CORRECTIONS TO "A UNIFIED FIXED POINT THEORY OF MULTIMAPS ON TOPOLOGICAL VECTOR SPACES"

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ABSTRACT. This is to correct Section 4 of our previous work [1].

Section 4 of our previous work [1] is incorrectly stated and our aim in this note is to replace the first part of Section 4 (from the beginning to the line 23 of page 815) by the following:

4. New fixed point theorems for condensing multimaps

In this section, we deduce new theorems for condensing maps.

Let $X$ be a closed convex subset of a t.v.s. $E$ and $C$ a lattice with a least element, which is denoted by 0. A function $\Phi : 2^X \to C$ is called a measure of noncompactness on $X$ provided that the following conditions hold for any $A, B \in 2^X$:

1. $\Phi(A) = 0$ if and only if $A$ is relatively compact;
2. $\Phi(\overline{A}) = \Phi(A)$; and
3. $\Phi(A \cup B) = \max\{\Phi(A), \Phi(B)\}$.

It follows that $A \subseteq B$ implies $\Phi(A) \leq \Phi(B)$.

The above notion is a generalization of the set-measure $\gamma$ and the ball-measure $\chi$ of noncompactness defined in terms of a family of seminorms or a norm.

For a measure $\Phi$ of noncompactness on $E$, a map $T : X \to E$ is said to be $\Phi$-condensing provided that if $A \subseteq X$ and $\Phi(A) \leq \Phi(T(A))$, then $A$ is relatively compact; that is, $\Phi(A) = 0$.

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From now on, we assume that $\Phi$ is a measure of noncompactness on the given set $X$ in a t.v.s. $E$ or on $E$ if necessary.

Note that any map defined on a compact set or any compact map is $\Phi$-condensing. Especially, if $E$ is locally convex, then a compact map $T : X \to E$ is $\gamma$- or $\chi$-condensing whenever $X$ is complete or $E$ is quasi-complete.

The following is well-known; for example, see Mehta et al. [1997].

**Lemma.** Let $X$ be a nonempty closed convex subset of a t.v.s. $E$ and $T : X \to X$ a $\Phi$-condensing map. Then there exists a nonempty compact convex subset $K$ of $X$ such that $T(K) \subset K$.

Note that even if $X$ is admissible, we can not say that $K$ is admissible in $E$. Therefore, we need the following concept:

A nonempty subset $X$ of a t.v.s. $E$ is said to be $q$-admissible if any nonempty compact convex subset $K$ of $X$ is admissible. We give some examples of $q$-admissible sets as follows:

1. Any nonempty locally convex subset of a t.v.s.
2. Any nonempty subset of a locally convex t.v.s.
3. Any nonempty subset of a t.v.s. $E$ on which its topological dual $E^*$ separates point. Note that any compact convex subset of such a space $E$ is affinely embeddable in a locally convex t.v.s.; see Weber [1992b].

It should be noted that an admissible t.v.s. (in the sense of Klee [1960]) and a $q$-admissible t.v.s. can be also defined.

From Theorem 1 and Lemma, we have the following:

**Theorem 2.** Let $X$ be a $q$-admissible closed convex subset of a t.v.s. $E$. Then any $\Phi$-condensing map $F \in \mathcal{B}(X, X)$ has a fixed point.

**Proof.** By Lemma, there is a nonempty compact convex subset $K$ of $X$ such that $F(K) \subset K$. Since $F \in \mathcal{B}(X, X)$, there exists a closed map $\Gamma \in \mathcal{B}(K, K)$ such that $\Gamma(x) \subset F(x)$ for all $x \in K$. Since $\Gamma$ is compact and $K$ is admissible, by Corollary 1.1, it has a fixed point $x_0 \in K$; that is, $x_0 \in \Gamma(x_0) \subset F(x_0)$. This completes our proof. $\square$
COROLLARY 2.1. Let \( X \) be a \( q \)-admissible closed convex subset of a t.v.s. \( E \). Then any closed \( \Phi \)-condensing map \( F \in \mathcal{B}(X,X) \) has a fixed point.

COROLLARY 2.2. Let \( X \) be a \( q \)-admissible closed convex subset of a t.v.s. \( E \). Then any \( \Phi \)-condensing map \( F \in \mathcal{B}^\sigma(X,X) \) has a fixed point.

In the remainder of this section, we list more than ten papers in chronological order, from which we can deduce particular forms of Theorem 2.

Darbo [1955]: Recall that Kuratowski defined the measure of noncompactness, \( \alpha(A) \), of a bounded subset \( A \) of a metric space \( (X,d) \):

\[
\alpha(A) = \inf\{\varepsilon > 0 : A \text{ can be covered by a finite number of sets of diameter less than or equal to } \varepsilon\}. 
\]

Let \( T : X \to X \) be a continuous map. Darbo calls \( T \) an \( \alpha \)-\textit{contraction} if given any bounded set \( A \) in \( X \), \( T(A) \) is bounded in \( X \) and

\[
\alpha[T(A)] \leq k\alpha(A),
\]

where the constant \( k \) fulfills the inequality \( 0 \leq k < 1 \).

Darbo [1955] showed that if \( G \) is a closed, bounded, convex subset of a Banach space \( X \) and \( T : G \to G \) is an \( \alpha \)-contraction, then \( T \) has a fixed point.

Sadovskii [1967]: Introduced the notion of condensing maps in Banach spaces and obtained a form of Corollary 2.1 extending the above result of Darbo.

\textit{This is the end of our corrections.}

REMARKS. 1. Our failure in [1] is mainly based on the unjustified fact that every admissible set is \( q \)-admissible. It would be interesting to prove or disprove this statement.

2. Until now, the results in Section 4 were used for locally convex t.v.s. only. There exists a measure of noncompactness on a certain subset in a more general t.v.s.
3. Similarly, in our another previous work [2, Theorems 3 and 4], the admissibility of $X$ should be replaced by the $q$-admissibility. Moreover, in [3, Theorem 1], $\text{cl} f(D)$ should be replaced by $D$. Further, each of [3, Theorems 2-4] can be slightly improved by replacing the admissibility of $K$ by that of $D$.

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References


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