But we know that $\dimm((X \cup Z \cup B) \times (X \cup Z \cup B) - (X \times Z) - (Z \times B)) = 2$, thus we have $(s - U) + (t - V) \leq \dimm(P \ast Q) \leq (s - U) + (t - V) + 2$. □

**Corollary 3.3.** Let $P = (X, Y, I_P)$ and $Q = (A, B, I_Q)$ be bipartite ordered sets with $\dimm(P) = s$ and $\dimm(Q) = t$ and let $\mathcal{F}$ and $\mathcal{G}$ be max-min $R$- and $C$-irreducible realizers of $X \times Z$ and $Z \times B$, respectively, where $X = \min(P \ast Q), Z = \midm(P \ast Q)$ and $B = \max(P \ast Q)$. Then we have the following properties:

1. If $\dimm(P) = \dimm_1(P) + 1$ and $\dimm(Q) = \dimm_1(Q)$, then there is $G_0 \in \mathcal{G}$ such that $s + t - (U + V) - 1 \leq \dimm(P \ast Q) \leq s + t -(U + V) + 1$, where $U + V$ is the max-min integer of $\mathcal{F}$ and $\mathcal{G} - \{G_0\}$.
2. If $\dimm(P) = \dimm_1(P)$ and $\dimm(Q) = \dimm_1(Q) + 1$, then there is $F_0 \in \mathcal{F}$ such that $s + t - (U + V) - 1 \leq \dimm(P \ast Q) \leq s + t -(U + V) + 1$, where $U + V$ is the max-min integer of $\mathcal{F} - \{F_0\}$ and $\mathcal{G}$.
3. If $\dimm(P) = \dimm_1(P) + 1$ and $\dimm(Q) = \dimm_1(Q) + 1$, then there are $F_0 \in \mathcal{F}$ and $G_0 \in \mathcal{G}$ such that $s + t - (U + V) - 2 \leq \dimm(P \ast Q) \leq s + t -(U + V)$, where $U + V$ is the max-min integer of $\mathcal{F} - \{F_0\}$ and $\mathcal{G} - \{G_0\}$.

**Proof.** (1) Let $\dimm(P) = \dimm_1(P) + 1$ and $\dimm(Q) = \dimm_1(Q)$ and let $\mathcal{F}$ and $\mathcal{G}$ be max-min $R$- and $C$-irreducible realizers of $X \times Z$ and $Z \times B$, respectively, where $X = \min(P \ast Q), Z = \midm(P \ast Q)$ and $B = \max(P \ast Q)$. Since $\dimm(P) = \dimm_1(P) + 1$, it follows that there is a Ferrers relation $F$ such that $F \cap (X \times Z) = \emptyset$. Then there is a Ferrers relation $G_0 \in \mathcal{G}$ such that $F \cup G_0$ contained a Ferrers relation in $P \ast Q$. Since $|\mathcal{F}| = s - 1$ and $|\mathcal{G} - \{G_0\}| = t - 1$, it follows from Theorem 3.2 that there is the max-min integer $U + V$ of $\mathcal{F}$ and $\mathcal{G} - \{G_0\}$ such that $s + t - 1 - (U + V) \leq \dimm(P \ast Q) \leq s + t + 1 -(U + V)$. By the similar method in (1), we obtain results (2) and (3). □
Lemma 3.4. Let $P = (X, Y, I_P)$ and $Q = (A, B, I_Q)$ be bipartite ordered sets with $\dim(P) = s, \dim(Q) = t$ and $s, t \geq 2$. Then we have $\dim(P \ast Q) \geq s + t - 2 - \lfloor \frac{s+t}{3} \rfloor$.

Proof. Let $X = \min(P \ast Q) = \{x_1, x_2, \cdots, x_n\}$, $Z = \mid(P \ast Q) = \{z_{uv} | 1 \leq u \leq l \text{ and } 1 \leq v \leq m\}$ and $B = \max(P \ast Q) = \{b_1, b_2, \cdots, b_k\}$. By Theorem 2.1, we have two cases as follows:

$$\dim(P) = \dim_I(P) + 1 \text{ or } \dim(P) = \dim_I(P),$$
$$\dim(Q) = \dim_I(Q) + 1 \text{ or } \dim(Q) = \dim_I(Q).$$

Case 1. $\dim(P) = \dim_I(P)$ and $\dim(Q) = \dim_I(Q)$.

Let $\mathcal{F} = \{F_1, F_2, \cdots, F_r\}$ and $\mathcal{G} = \{G_1, G_2, \cdots, G_l\}$ be realizers of $(X \cup Z) \times (X \cup Z)$ and $(Z \cup B) \times (Z \cup B)$, respectively, and let $\mathcal{H} = \{H_1, H_2, \cdots, H_m\}$ be an arbitrary realizer of $P \ast Q$. Note that, for each $F_i \in \mathcal{F}$ and $G_j \in \mathcal{G}$, there are $(x_i, y_j) \in X \times Y - I_P$ and $(a_i, b_j) \in A \times B - I_Q$ such that \{(x_i, z_{uv}) | 1 \leq j \leq m\} $\subseteq F_i, \{(z_{iv}, b_j) | 1 \leq i \leq l\} $\subseteq G_j$ and $R(F_i) \cap C(G_j) \neq \emptyset$. Then, for each $H$ of $\mathcal{H}$, there do not exist $F_{i_0} \in \mathcal{F}$ and $G_{j_0} \in \mathcal{G}$ such that $F_{i_0} \cup G_{j_0} \subseteq H$. For all $E \subseteq (X \cup Z \cup B) \times (X \cup Z \cup B) - I$, we see that

(i) $\text{fdim}\left(\bigcup_{i=1}^{3r} D_i \cup E\right) \geq 2r$ for all $D_i \in \mathcal{F} \cup \mathcal{G}$ if $s + t = 3r$,

(ii) $\text{fdim}\left(\bigcup_{i=1}^{3r+1} D_i \cup E\right) \geq 2r + 1$ for all $D_i \in \mathcal{F} \cup \mathcal{G}$ if $s + t = 3r + 1$,

(iii) $\text{fdim}\left(\bigcup_{i=1}^{3r+2} D_i \cup E\right) \geq 2r + 2$ for all $D_i \in \mathcal{F} \cup \mathcal{G}$ if $s + t = 3r + 2$.

Hence we have $\dim(P \ast Q) \geq s + t - \lfloor \frac{s+t}{3} \rfloor$.

Case 2. $\dim(P) = \dim_I(P) + 1$ and $\dim(Q) = \dim_I(Q)$.

Let $\{F_1, F_2, \cdots, F_s\}$ and $\{G_1, G_2, \cdots, G_l\}$ be realizers of $(X \cup Z) \times (X \cup Z)$ and $(Z \cup B) \times (Z \cup B)$, respectively, and let $\mathcal{H} = \{H_1, H_2, \cdots, H_m\}$ be arbitrary realizer of $P \ast Q$. Since $\dim(P) = \dim_I(P) + 1$, without loss of generality, we may assume that there is at most one element $F$ of $\{F_1, F_2, \cdots, F_s\}$, say $F = F_s$, such that $F_s \cap (X \times Z) = 0$. Then there is a Ferrers relation $H \in \mathcal{H}$ such that $F_s \cup G_{j_0} \subseteq H$ for some $j_0 \in \{1, 2, \cdots, t\}$.
Let $\mathcal{F} = \{F_1, F_2, \ldots, F_{s-1}\}$ and let $\mathcal{G} = \{G_1, G_2, \ldots, G_{t-1}\}$ with $G_t = G_{j_0}$. Note that for each $F_i \in \mathcal{F}$ and $G_j \in \mathcal{G}$, there are $(x_i, y_u) \in X \times Y - I_P$ and $(a_{sv}, b_j) \in A \times B - I_Q$ such that $\{(x_i, z_{uv}) \mid 1 \leq j \leq m, 1 \leq i \leq l\} \subseteq F_i \cap \{(z_{iv}, b_j) \mid 1 \leq i \leq l\} \subseteq G_j$ and $R(F_i) \cap C(G_j) \neq \emptyset$. Then as in case 1, for each element $H$ of $\mathcal{H}$, there do not exist $F_{i_0} \in \mathcal{F}$ and $G_{j_0} \in \mathcal{G}$ such that $F_{i_0} \cup G_{j_0} \subseteq H$. By the same method in Case 1, for all $E \subseteq (X \cup Z \cup B) \times (X \cup Z \cup B) - I, \dim(\bigcup_{E \in \mathcal{G}} D \cup E) \geq (s-1) + (t-1) - \left[\frac{(s-1) + (t-1)}{3}\right] + 1 = s + t - 1 - \left[\frac{s+t-2}{3}\right]$. Hence we have $\dim(P \ast Q) \geq (s-1) + (t-1) - \left[\frac{(s-1) + (t-1)}{3}\right] + 1 = s + t - 1 - \left[\frac{s+t-2}{3}\right]$. 

**Case 3.** $\dim(P) = \dim_I(P)$ and $\dim(Q) = \dim_I(Q) + 1$.

By symmetry to the Case 2, we get $\dim(P \ast Q) \geq s + t - 1 - \left[\frac{s+t-2}{3}\right]$. 

**Case 4.** $\dim(P) = \dim_I(P) + 1$ and $\dim(Q) = \dim_I(Q) + 1$.

Let $\{F_1, F_2, \ldots, F_s\}$ and $\{G_1, G_2, \ldots, G_t\}$ be realizers of $(X \cup Z) \times (X \cup Z)$ and $(Z \cup B) \times (Z \cup B)$, respectively, and let $\mathcal{H} = \{H_1, H_2, \ldots, H_w\}$ be arbitrary realizer of $P \ast Q$. Since $\dim(P) = \dim_I(P) + 1$, without loss of generality, we may assume that there is at most one element $F$ of $\{F_1, F_2, \ldots, F_s\}$, say $F = F_s$, such that $F_s \cap (X \times Z) = \emptyset$. Similarly, since $\dim(Q) = \dim_I(Q) + 1$, without loss of generality, we may assume that there is at most one element $G$ of $\{G_1, G_2, \ldots, G_t\}$, say $G = G_t$ such that $G_t \cap (Z \times B) = \emptyset$. Then there are Ferrers relations $H, H' \in \mathcal{H}$ such that $F_s \cup G_j \subseteq H$ and $F_i \cup G_t \subseteq H'$ for some $i_0$ and $j_0$ with $1 \leq i_0 \leq s - 1$ and $1 \leq j_0 \leq t - 1$. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_{s-2}\}$ and $\mathcal{G} = \{G_1, G_2, \ldots, G_{t-2}\}$ with $F_{i_0} = F_{s-1}$, $G_{j_0} = G_{t-1}$. Note that for each $F_i \in \mathcal{F}$ and $G_j \in \mathcal{G}$, there are $(x_i, y_u) \in X \times Y - I_P$ and $(a_{sv}, b_j) \in A \times B - I_Q$ such that $\{(x_i, z_{uv}) \mid 1 \leq j \leq m, 1 \leq i \leq l\} \subseteq F_i \cap \{(z_{iv}, b_j) \mid 1 \leq i \leq l\} \subseteq G_j$ and $R(F_i) \cap C(G_j) \neq \emptyset$. Then as in case 1, for each element $H$ of $\mathcal{H}$, there do not exist $F_i \in \mathcal{F}$ and $G_j \in \mathcal{G}$ such that $F_i \cup G_j \subseteq H$. By the same method in Case 1, for all $E \subseteq (X \cup Z \cup B) \times (X \cup Z \cup B) - I, \dim(\bigcup_{E \in \mathcal{G}} D \cup E) \geq (s-2) + (t-2) - \left[\frac{s-2 + t-2}{3}\right] + 2 = s + t - 2 - \left[\frac{s+t-4}{3}\right]$. Thus we have $\dim(P \ast Q) \geq (s-2) + (t-2) - \left[\frac{s-2 + t-2}{3}\right] + 2 = s + t - 2 - \left[\frac{s+t-4}{3}\right]$. 

By Cases 1, 2, 3 and 4, we get $\dim(P \ast Q) \geq s + t - 2 - \left[\frac{s+t-4}{3}\right]$. □

Consider the two $n$- and $k$-dimensional standard ordered sets $S_n$ and $S_k$ for $n, k \geq 3$. Let $\min(S_n \ast S_k) = \{x_1, x_2, \ldots, x_n\}$, $\max(S_n \ast S_k) = \{b_1, b_2, \ldots, b_k\}$ and $\text{mid}(S_n \ast S_k) = \{z_{ij} \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq k\}$. Then we have the following properties:

(i) $\min(S_n \ast S_k) < \max(S_n \ast S_k)$. 

(ii) $x_i < z_{uv}$ and $z_{uv} < b_j$ in $S_n \ast S_k$ for all $1 \leq i, u \leq n$ and $1 \leq j, v \leq k$ with $i \neq u$ and $j \neq v$.

(iii) $x_i \parallel z_{ij}$ and $z_{ij} \parallel b_j$ in $S_n \ast S_k$ for all $1 \leq i \leq n$ and $1 \leq j \leq k$.

Now for all $n, k \geq 2$, let $P_{n,k} = S_n \ast S_k$ (See $S_3 \ast S_3$ in Figure 1). We will see that $H_1 = H_{11}, H_2 = H_{12}, H_3 = H'_{11}, H_4 = H'_{12}$ and $H_5 = H_m(m = 2s + t + 2), H_6 = H_m(m = n - 1)$ in Example 2 and Case 2 of the proof of Theorem 3.5. In general, we have the following theorem:

**Theorem 3.5.** Let $n$ and $k$ be the positive integers with $n \geq k \geq 2$, we have

$$\dim(P_{n,k}) = \begin{cases} n & \text{if } n \geq 2k \\ n + k - \left\lceil \frac{n+k}{3} \right\rceil & \text{if } n < 2k. \end{cases}$$

**Proof.** Let $X = \min(P_{n,k}) = \{x_1, x_2, \ldots, x_n\}$, $B = \max(P_{n,k}) = \{b_1, b_2, \ldots, b_k\}$ and $Z = \mid\mid(P_{n,k}) = \{z_{ij} \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq k\}$. Then $X < B$ and $x_i \parallel z_{ij}$ and $z_{ij} \parallel b_j$ in $P_{n,k}$ for $1 \leq i \leq n$ and $1 \leq j \leq k$. Consider the Ferrers relations $F_i = \{(x_i, z) \mid z \in Z \text{ with } x_i \not\preceq z\}$ and $G_j = \{(z, b_j) \mid z \in Z \text{ with } z \not\preceq b_j\}$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$. Then $\mathcal{F} = \{F_1, F_2, \ldots, F_n\}$ and $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ are realizers of $X \times Z$ and $Z \times B$, respectively. Then, for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$, we have

$$C(F_i) = \{x_i\} \text{ and } R(F_i) = \{z_{i1}, z_{i2}, \ldots, z_{ik}\}$$

$$C(G_j) = \{z_{1j}, z_{2j}, \ldots, z_{nj}\} \text{ and } R(G_j) = \{b_j\}.$$  

**Case 1.** $n \geq 2k$.  

Let $\mathcal{F} = \{F_1, F_n, F_i, F_{k+i-1}, F_{2k}, \ldots, F_{n-1} \mid 2 \leq i \leq k\}$. We will construct distinct $n$-Ferrers relations in $P_{n,k} \times P_{n,k} - I$ as follows:

(i) $H_1 = F_1 \cup \{\{z_{1l}, b_1\} \mid 2 \leq l \leq k\} \cup$

$$\{((Z \cup B) \times R(F_i)) \cup ((X \cup Z \cup B) \times (X - \{x_i\})) - \{\{(z_{iv}, z_{iv}) \mid 1 \leq u \leq v \leq k\} - \{(x_i, x_j) \mid 1 \leq j \leq i \leq n\},$$

$H_n = F_n \cup \{\{z_{1l}, b_1\} \cup ((X \cup Z \cup B) \times (X - \{x_n\}) \cup$

$$\{(z_{ij}, z_{uv}) \mid i < u \text{ or } i = u, 1 \leq j < v \leq k \text{ with } 1 \leq i, u \leq n\} -$$

$$\{(x_i, x_j) \mid 1 \leq i \leq j \leq n\}.$$
(ii) For each \( i \) with \( 2 \leq i \leq k \), we construct the two Ferrers relations \( H_i \) and \( H_{k+i-1} \) in \( P_{n,k} \times P_{n,k} - I \). Then there are \( F_i, F_{k+i-1} \in \mathcal{F} \) and \( G_i \in \mathcal{G} \) such that

\[
H_i = F_i \cup \{ (G_i - \{(z_{ii}, b_i)\}) \} \cup \{ (b_j, b_i) \mid 1 \leq j \leq k \text{ with } i \neq j \} \cup \\
( (Z \cup B) \times R(F_i) ) \cup \{ (z_{ni}, z_{nu}) \mid 1 \leq v \leq i - 1 \} - \\
\{ (z_{ij}, z_{iv}) \mid 1 \leq j \leq v \leq k \},
\]

\[
H_{k+i-1} = F_{k+i-1} \cup \{ (z_{ii}, b_i) \} \cup \{ (Z \cup B) \times R(F_{k+i-1}) \} - \\
\{ (z_{k+i-1,j}, z_{k+i-1,v}) \mid 1 \leq j \leq v \leq k \}.
\]

(iii) For all \( m \) with \( 2k \leq m \leq n - 1 \), we construct the Ferrers relation \( H_m \) in \( P_{n,k} \times P_{n,k} - I \). Then there are \( F_m \in \mathcal{F} \) such that

\[
H_m = F_m \cup \{ (Z \cup B) \times R(F_m) \} \cup \{ (z_{nm}, z_{nu}) \mid 1 \leq l \leq m - 1 \} - \\
\{ (z_{mj}, z_{mu}) \mid 1 \leq j \leq l \leq k \}.
\]

Note that \( R(F_i) \cap R(F_{k+i-1}) = \emptyset \) and \( R(F_i) \cap C(G_j) = \{ z_{ij} \} \) for all \( F_i, F_{k+i-1} \in \mathcal{F} \) and \( G_j \in \mathcal{G} \). Then \( H_1, H_n, H_i, H_{k+i-1} (2 \leq i \leq k) \) and \( H_m (2k \leq m \leq n - 1) \) are Ferrers relations in \( P_{n,k} \times P_{n,k} \). Furthermore, \( (H_1 \cup H_n) \cup \bigcup_{i=2}^{k} (H_i \cup H_{k+i-1}) \cup \bigcup_{m=2k}^{n-1} H_m = (X \cup Z \cup B) \times (X \cup Z \cup B) - I \) and hence \( \dim(P_{n,k}) \leq 2k + (n - 2k) = n \). Since \( S_n \subset P_{n,k} - B \), it follows that \( \dim(P_{n,k}) \geq \dim(P_{n,k} - B) \geq n \). Hence we conclude that \( \dim(P_{n,k}) = n \).

**Case 2.** \( k \leq n < 2k \).

Let \( s \) and \( t \) be positive integers with

\[
s = 2\left[ \frac{n + k}{3} \right] - k \text{ and } t = k - \left[ \frac{n + k}{3} \right].
\]

We will construct the distinct \( n + k - \left[ \frac{n + k}{3} \right] \)-Ferrers relations in \( P_{n,k} \times P_{n,k} - I \) as follows:

(i) \( H_{11} = F_1 \cup \{ (G_k - \{(z_{1k}, b_k)\}) \} \cup \{ (X \cup Z \cup B) \times (X - \{x_1\}) \} \cup \\
(Z \cup B) \times R(F_1) - \{ (x_i, x_j) \mid 2 \leq i \leq j \leq n \} - \\
\{ (z_{iu}, z_{iv}) \mid 1 \leq u \leq v \leq k \},
\]

\[
H_{12} = F_2 \cup \{ (z_{1k}, b_k) \} \cup \{ (X \cup Z \cup B - \{x_1\}) \times (X - \{x_2\}) \} \cup B \times B \cup \\
( (Z \cup B - R(F_1)) \times (Z - R(F_1)) ) \cup \{ (z_{1k}) \times (Z - R(F_1)) \} - \\
\{ (x_i, x_j) \mid 3 \leq j \leq i \leq n \} - \{ (b_i, b_j) \mid 1 \leq j \leq i \leq k \} - \\
\{ (z_{ij}, z_{uv}) \mid i < u \text{ or } u = i, 2 \leq j \leq v \leq k \text{ with } 1 \leq i, u \leq n \}.
\]
The dimension of the convolution of bipartite ordered sets

\[ H'_{11} = G_1 \cup \left( \bigcup_{i=1}^{n-2} (R(F_{i+1}) \cup \cdots \cup R(F_i)) \times \{z_{i+1,2}\} \right) \cup B \times B \cup \left( (C(G_1) \cup B) \times C(G_2) \right) - \\
\{(z_{nu}, z_{nu}) \mid 1 \leq v \leq u \leq k\} - \{(b_i, b_j) \mid 1 \leq i \leq j \leq k\}, \]

\[ H'_{12} = G_2 \cup \left( \bigcup_{i=1}^{n-2} (R(F_{i+1}) \cup \cdots \cup R(F_i)) \times \{z_{i+1,2}\} \right) \cup \left( (Z - R(F_n)) \times \{z_{n1}\} \right) \cup \\
\left( (Z - R(F_n)) \times \{Z - R(F_n)\} \right) - \\
\{(z_{ij}, z_{uw}) \mid u \leq i \text{ or } i = u, 1 \leq v \leq j \leq k \text{ with } 1 \leq i, u \leq n\} - \\
\left[ Z \times (C(G_2) \cup \{z_{n3}, z_{n4}, \ldots, z_{nk}\}) \right]. \]

(ii) For \( i = 2, 3, \ldots, s + 1 \), let

\[ H_{i1} = F_{i+1} \cup \left( G_{2t+i-1} \cup \{z_{i+1,2t+i-1}, b_{2t+i-1}\} \right) \cup \\
\left[ (C(G_{2t+i-1}) - \{z_{i+1,2t+i-1}\}) \times R(F_{i+1}) \right], \]

\[ H_{i2} = F_{i+1} \cup \left( \{z_{i+1,2t+i-1}, b_{2t+i-1}\} \right) \cup \left( z_{i+1,2t+i-1} \right) \times R(F_{i+1}). \]

(iii) For \( j = 2, 3, \ldots, t \), let

\[ H'_{j1} = G_{j+1} \cup \left( F_{2s+j+1} \cup \{z_{2s+j+1}, z_{2s+j+1}\} \right) \cup \\
\left[ (C(G_{j+1}) \times R(F_{2s+j+1}) - \{z_{2s+j+1}\}) \right], \]

\[ H'_{j2} = G_{j+1} \cup \left( \{z_{2s+j+1}, z_{2s+j+1}\} \right) \cup \left[ C(G_{j+1}) \times \{z_{2s+j+1}\} \right]. \]

(iv) For all \( m = 2s + t + 2, 2s + t + 3, \ldots, n - 1 \), let \( H_m = F_m \).

Note that

\[(*) \quad \cdots \quad R(F_i) \cap R(F_j) = \emptyset, C(G_u) \cap C(G_v) = \emptyset \quad \text{and} \quad R(F_i) \cap C(G_u) = \{z_{iu}\} \]

for all \( i, j = 1, 2, \ldots, s \) with \( i \neq j \) and \( u, v = 1, 2, \ldots, t \) with \( u \neq v \).

Then \( H_{i1}, H_{i2}(i = 1, 2, \ldots, s+1), H'_{j1}, H'_{j2}(j = 1, 2, \ldots, t) \) and \( H_{m}(m = 2s + t + 2, 2s + t + 3, \ldots, n - 1) \) are Ferrers relations in \( P_{n,k} \times P_{n,k} \) and

\[ \bigcup_{i=1}^{s+1}(H_{i1} \cup H_{i2}) \cup \bigcup_{j=1}^{t}(H'_{j1} \cup H'_{j2}) \cup \bigcup_{r=2s+t+2}^{n-1} H_r = P_{n,k} \times P_{n,k} - I. \]

Note that \( 2 \times (s + 1) + 2 \times t + (n - 1 - 2s - t - 1) = n + k - \left\lceil \frac{n+k}{3} \right\rceil \).

Hence we have \( \dim(P_{n,k}) \leq n + k - \left\lceil \frac{n+k}{3} \right\rceil \).

From \((*)\), it follows by Theorem 2.2 that \( \text{fdim}((X \cup Z) \times (X \cup Z)) = \dim_l(X, Z, I) \) and \( \text{fdim}((Z \cup B) \times (Z \cup B)) = \dim_l(Z, B, I) \). By Theorem
3.2, we have \( \dim(P_{n,k}) \geq n + k - \lfloor \frac{n+k}{3} \rfloor \). We conclude that
\[
\dim(P_{n,k}) = n + k - \lfloor \frac{n+k}{3} \rfloor.
\]
\( \Box \)

ACKNOWLEDGEMENT. The author wishes to thank H. C. Jung for help in the writing of this paper.

References


Department of Mathematics
Sogang University
Seoul 121-742, Korea
*E-mail*: bae@math.sogang.ac.kr