STATIONARY SOLUTIONS FOR
ITERATED FUNCTION SYSTEMS
CONTROLLED BY STATIONARY PROCESSES

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ABSTRACT. We consider a class of discrete parameter processes on
a locally compact Banach space $S$ arising from successive com-
positions of strictly stationary random maps with state space $C(S, S)$,
where $C(S, S)$ is the collection of continuous functions on $S$ into
itself. Sufficient conditions for stationary solutions are found. Exis-
tence of $p$th moments and convergence of empirical distributions for
trajectories are proved.

1. Introduction

Let $S$ be a locally compact Banach space with a norm $| \cdot |$ and let
$C(S, S)$ be the set of all continuous functions from $S$ into itself. En-
dow $C(S, S)$ with the compact open topology and $C(S, S)$ is a complete
separable metric space.

Let $(\Omega, \mathcal{F}, P)$ be a probability space on which is defined a sequence
of stationary random maps $\Gamma_0, \Gamma_1, \Gamma_2, \cdots$ taking values on $C(S, S)$.

In this paper we consider the following sequence of process $\{X_n : n \geq 0\}$ on $S$,

\begin{equation}
X_0, \quad X_{n+1} = \Gamma_n(X_n), \quad (n \geq 0)
\end{equation}

where $X_0$ is arbitrarily prescribable $S$-valued random variable indepen-
dent of $\{\Gamma_n : n \geq 0\}$.

Stationarity of model (1.1) is of importance in statistical inference
and sufficient conditions for stationarity for various models have been

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nology, 1997.
found in many papers (see, e.g. [2], [3], and [6]-[15]). If we assume that \( \{ \Gamma_n : n \geq 0 \} \) are independent and identically distributed, then \( \{ X_n \} \) given by (1.1) is a Markov chain. For Markov chains, stationary condition can be found by looking for conditions for ergodicity and in this case, Markov property can be used to investigate the problems. In [11] and references therein, sufficient conditions for ergodicity of \( \varphi \)-irreducible Markov chains are obtained, by using, so called Tweedie's criterion. Ergodicity of Markov chains without \( \varphi \)-irreducibility assumption is studied in, for example, [3], [9] and [15]. Processes given by more general sequences, such as, stationary process, finite Markov chain, finite semi-Markov chain, regenerative process are investigated in [8], [2], [13], and [12], respectively. The process \( \{ X_n \} \) generated by a stationary process \( \{ \Gamma_n \} \) is considered in [8] and it is proved that negative Lyapunov exponent with some additional assumptions is sufficient for stationarity of \( \{ X_n \} \). Time reversal idea and Kingman's subadditive ergodic theorem are used for the proof in [8].

Our main objective in this paper is to find out conditions that ensure the existence of a unique stationary solution of (1.1) via the \( L^p \) contraction. Also finiteness of \( p \)th moment of the limiting distribution and convergence of the empirical distribution of the trajectories are obtained.

Since \( C(S, S) \) is a complete separable metric space, without loss of generality we may assume that \( \{ \Gamma_n : n \geq 0 \} \) extends backward in time to \( \{ \Gamma_n : -\infty < n < \infty \} \). For solving our problems, it is more convenient to deal with the equation

\[
X_{n+1} = \Gamma_n(X_n), \quad n \in \mathbb{Z}
\]

where \( \mathbb{Z} \) denotes the set of all integers.

For \( S \)-valued random variable \( X : \Omega \to S \), define

\[
\| X \|_p = \begin{cases} 
(E |X|^p)^{1/p} & \text{if } 1 \leq p < \infty \\
E |X|^p & \text{if } 0 < p < 1 \\
\text{ess. sup. } |X| & \text{if } p = \infty 
\end{cases}
\]

We write \( X \in L^p \) if \( \| X \|_p < \infty \).

Note that for \( S \)-valued random variables \( X \) and \( Y \),

\[
\| X + Y \|_p \leq \| X \|_p + \| Y \|_p, \quad \forall p \in (0, \infty].
\]
We make the following assumption.

**Assumption A.** There exist \( p \in (0, \infty), \ x_0 \in S \) and some \( \delta > 0 \) such that for every compact set \( C \subset S \),

\[
\|\text{diam}(\Gamma_n \cdots \Gamma_0 C)\|_p \to 0 \quad \text{as} \quad n \to \infty
\]

and

\[
\sup_{n \geq 0} \|\Gamma_n \cdots \Gamma_1 \Gamma_0 x_0\|_{p+\delta} < \infty.
\]

Here we write \( \gamma x \) for the value of the map \( \gamma \in C(S, S) \) at \( x \), and
\[
\gamma_n \cdots \gamma_1 \gamma_0
\]
for the compositions of the maps \( \gamma_0, \gamma_1, \ldots, \gamma_n \). Also write for \( B \subset S \), \( \text{diam}(\gamma B) = \sup\{|\gamma x - \gamma y| : x, y \in B\} \).

2. Main Results

For any \( x \in S, \ -\infty < k < \infty, \ n \geq 1 \), we define

\[
X_{k,n}(x) = \Gamma_{k-1} \Gamma_{k-2} \cdots \Gamma_{k-n} x.
\]

Our first main result is:

**Theorem 1.** Suppose that \( \{\Gamma_n\} \) is a stationary process in \( C(S, S) \) satisfying the Assumption A. Then there exists a process \( \{Y_k : -\infty < k < \infty\} \) such that

1. for any \( x \in S \), \( X_{k,n}(x) \) converges in \( L^p \) to \( Y_k \) as \( n \to \infty \) and the distribution of \( Y_k \) is independent of \( x \),
2. if we take \( X_0 = Y_0 \), then \( \{Y_k : k \geq 0\} \) is a unique stationary solution of the equation (1.1), and
3. for any \( x \in S \), \( \Gamma_n \Gamma_{n-1} \cdots \Gamma_0 x \) converges in distribution to \( Y_0 \) as \( n \to \infty \).

To prove the theorem 1, we start with the following lemma:

**Lemma 1.** Let the Assumption A hold. Then for any \( k \in \mathbb{Z} \),
\( \{X_{k,n}(x_0) : n \geq 1\} \) is Cauchy in \( L^p \).

**Proof.** For \( \alpha > 0 \), we define \( h_\alpha : S \to S \) by

\[
h_\alpha(x) = \begin{cases} x & \text{if } |x| < \alpha \\ 0 & \text{if } |x| \geq \alpha. \end{cases}
\]
Then
\begin{align}
(2.2) \quad & \|X_{k,n}(x_0) - X_{k,n+m}(x_0)\|_p \\
& \leq \|X_{k,n}(x_0) - h_\alpha(X_{k,n}(x_0))\|_p + \|h_\alpha(X_{k,n}(x_0)) - h_\alpha(X_{k,n+m}(x_0))\|_p \\
& \quad + \|h_\alpha(X_{k,n+m}(x_0)) - X_{k,n+m}(x_0)\|_p.
\end{align}

Let three terms on the righthand side of (2.2) be denoted by $I_1$, $I_2$ and $I_3$ respectively.

We can easily show that
\begin{align}
(2.3) \quad & E|X_{k,n}(x_0) - h_\alpha(X_{k,n}(x_0))|^p = E \left[ |X_{k,n}(x_0)|^p I_{\{|X_{k,n}(x_0)| \geq \alpha\}} \right] \\
& \leq \frac{1}{\alpha^\delta} E|X_{k,n}(x_0)|^{p+\delta}
\end{align}
and
\begin{align}
(2.4) \quad & E|h_\alpha(X_{k,n}(x_0)) - h_\alpha(X_{k,n+m}(x_0))|^p \\
& \leq E \left[ |X_{k,n}(x_0) - X_{k,n+m}(x_0)|^p I_{\{|X_{k,n}(x_0)| < \alpha, |X_{k,n+m}(x_0)| \geq \alpha\}} \right] \\
& \quad + \alpha^p P(|X_{k,n}(x_0)| \geq \alpha) + \alpha^p P(|X_{k,n+m}(x_0)| \geq \alpha).
\end{align}

From (1.5) and stationarity of $\{\Gamma_n\}$, we have for any $k \in \mathbb{Z}$,
\begin{align}
(2.5) \quad & \frac{1}{\alpha^\delta} E|X_{k,n}(x_0)|^{p+\delta} \longrightarrow 0
\end{align}
and
\begin{align}
(2.6) \quad & \alpha^p P(|X_{k,n}(x_0)| \geq \alpha) \longrightarrow 0
\end{align}
as $\alpha \to \infty$ uniformly in $n$, $n \geq 1$.

Now let $\epsilon > 0$ be given. Then from (2.3)-(2.6), we may choose $\alpha > 0$ sufficiently large that for any $k \in \mathbb{Z}$,
\begin{align}
(2.7) \quad & I_1 + I_3 + \alpha^p P(|X_{k,n}(x_0)| \geq \alpha) + \alpha^p P(|X_{k,n+m}(x_0)| \geq \alpha) < \frac{\epsilon}{2},
\end{align}
uniformly in $m$ and $n$ ($n, m \geq 1$).

For $\alpha$ satisfying (2.7) above, we take $M \gg 2\alpha$ so that for any $k \in \mathbb{Z}$,
\begin{align}
(2.8) \quad & (2\alpha)^p P(|\Gamma_{k-n-1} \cdots \Gamma_{k-n-m} x_0| > M) \leq (2\alpha)^p \sup_n \frac{\|\Gamma_n \cdots \Gamma_0 x_0\|_{p+\delta}}{M^{p+\delta}} < \frac{\epsilon}{4},
\end{align}
for all positive integers $m$ and $n$. 
Take $B_M = \{ x \in S : |x| \leq M \}$. Then we may assume, without loss of generality, that $x_0 \in B_M$ and hence

\begin{equation}
E[|X_{k,n}(x_0) - X_{k,n+m}(x_0)|^p I_{\{|X_{k,n}(x_0)| < \alpha, |X_{k,n+m}(x_0)| < \alpha\}}]
\leq E[|X_{k,n}(x_0) - X_{k,n+m}(x_0)|^p I_{\{\Gamma_{k-n-1} \cdots \Gamma_{k-n-m} x_0 \leq M\}}]
+ E[|X_{k,n}(x_0) - X_{k,n+m}(x_0)|^p I_{\{|X_{k,n}(x_0)| < \alpha, |X_{k,n+m}(x_0)| < \alpha\}} I_{\{\Gamma_{k-n-1} \cdots \Gamma_{k-n-m} x_0 > M\}}]
\leq E[\text{diam}(\Gamma_{k-n-1} \cdots \Gamma_{k-n} B_M)]^p
+ (2\alpha)^p P(|\Gamma_{k-n-1} \cdots \Gamma_{k-n-m} x_0| > M).
\end{equation}

Moreover, by (1.4), there exist $n_0 = n_0(\epsilon)$ such that for any $n$, $n > n_0$

\begin{equation}
||\text{diam}(\Gamma_{k-1} \cdots \Gamma_{k-n} B_M)||_p < \frac{\epsilon}{4}, \quad \forall k \in \mathbb{Z}.
\end{equation}

When $0 < p < 1$, combining (2.2)-(2.10), we have for each $k$, if $n > n_0$,

\begin{equation}
\| X_{k,n}(x_0) - X_{k,n+m}(x_0) \|_p < \epsilon, \quad m = 1, 2, 3, \cdots.
\end{equation}

For the case $1 \leq p < \infty$, in the same manner as above, there are $\alpha > 0$ and $M \gg 2\alpha$ such that for any $k$ in $\mathbb{Z}$,

\begin{equation}
I_1 + I_3 + \alpha \{ P(|X_{k,n}(x_0)| \geq \alpha) \}^{\frac{1}{p}} + \alpha \{ P(|X_{k,n+m}(x_0)| \geq \alpha) \}^{\frac{1}{p}} < \frac{\epsilon}{2},
\end{equation}

and

\begin{equation}
2\alpha \{ P(|\Gamma_{k-n-1} \cdots \Gamma_{k-n-m} x_0| > M) \}^{\frac{1}{p}} \leq \frac{\epsilon}{4},
\end{equation}

for all positive integers $m$ and $n$, and we obtain (2.11), by using (2.10), (2.12) and (2.13).

Now we assume that $p = \infty$. Then by (1.5) there exists $K_0$ such that $\sup_n \| \Gamma_n \cdots \Gamma_0 x_0 \|_{\infty} \leq K_0 < \infty$. If we take $K = \max\{|x_0|, K_0\}$, then by (1.4), for $k \in \mathbb{Z}$,

\begin{equation}
\| X_{k,n}(x_0) - X_{k,n+m}(x_0) \|_{\infty} \leq \| \text{diam}(\Gamma_{k-1} \cdots \Gamma_{k-n} B_K) \|_{\infty} \to 0
\end{equation}
as $n \to \infty$, uniformly in $m$.

Hence for any $p$, $0 < p \leq \infty$, $\{ X_{k,n}(x_0) \}$ is Cauchy in $L^p$.

We may now prove the theorem 1.

Proof of theorem 1. (1) By completeness of $L^p$ space, $0 < p \leq \infty$
and above lemma, for each k, \( \{X_{k,n}(x_0)\} \) converges in \( L^p \) to some random variable in \( L^p \), say \( Y_k \). But for any \( x \in S \),

\[
\|X_{k,n}(x) - Y_k\|_p \leq \|X_{k,n}(x) - X_{k,n}(x_0)\|_p + \|X_{k,n}(x_0) - Y_k\|_p \to 0
\]
as \( n \to \infty \), by assumption (1.4) and lemma 1. Hence \( L^p \)-limit of \( X_{k,n}(x) \) is independent of \( x \).

(2) Let \( k \) be fixed. Since \( X_{k,n}(x_0) \to Y_k \) in \( L^p \), there exists a subsequence \( \{X_{k,n_j}(x_0) : j \geq 0\} \) such that \( X_{k,n_j}(x_0) \to Y_k \) a.s. as \( j \to \infty \). Since \( \{\Gamma_k(X_{k,n_j}(x_0)) : j \geq 0\} \) is a subsequence of \( \{X_{k+1,n}(x_0) : n \geq 1\} \), by uniqueness of \( L^p \)-limit, \( \Gamma_k(X_{k,n_j}(x_0)) \) and \( X_{k+1,n}(x_0) \) have the same limit, \( Y_{k+1} \), i.e.,

\[
(2.15) \quad \Gamma_k(X_{k,n_j}(x_0)) \to Y_{k+1} \text{ in } L^p \text{ as } j \to \infty.
\]

On the other hand, each \( \gamma \in C(S,S) \), \( \gamma(X_{k,n_j}) \to \gamma(Y_k) \) a.s. and hence

\[
(2.16) \quad \Gamma_k(X_{k,n_j}(x_0)) \to \Gamma_k(Y_k) \text{ a.s.}
\]

From (2.15) and (2.16) and uniqueness of the limit, \( Y_{k+1} = \Gamma_k(Y_k) \) a.s., which says that \( Y_k \) is a solution of (1.1).

Stationarity of \( \{Y_k\} \) follows from that of \( \{\Gamma_k\} \).

(3) For \( \forall x \in S \), \( \Gamma_n \cdots \Gamma_0 x = \Gamma_{-1} \Gamma_{-2} \cdots \Gamma_{-n-1} x \to Y_0 \) in \( L^p \) as \( n \to \infty \), from which \( \Gamma_n \cdots \Gamma_0 x \to Y_0 \) in distribution as \( n \to \infty \).

**Theorem 2.** Suppose that the Assumption A holds. In addition, assume for any compact set \( C \subset S \), \( \text{diam}(\Gamma_n \cdots \Gamma_0 C) \to 0 \) a.s. as \( n \to \infty \). Then for every \( x \in S \),

\[
(2.17) \quad \frac{1}{n} \sum_{i=0}^{n-1} f(\Gamma_i \cdots \Gamma_0 x) \to E(f(Y_0)|I) \text{ a.s}
\]
as \( n \to \infty \) for every bounded continuous real valued function \( f \) on \( S \), where \( I \) is an invariant \( \sigma \)-field of \( \{\Gamma_n\} \).

**Proof.** Since \( E|f(Y_0)| < \infty \), by ergodic theorem,

\[
(2.18) \quad \frac{1}{n} \sum_{i=0}^{n-1} f(\Gamma_i \cdots \Gamma_0 Y_0) \to E(f(Y_0)|I) \text{ a.s.}
\]
as \( n \to \infty \). For any given \( x \in S \), choose \( r > 0 \) such that \( |x| < r \) and then define \( \Omega_r = \{\omega \in \Omega : |Y_0(\omega)| \leq r\} \). Define \( Y_0^r \) by \( Y_0^r = Y_0 \) on \( \Omega_r \) and 0 on
Stationary solutions for iterated function systems

\( \Omega - \Omega_r \). Clearly, on \( \Omega_r \),
\[
\frac{1}{n} \sum_{i=0}^{n-1} f(\Gamma_i \cdots \Gamma_0 Y_0^r) = \frac{1}{n} \sum_{i=0}^{n-1} f(\Gamma_i \cdots \Gamma_0 Y_0).
\]
Moreover,
\[
(2.19) \quad |\Gamma_n \cdots \Gamma_0 Y_0^r - \Gamma_n \cdots \Gamma_0 x| \leq \text{diam}(\Gamma_n \cdots \Gamma_0 B_r) \to 0 \quad a.s.
\]
as \( n \to \infty \).

Therefore on \( \Omega_r \), for every bounded uniformly continuous function \( f \),
\[
(2.20) \quad \frac{1}{n} \sum_{i=0}^{n-1} f(\Gamma_i \cdots \Gamma_0 x) \to E(f(Y_0)|I) \quad a.s.
\]
But from the fact that every continuous function defined on a compact set is uniformly continuous and the assumption \( \text{diam}(\Gamma_n \cdots \Gamma_0 B_r) \to 0 \) as \( n \to \infty \), (2.20) holds for every bounded continuous function, which together with \( P(\Omega_r) \to 1 \) implies the conclusion.

For \( \gamma \in C(S,S) \), define a generalized norm
\[
l(\gamma) = \sup_{x \neq y} \frac{|\gamma(x) - \gamma(y)|}{|x - y|}.
\]

Let \( \text{Lip}(S,S) = \{ \gamma \in C(S,S) \mid l(\gamma) < \infty \} \). Then a function \( \gamma \in \text{Lip}(S,S) \) is called a Lipschitzian map on \( S \) to \( S \).

**Remark.** Suppose that \( \{ \Gamma_n \} \) is a sequence of independent and identically distributed random elements taking values on \( \text{Lip}(S,S) \). Then \( \{ X_n \} \) obtained recursively by (1.1) is a Markov chain and using Markov property, it is proved that assumptions \( E \log^+ l(\Gamma_1) < 0 \) and \( E \log^+ |x_0 - \Gamma_1(x_0)| < \infty \) for some \( x_0 \) in \( S \) are sufficient for ergodicity (stationarity) of \( X_n \) (see, e.g., [8],[9],[12]).

For following corollaries, we assume that \( \{ \Gamma_n : n \geq 0 \} \) is a sequence of stationary processes in \( \text{Lip}(S,S) \).

**Corollary 1.** Suppose that \( \{ \Gamma_n \} \) is a stationary process in \( \text{Lip}(S,S) \). If there exists \( p \in (0, \infty) \) such that \( \sup_{n \geq 0} ||\Gamma_n \cdots \Gamma_0 x_0||_p < \infty \) for some \( x_0 \) and \( ||l(\Gamma_n \cdots \Gamma_0)||_p \to 0 \) as \( n \to \infty \), then the conclusions of theorem 1 hold with \( p', 0 < p' < p \).

**Proof.** For given \( 0 < p < \infty \), choose \( p' > 0 \) and \( \delta > 0 \), such that \( p = p' + \delta \). Note that \( |\gamma x - \gamma y| \leq l(\gamma)|x - y|, \forall \gamma \in \text{Lip}(S,S) \) and
\[
(E||l(\Gamma_n \cdots \Gamma_0)||_{p'})^\delta \leq E(||l(\Gamma_n \cdots \Gamma_0)||_{p})^\delta, \quad 0 < p' < p.
\]
Since for every compact set \( C \subset S \),
\[
\|\operatorname{diam}(\Gamma_n \cdots \Gamma_0 C)\|_{p'} = \|l(\Gamma_n \cdots \Gamma_0)\operatorname{diam}C\|_{p'} \to 0 \quad \text{as} \quad n \to \infty,
\]
(1.4) and (1.5) hold with \( p = p' \) and hence the conclusion follows.

**Corollary 2.** Suppose that for a stationary process \( \{\Gamma_n\} \) in \( \operatorname{Lip}(S, S) \), \( \sum_{n=0}^{\infty} \|l(\Gamma_n \cdots \Gamma_0)\|_p < \infty \), for some \( p \in (0, \infty) \). If \( \|\Gamma_0 x_0\|_r < \infty \) for some \( x_0 \in S \) and \( r > p \), then the conclusions in theorem 1 hold with \( p = p', 0 < p' < p \).

**Proof.** For given \( p > 0, \), \( 0 < p' < p \), and \( \delta > 0 \) be such that \( p = p' + \delta \). For any compact set \( C \subset S \), we have
\[
\|\operatorname{diam}(\Gamma_n \cdots \Gamma_0 C)\|_{p'} \leq \|\operatorname{diam}C \cdot l(\Gamma_n \cdots \Gamma_0)\|_{p'} \to 0,
\]
as \( n \to \infty \). Apply the Hölder’s inequality to get
\[
\|\Gamma_0 \cdots \Gamma_{n-1} x_0 - \Gamma_0 \cdots \Gamma_{n+1} x_0\|_{p' + \delta} \leq \|l(\Gamma_0 \cdots \Gamma_{n+1}) \cdot |\Gamma_{n-1} x_0 - x_0|\|_{p' + \delta} \leq \|l(\Gamma_0 \cdots \Gamma_{n+1})\|_{p' + \delta} \cdot \|\Gamma_{n-1} x_0 - x_0\|_r
\]
where \( \|\Gamma_{n-1} x_0 - x_0\|_r \leq K = \|\Gamma_{n-1} x_0\|_r + \|x_0\|_r < \infty \) with \( r = \frac{p(2p-\delta)}{\delta} \).

Now from stationarity of \( \{\Gamma_n\} \) and (2.22), we have
\[
\|\Gamma_0 \cdots \Gamma_0 x_0\|_{p' + \delta} = \|\Gamma_0 \cdots \Gamma_{n-1} x_0\|_{p' + \delta} \leq \|\Gamma_0 \cdots \Gamma_{n-1} x_0 - \Gamma_0 \cdots \Gamma_{n-1} x_0\|_{p' + \delta} + \|\Gamma_0 \cdots \Gamma_{n+1} x_0 - \Gamma_0 \cdots \Gamma_{n+1} x_0\|_{p' + \delta} + \cdots + \|\Gamma_0 \cdots \Gamma_{n-1} x_0 - \Gamma_0 x_0\|_{p' + \delta} + \|\Gamma_0 x_0\|_{p' + \delta} \leq K \left( \sum_{k=0}^{n-1} \|l(\Gamma_{k} \cdots \Gamma_0)\|_{p' + \delta} + 1 \right),
\]
and hence
\[
\sup_n \|\Gamma_0 \cdots \Gamma_0 x_0\|_{p' + \delta} < \infty.
\]
If \( r > p \), then \( r \) can be written by \( r = \frac{p(2p-\delta)}{\delta} \) for some \( \delta > 0, 0 < \delta < p \), and hence we can find \( p' > 0, 0 < p' < p \) satisfying \( p = p' + \delta \). Therefore by (2.21) and (2.23), proof is completed.

**Remark.** The assumptions in corollary 2 can be weakened as follows: there exists a positive integer \( m_0 \) such that \( \sum_{n=0}^{\infty} \|l(\Gamma_{n-1} \cdots \Gamma_0)\|_p < \infty \) for some \( p \in (0, \infty) \) and \( \sup_{0 \leq n < m_0} \|\Gamma_0 \cdots \Gamma_0 x_0\|_r < \infty \) for some
\( x_0 \in S \), and \( r > p \). The proof follows essentially the same line of corollary 2.

**Remark.** Let \( x \mapsto A_n x + b_n \) be random stationary affine maps of \( S \) into itself. Here \( A_n \) denotes a random linear operator on \( S \), \( b_n \) a random vector in \( S \). Consider the sequence \( X_{n+1} = A_n X_n + b_n \). Then \( \| \Gamma_n \cdots \Gamma_1 \|_L = \| A_n \cdots A_1 \| \), where \( \| \cdot \| \) denotes the operator norm. Hence if \( \sum_{n=1}^{\infty} \| A_n \cdots A_1 \|_p < \infty \) for some \( p > 0 \) and \( E|b_0|^r < \infty, \forall r > p \), then there exists a stationary solution \( \{ Y_n \} \) such that \( X_n \) converges to \( Y_0 \) in \( L^{p'} \), \( 0 < p' < p \).

**References**


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