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SOME REMARKS ON COASSOCIATED PRIMES

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ABSTRACT. The purpose of this paper is to develop the theory of coassociated primes and to investigate Melkersson's question [8].

0. Introduction

Let R be a commutative ring with identity. There have been four attempts to dualize the theory of associated prime ideals by I. G. MacDonald [6], L. Chambless [4], H. Zöschinger [14] and S. Yassemi [12]. In [12], it is shown that, in the case the ring R is Noetherian, these definitions are equivalent. However Yassemi's definition is "in some sense" the best one because it constructs a connection between the theory of associated prime ideals and its dual through the Matlis duality. In section one, we give a new description of Yassemi's definition by using the concept of finitely embedded modules (the dual notion of finitely generated). Next, we will prove some dual results and provide some counterexamples.

Let R be a Noetherian ring and let M be a representable module of finite Goldie dimension. (If the module has a secondary representation, we simply call it representable.) Melkersson [8] conjectured that, for any flat R-module F, the module $\mathrm{Hom}_R(F, M)$ is representable. In section 2, we give an affirmative answer to this question in a special case. Also, in general, we show that the set of coassociated prime ideals of $\mathrm{Hom}_R(F, M)$ is finite. This is an evidence which supports the conjecture.

Throughout, we use $\mathrm{Max}(R)$ to denote the set of all maximal ideals of R. For any ring homomorphism $f : R \longrightarrow S$, we let $f^* : \mathrm{Spec}(S) \longrightarrow \mathrm{Spec}(R)$ be the induced map.

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1. Some Dual Results

It is easy to see that, for an R-module M, the set $\mathrm{Ass}_R(M)$ of associated prime ideals of M is equal to

$$\{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} = (0 :_R N) \text{ for some finitely generated submodule } N \text{ of } M\}.$$

The definition of the dual notion of "finitely generated" is given separately by P. Vámos [11] and F. W. Anderson and K. R. Fuller [1]. It follows from [**1**, Proposition 10.7] and [11, Lemma 1] that these definitions coincide. We use Vámos's definition. Let, for an R-module $M, E(M)$ denote its injective envelope. An R-module M is said to be finitely embedded (f.e.) if

$$E(M) = E(R/\mathfrak{m}_1) \oplus E(R/\mathfrak{m}_2) \oplus \cdots \oplus E(R/\mathfrak{m}_k),$$

where each $\mathfrak{m}_i$ is a maximal ideal of R. It is natural, for an R-module M, to define the set $\mathrm{Coass}_R(M)$ (resp. $\widetilde{\mathrm{Coass}}_R(M)$of coassociated (resp. weakly coassociated) prime ideals of M as follows.

DEFINITION 1.1. $\mathrm{Coass}_R(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} = (0 :_R L)$, for some f.e. homomorphic image L of $M\}$ (resp. $\widetilde{\mathrm{Coass}}_R(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p}$ is minimal over $(0 :_R L)$, for some f.e. homomorphic image L of $M\}$.

The following lemma shows that this definition is equivalent to Yassemi's one ([12], [13]).

LEMMA 1.2. *Let M be an R-module. Then*
(i) $\mathfrak{p} \in \mathrm{Coass}_R(M)$ *if and only if there exists a homomorphic image K of M such that $K \subseteq E(R/\mathfrak{m})$ for some maximal ideal $\mathfrak{m}$ and $\mathfrak{p} = (0 :_R K)$.*
(ii) $\mathfrak{p} \in \widetilde{\mathrm{Coass}}_R(M)$ *if and only if there exists a homomorphic image K of M such that $K \subseteq E(R/\mathfrak{m})$ for some maximal ideal $\mathfrak{m}$ and $\mathfrak{p}$ is minimal over $(0 :_R K)$.*

Proof. (i) The proof of [12, Lemma 1.5] shows that if L is a f.e. R-module with $(0 :_R L) = \mathfrak{p}$, a prime ideal of R, then there exists a homomorphic image K of L such that $(0 :_R K) = \mathfrak{p}$ and $K \subseteq E(R/\mathfrak{m})$ for some maximal ideal of $\mathfrak{m}$ of R. The converse follows from the fact that if $E \neq 0$ is an indecomposable injective module, then it is an injective envelope of every non-zero submodule of itself (see [10, Proposition 2.28]).
(ii) Similar to (i). □

We summarize some important properties of coassociated (resp. weakly coassociated) prime ideals from [12] and [13] in the following fact and we may use them without further comment.

FACT 1.3. Let M be an R-module.
(i) $\mathfrak{p} \in \mathrm{Coass}_R(M)$ (resp. $\mathfrak{p} \in \widetilde{\mathrm{Coass}}_R(M)$) if and only if there exists $\mathfrak{m} \in \mathrm{Max}(R)$ such that $\mathfrak{p} \in \mathrm{Ass}_R(\mathrm{Hom}_R(M, E(R/\mathfrak{m})))$ (resp. $\mathfrak{p} \in \widetilde{\mathrm{Ass}}_R(\mathrm{Hom}_R(M, E(R/\mathfrak{m})))$ (here, for an R-module M, $\widetilde{\mathrm{Ass}}_R(M)$ denotes the set of Bourbaki's weakly associated prime ideals of M).
(ii) $\widetilde{\mathrm{Coass}}_R(M) \neq \phi$ if and only if $M \neq 0$.
(iii) If R is a Noetherian ring, then $\widetilde{\mathrm{Coass}}_R(M) = \mathrm{Coass}_R(M)$.
(iv) If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence of R-modules, then

$$\mathrm{Coass}_R(M'') \subseteq \mathrm{Coass}_R(M) \subseteq \mathrm{Coass}_R(M') \cup \mathrm{Coass}_R(M'')$$

and

$$\widetilde{\mathrm{Coass}}_R(M'') \subseteq \widetilde{\mathrm{Coass}}_R(M) \subseteq \widetilde{\mathrm{Coass}}_R(M') \cup \widetilde{\mathrm{Coass}}_R(M'').$$

(v) If M is representable, then $\widetilde{\mathrm{Coass}}_R(M) = \mathrm{Att}_R(M)$.

We need the following lemma to prove 1.5 and 1.7.

LEMMA 1.4. *Let S be a commutative Noetherian ring and $f : R \longrightarrow S$ a ring homomorphism. Let M be an S-module. Then* $\mathrm{Coass}_R(M) = \widetilde{\mathrm{Coass}}_R(M)$.

Proof. It easily follows from 1.3 (i) and [5, Lemma 1.1]. □

It is well-known that if A is an Artinian R-module, then $\mathrm{Ass}_R(A) = \mathrm{Supp}_R(A) \subseteq \mathrm{Max}(R)$. The third part of the following lemma provides the dual of this fact.

LEMMA 1.5. (i) $\mathrm{Coass}_R(R) = \mathrm{Max}(R)$.
(ii) *If M is a finitely generated R-module, then* $\mathrm{Coass}_R(M) \subseteq \mathrm{Max}(R)$.
(iii) *If N is a Noetherian R-module, then*

$$\mathrm{Coass}_R(N) = \mathrm{Cosupp}_R(N) \subseteq \mathrm{Max}(R),$$

where, for an R-module M, $\mathrm{Cosupp}_R(M)$ is the set of prime ideals $\mathfrak{p}$ such that there exists a f.e. homomorphic image L of M with $\mathfrak{p} \supseteq (0 :_R L)$ (see [12, 2.1]).

Proof. (i) It is clear that $\mathrm{Max}(R) \subseteq \mathrm{Coass}_R(R)$. Let $\mathfrak{p} \in \mathrm{Coass}_R(R)$. Then there exists an ideal $\mathfrak{a}$ of R such that $(0 :_R R/\mathfrak{a}) = \mathfrak{p}$, and $R/\mathfrak{a} \subseteq E(R/\mathfrak{m})$ for some $\mathfrak{m} \in \mathrm{Max}(R)$. Thus $\mathfrak{a} = \mathfrak{p}$ and, by [10, Proposition 2.28], $E(R/\mathfrak{a}) \cong E(R/\mathfrak{m})$. Therefore, by [10, Corollary of Lemma 2.31], $\mathfrak{p} = \mathfrak{m}$.
(ii) Since M is a homomorphic image of a direct sum of finite copies of R, the claim follows from (i) and 1.3 (iv).
(iii) It is clear that $\mathrm{Coass}_R(N) \subseteq \mathrm{Cosupp}_R(N)$. Let $\mathfrak{p} \in \mathrm{Cosupp}_R(N)$. Then there exists $\mathfrak{q} \in \widetilde{\mathrm{Coass}}_R(N)$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. Set $S = R/(0 :_R N)$. It follows from 1.4 that $\mathrm{Coass}_R(N) = \widetilde{\mathrm{Coass}}_R(N)$. Thus the conclusion follows from (ii). □

For an Artinian R-module A, $\mathrm{Ass}_R(A)$ is finite. But 1.5 (i) provides examples of Noetherian modules with an infinite number of coassiated prime ideals. In fact, in the first draft of this paper, we have proved 1.5 (i) only for $\mathbb{Z}$, the ring of integers, but S. Yassemi informed us that our argument can be applied to any commutative ring. Also, note that 1.5 (ii) extends [12, Lemma 4.5].

PROPOSITION 1.6. ([9, Proposition 4.1]) *Let $f : R \longrightarrow S$ be a ring homomorphism and M be an S-module. Suppose that M is representable as an S-module, then M is representable as an R-module and* $\mathrm{Att}_R(M) = f^*(\mathrm{Att}_S(M))$.

THEOREM 1.7. *Let $f : R \longrightarrow S$ be a ring homomorphism and M be an S-module.*
(i) *If S is Noetherian, then $f^*(\mathrm{Coass}_S(M)) \subseteq \mathrm{Coass}_R(M)$.*
(ii) *If R and S are both Noetherian and for any proper R-submodule N of M, $SN \neq M$, then $f^*(\mathrm{Coass}_S(M)) = \mathrm{Coass}_R(M)$.*
(iii) *If f is surjective, then $f^*(\mathrm{Coass}_S(M)) = \mathrm{Coass}_R(M)$.*

Proof. (i) Let $\mathfrak{p} \in f^*(\mathrm{Coass}_S(M))$. Then there exists an Artinian S-homomorphic image L of M and a prime ideal $\mathfrak{q}$ of S such that $(0 :_S L) = \mathfrak{q}$ and $\mathfrak{p} = f^{-1}(\mathfrak{q})$. Consequently, by [6, 2.2], $\mathfrak{q} \in \mathrm{Att}_S(L)$. Since L is representable as an S-module, it follows from 1.6, 1.3 and 1.4 that

$$\mathrm{Coass}_R(L) = \widetilde{\mathrm{Coass}}_R(L) = \mathrm{Att}_R(L) = f^*(\mathrm{Att}_S(L)).$$

Therefore $\mathfrak{p} \in \mathrm{Coass}_R(L) \subseteq \mathrm{Coass}_R(M)$.
(ii) Let $\mathfrak{p} \in \mathrm{Coass}_R(M)$. Then there is an R-submodule N of M such that M/N is an Artinian $\mathfrak{p}$-secondary module. It follows from the exact

sequence

$$M/N \longrightarrow M/SN \longrightarrow 0$$

that $\mathrm{Coass}_R(M/SN) \subseteq \mathrm{Coass}_R(M/N) = \{\mathfrak{p}\}$. Since $M \neq SN$, $\mathrm{Coass}_R(M/SN) \neq \phi$ and so $\mathrm{Coass}_R(M/SN) = \{\mathfrak{p}\}$. As M/SN is an Artinian S-module, by 1.3 and 1.6,

$$\mathrm{Coass}_R(M/SN) = f^*(\mathrm{Coass}_S(M/SN)) \subseteq f^*(\mathrm{Coass}_S(M)).$$

Therefore $\mathfrak{p} \in f^*(\mathrm{Coass}_S(M))$.
(iii) A subset N of M is an R-submodule of M if and only if it is an S-submodule of M so the result follows from [11, Proposition 1*]. □

Now, we give an example to show that the inclusion in Theorem 1.7 (i) might be strict.

EXAMPLE 1.8. Let R be a Noetherian integral domain and $\mathfrak{p}$ be a prime ideal such that $\mathfrak{p} \notin \mathrm{Max}(R)$ and $\mathfrak{p}$ is not a minimal prime ideal. Let $f : R \longrightarrow R_\mathfrak{p}$ be the natural map. If $\mathrm{Coass}_R(R_\mathfrak{p}) = f^*(\mathrm{Coass}_{R_\mathfrak{p}}(R_\mathfrak{p}))$, then it follows from 1.5(i) that $\mathrm{Coass}_R(R_\mathfrak{p}) = \{\mathfrak{p}\}$. But, by [15, Folgerung 4.7]

$$\mathrm{Coass}_R(R_\mathfrak{p}) = \{\mathfrak{q} \in \mathrm{Spec}(R) : \mathfrak{q} \subseteq \mathfrak{p}\}.$$

This is a contradiction.

Let M be an R-module and Y be a subset of $\mathrm{Ass}_R(M)$. There is a submodule N of M such that $\mathrm{Ass}_R(N) = Y$ and $\mathrm{Ass}_R(M/N) = \mathrm{Ass}_R(M) - Y$. (See e.g. [3, Ch. IV, §1, Proposition 4]) However, the following example shows that this property does not hold for coassociated prime ideals.

EXAMPLE 1.9. Let $(R, \mathfrak{m})$ be a Noetherian local ring with $\dim R \geq 2$. Let Z be the set of height one prime ideals of R. Then Z is a infinite discrete subset of $\mathrm{Spec}(R)$. Let Y be a countable infinite subset of Z. Since $(0 :_{E(R/\mathfrak{p})} \mathfrak{p})$ is $\mathfrak{p}$-secondary, $\mathrm{Coass}_R(0 :_{E(R/\mathfrak{p})} \mathfrak{p}) = \{\mathfrak{p}\}$. Let $M = \oplus_{\mathfrak{p} \in \mathrm{Spec}(R)}(0 :_{E(R/\mathfrak{p})} \mathfrak{p})$. Then $\mathrm{Coass}_R(M) = \mathrm{Spec}(R)$. But it follows from [15, Folgerung 1.6] that there is no submodule N of M with $\mathrm{Coass}_R(N) = Y$.

2. On a Question of Melkersson

In this section, R will denote a Noetherian ring. An R-module M is said to have finite Goldie dimension, if M does not contain an infinite direct sum of non-zero submodules, or equivalently $E(M)$ decomposes as a finite direct sum of indecomposable injective submodules.

The following result gives an affirmative answer to Melkersson's question [8], in a special case.

THEOREM 2.1. *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring and $\dim R \leq 1$. Suppose the R-module M is representable and it has finite Goldie dimension. Then, for any flat R-module F, the module $\mathrm{Hom}_R(F, M)$ is representable.*

Proof. It follows from [2, Theorem 4.1] that $E(R/\mathfrak{p})$ is reflexive for all $\mathfrak{p} \in \mathrm{Spec}(R)$. Since any finite direct sum of reflexive modules and any submodule of a reflexive module is also reflexive, the module M is reflexive. That is, if $N = \mathrm{Hom}_R(M, E(R/\mathfrak{m}))$, then $M \cong \mathrm{Hom}_R(N, E(R/\mathfrak{m}))$. By [13, Theorem 5.1 (d)], the zero submodule of N has a primary decomposition. We have $\mathrm{Hom}_R(F, M) \cong \mathrm{Hom}_R(N, \mathrm{Hom}_R(F, E(R/\mathfrak{m}))$, so [13, Theorem 5.1(b)] implies that $\mathrm{Hom}_R(F, M)$ is representable. □

We know that, for a representable R-module M, the set $\mathrm{Coass}_R(M)$ is finite. The following result shows that $\mathrm{Coass}_R(\mathrm{Hom}_R(F, M))$ is finite.

THEOREM 2.2. *Suppose that the module M is representable and that it has finite Goldie dimension. Then, for any flat R-module F, the set $\mathrm{Coass}_R(\mathrm{Hom}_R(F, M))$ is finite.*

Proof. Let $\mathrm{Ass}_R(M) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$. The proof is by induction on n. If $n = 1$, then M has the structure of an Artinian module over $R_{\mathfrak{p}_1}$ and so $\mathrm{Hom}_R(F, M) \cong \mathrm{Hom}_{R_{\mathfrak{p}_1}}(F \otimes_R R_{\mathfrak{p}_1}, M)$. Hence, in this case, the result follows from [8, Proposition 5.3] and 1.6.

Now, let $n > 1$ and $\mathfrak{p}_1$ be a maximal element of $\mathrm{Ass}_R(M)$. Then there is $x \in \mathfrak{p}_1 - \bigcup_{i=2}^{n} \mathfrak{p}_i$. Set $S = \{x^i : i \in \mathbb{N}_0\}$. By [8, Theorem 2.1], $\Gamma_{Rx}(M)$ is representable and it follows from [7, Theorem 18.4] that $S^{-1}M$ has finite Goldie dimension, $\mathrm{Ass}_R(\Gamma_{Rx}(M)) = \{\mathfrak{p}_1\}$ and $\mathrm{Ass}_R(S^{-1}M) = \{\mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$. Therefore, by [8, Theorem 5.1], it follows from the canonical exact sequence

$$0 \longrightarrow \Gamma_{Rx}(M) \longrightarrow M \longrightarrow S^{-1}M \longrightarrow 0$$

that the sequence

$$0 \longrightarrow \mathrm{Hom}_R(F, \Gamma_{Rx}(M)) \longrightarrow \mathrm{Hom}_R(F, M) \longrightarrow \mathrm{Hom}_R(F, S^{-1}M) \longrightarrow 0$$

is exact. Hence

$$\mathrm{Coass}_R(\mathrm{Hom}_R(F,M)) \subseteq \mathrm{Coass}_R(\mathrm{Hom}_R(F,\Gamma_{Rx}(M)))$$
$$\cup\, \mathrm{Coass}_R(\mathrm{Hom}_R(F,S^{-1}M))$$

and the claim follows by induction. □

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