ANALYTIC TYPES OF THE SURFACE SINGULARITIES
DEFINED BY SOME WEIGHTED HOMOGENEOUS
POLYNOMIALS

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ABSTRACT. We classify analytically surface singularities defined by
some weighted homogeneous polynomials which are topologically
equivalent to the type $z_0^a + z_1^k + z_2^l = 0$.

1. Introduction

It is well known by Theorem 2.8 ([5]) that surface singularities defined
by weighted homogeneous polynomials can be classified topologically by
seven classes.

The aim in this paper is to classify analytically isolated surface singular-
ities defined by some weighted homogeneous polynomials, which are
topologically equivalent to the type $z_0^a + z_1^k + z_2^l = 0$.

Let $\mathcal{O}_{n+1}$ or $\mathbb{C}\{z_1, \ldots, z_n\}$ be the ring of convergent power series
at the origin in $\mathbb{C}^{n+1}$ and $f, g \in \mathcal{O}_{n+1}$. Then the natural question arises:
What is the concrete criterion for $f$ and $g$ to have the same analytic
type?

It is known by Theorem 2.6 ([3]) that two germs of complex analytic
hypersurface singularities defined by $f$ and $g$ with isolated singular
points at the origin in $\mathbb{C}^{n+1}$ are analytically equivalent if and only if their
moduli algebra $\mathcal{O}_{n+1}/(f, \Delta f)$ and $\mathcal{O}_{n+1}/(g, \Delta g)$ are isomorphic as a $\mathbb{C}$-

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paper are contained in the author's doctoral dissertation written under the guidance
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theorem, it is still difficult to find a concrete criterion for analytic equivalence between two surfaces with isolated singular points at the origin.

By Theorem 2.7 ([5]), if $V = \{ z \in \mathbb{C}^{n+1} : f(z) = 0 \}$ and $W = \{ z \in \mathbb{C}^{n+1} : g(z) = 0 \}$ are surface singularities at the origin defined by weighted homogeneous polynomials $f$ and $g$ with the same weights, then $V$ and $W$ are topologically equivalent. But, for the analytic case, $V$ and $W$ may not be analytically equivalent, even though they have the same weights.

By the above motivation, we find a necessary and sufficient condition for given four different types of some surface singularities, which are topologically equivalent to the type $z_0^n + z_1^k + z_2^l = 0$, to be analytically equivalent.

2. Definitions and Known Preliminaries

Let $\mathcal{O}_{n+1}$ be the ring of germs of holomorphic functions at the origin in $\mathbb{C}^{n+1}$ and $f(z_0, \ldots, z_n)$ and $g(z_0, \ldots, z_n)$ are in $\mathcal{O}_{n+1}$ which have isolated singular points at the origin in $\mathbb{C}^{n+1}$.

DEFINITION 2.1. Let $V = \{ z \in \mathbb{C}^{n+1} : f(z) = 0 \}$ and $W = \{ z \in \mathbb{C}^{n+1} : g(z) = 0 \}$ be germs of complex hypersurfaces with isolated singularity at the origin. $f$ and $g$ are said to have the same analytic type of singularity at the origin, if there is a germ at the origin of biholomorphism $\psi : (U_1, O) \to (U_2, O)$ such that $\psi(V) = W$ and $\psi(O) = O$ where $U_1$ and $U_2$ are open subsets in $\mathbb{C}^{n+1}$, that is, $f \circ \psi = u g$ where $u$ is a unit in $\mathcal{O}_{n+1}$. Then we write $f \approx g$. If not, we write $f \not\approx g$.

DEFINITION 2.2. Two germs of holomorphic functions $f, g : (\mathbb{C}^{n+1}, O) \to (\mathbb{C}, O)$ are called right equivalent if there exists a biholomorphism $\varphi : (\mathbb{C}^{n+1}, O) \to (\mathbb{C}^{n+1}, O)$ such that $f = g \circ \varphi$.

DEFINITION 2.3. $f(z_0, \ldots, z_n)$ is called a weighted homogeneous polynomial with weights $(\omega_0, \ldots, \omega_n)$, where $\omega_0, \ldots, \omega_n$ are fixed positive rational numbers, if it can be expressed as a linear combination of monomials $z_0^{\omega_0} z_1^{i_1} \cdots z_n^{i_n}$ for which $\frac{i_0}{\omega_0} + \cdots + \frac{i_n}{\omega_n} = 1$.

DEFINITION 2.4. $f \in \mathcal{O}_{n+1}$ is called quasihomogeneous if $f \approx g$ for some weighted homogeneous polynomial $g$. 
THEOREM 2.5 ([4]). If \((V, O)\) and \((W, O)\) be germs of isolated hypersurface singularities at the origin in \(\mathbb{C}^{n+1}\) defined by weighted homogeneous polynomials \(f\) and \(g\) respectively, then \((V, O)\) and \((W, O)\) are analytically equivalent if and only if \(f\) and \(g\) are right equivalent. That is, there exists a biholomorphism \(\varphi : (\mathbb{C}^{n+1}, O) \rightarrow (\mathbb{C}^{n+1}, O)\) such that \(f \circ \varphi = g\).

THEOREM 2.6 ([3]). Suppose that \(V = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} : f(z_0, \ldots, z_n) = 0\}\) and \(W = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} : g(z_0, \ldots, z_n) = 0\}\) have the isolated singular point at the origin. Then the following conditions are equivalent.

(i): \(f \approx g\).

(ii): \(A(f)\) is isomorphic to \(A(g)\) as a \(\mathbb{C}\)-algebra where \(A(f) = n+1O/(f, \Delta(f))\), \(A(g) = n+1O/(g, \Delta(g))\) and \((f, \Delta(f))\) is the ideal in \(n+1O\) generated by \(f, \partial f/\partial z_0, \ldots, \partial f/\partial z_n\).

(iii): \(B(f)\) is isomorphic to \(B(g)\) as a \(\mathbb{C}\)-algebra where \(B(f) = n+1O/(f, m\Delta(f))\), \(B(g) = n+1O/(g, m\Delta(g))\) and \((f, m\Delta(f))\) is the ideal in \(n+1O\) generated by \(f\) and \(z_i \partial f/\partial z_i\) for all \(i, j = 0, 1, \ldots, n\).

THEOREM 2.7 ([5]). Suppose that \(f(z_0, z_1, z_2)\) and \(g(z_0, z_1, z_2)\) are weighted homogeneous polynomials with the same weights \((\omega_0, \omega_1, \omega_2)\). If \(f\) and \(g\) have isolated singularities at the origin in \(\mathbb{C}^3\), then \(f\) is topologically equivalent to \(g\).

THEOREM 2.8 ([5]). Let \((V, 0)\) and \((W, 0)\) be two isolated quasi-homogeneous surface singularities having the same topological type. Then \((V, 0)\) is connected to \((W, 0)\) by a family of constant topological type. In fact \((V, 0)\) is connected to one of the followings:

Class I. \(V(a_0, a_1, a_2; 1) = \{z_0^{a_0} + z_1^{a_1} + z_2^{a_2} = 0\}\)

Class II. \(V(a_0, a_1, a_2; 2) = \{z_0^{a_0} + z_1^{a_1} + z_2^{a_2} = 0\}\) where \(a_1 > 0\)

Class III. \(V(a_0, a_1, a_2; 3) = \{z_0^{a_0} + z_1^{a_1} z_2 + z_1 z_2^{a_2} = 0\}\) where \(a_1 > 0, a_2 > 0\)

Class IV. \(V(a_0, a_1, a_2; 4) = \{z_0^{a_0} + z_1^{a_1} z_2 + z_0 z_2^{a_2} = 0\}\) where \(a_0 > 0\)

Class V. \(V(a_0, a_1, a_2; 5) = \{z_0 z_1 + z_1^{a_1} z_2 + z_0 z_2^{a_2} = 0\}\)

Class VI. \(V(a_0, a_1, a_2; 6) = \{z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{a_2} = 0\}\) where \((a_0 - 1)(a_1 b_2 + a_2 b_1) = a_0 a_1 a_2\)

Class VII. \(V(a_0, a_1, a_2; 7) = \{z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1 z_2^{b_2} = 0\}\) where \(c(a_0 - 1)(a_1 b_2 + a_2 b_1) = a_2 (a_0 a_1 - 1)\)

THEOREM 2.9 ([1]). Let \(f\) and \(g\) be weighted homogeneous polynomials, which are not homogeneous, with isolated singularity at the origin.
in $\mathbb{C}^2$ such that $f \not\sim z_0^2 + z_1^2$ and $g \not\sim z_0^2 + z_1^2$. Then we may assume without loss of generality that analytically,

\begin{align*}
f &= z_0^{\delta_1}z_1^{\delta_2}f_1 \quad \text{with} \\
f_1 &= z_0^n + z_1^k + \sum_{i=1}^{d-1} A_i z_0^{(d-i)m_1} z_1^{i_1} \quad \text{and} \\
g &= z_0^{\varepsilon_1}z_1^{\varepsilon_2}g_1 \quad \text{with} \\
g_1 &= z_0^m + z_1^l + \sum_{j=1}^{e-1} B_j z_0^{(e-j)m_1} z_1^{j_1}
\end{align*}

where

(a): $2 \leq n < k$, $d = \gcd(n, k)$ with $n = dn_1$ and $k = dk_1$,
(b): $2 \leq m < l$, $e = \gcd(m, l)$ with $m = em_1$ and $l = el_1$,
(c): $\varepsilon_1, \varepsilon_2, \delta_1, \delta_2$ are either 1 or 0, respectively, and
(d): $A_i$ and $B_j$ are complex numbers for $1 \leq i < d - 1$ and $1 \leq j < e - 1$.

Also, we need to assume without loss of generality that

- if $\gcd(n, k) = n$, i.e., $n_1 = 1$, then $A_1 = 0$ and
- if $\gcd(m, l) = m$, i.e., $m_1 = 1$, then $B_1 = 0$.

As a conclusion, we get the following:

$f \approx g$ if and only if $\varepsilon_i = \delta_i$ for $i = 1, 2$ and $f_1 \approx g_1$ if and only if $\varepsilon_i = \delta_i$ for $i = 1, 2$ and $n = m, k = l$ and there is a complex number $\rho$ with $\rho^d = 1$ such that $A_i\rho^i = B_i$ for $i = 1, \ldots, d - 1$.

**Theorem 2.10** ([2]). Let $f = z_0^n + z_1^n + \sum_{i=1}^{s} a_i z_0^i z_1^{n-1} + \sum_{j=1}^{t} b_j z_0^j z_1^{n-j}$ and $g = z_0^m + z_1^l + \sum_{j=1}^{e} b_j z_0^j z_1^{m-1}$ be homogeneous polynomials with isolated singularity at the origin in $\mathbb{C}^2$ where $n \geq 2s + 3, n \geq 2t + 3$ and $n \geq 5$. Then $f \approx g$ if and only if there is a complex number $\rho$ with $\rho^n = 1$ such that $b_i = a_i\rho^i$ for $i = 1, \ldots, s = t$. Moreover, if $f = z_0^4 + z_1^4 + az_0 z_1^4$ and $g = z_0^4 + z_1^4 + b z_0^2 z_1^2$ have an isolated singularity at the origin, then $f \approx g$ if and only if $a^4 = b^4$.

3. Main Results

We find a concrete criterion to have the same analytic type for given two surfaces singularities at the origin, which are defined by some weighted homogeneous polynomials. Consider the four different types of singularities defined by some weighted homogeneous polynomials with isolated
singular points at the origin in \( \mathbb{C}^3 \), which are topologically equivalent to the type \( z_0^n + z_1^k + z_2^l = 0 \), as follows:

\[
T_0(z_0, z_1, z_2) = z_0^n + z_1^k + z_2^l;
\]

\[
T_1(z_0, z_1, z_2) = z_0^n + z_1^k + z_2^l + \sum_{\alpha, \beta} A_{\alpha, \beta} z_0^\alpha z_1^\beta \quad \text{with some } A_{\alpha, \beta} \neq 0;
\]

\[
T_2(z_0, z_1, z_2) = z_0^n + z_1^k + z_2^l + \sum_{\gamma, \delta} B_{\gamma, \delta} z_1^\gamma z_2^\delta \quad \text{with some } B_{\gamma, \delta} \neq 0;
\]

\[
T_3(z_0, z_1, z_2) = z_0^n + z_1^k + z_2^l + \sum_{\epsilon, \tau} C_{\epsilon, \tau} z_0^\epsilon z_2^\tau \quad \text{with some } C_{\epsilon, \tau} \neq 0;
\]

\[
T_4(z_0, z_1, z_2) = z_0^n + z_1^k + z_2^l + \sum_{\alpha, \beta, \gamma, \delta} D_{\alpha, \beta, \gamma, \delta} z_0^\alpha z_1^\beta z_2^\gamma z_2^\delta \quad \text{with some } D_{\alpha, \beta, \gamma, \delta} \neq 0.
\]

**Definition 3.1.** It is said that a weighted homogeneous polynomial \( f \) belongs to the type \( T_i \) if \( f \) can be written in the form of \( T_i \) for \( i = 0, 1, 2, 3, 4 \). In this case, we write \( f \in T_i \). Otherwise, \( f \not\in T_i \).

Note that the surface singularities defined by the above four different types of weighted homogeneous polynomials are topologically equivalent to the surface singularity defined by \( z_0^n + z_1^k + z_2^l = 0 \). It is a consequence of Theorem 2.7 ([5]). But, for the analytic case, we find the different results. Those are followings:

First, even though \( f \) and \( g \) belong to the same type in a sense Definition 3.1, \( f \) and \( g \) may not be analytically equivalent. If \( f \) and \( g \) are analytically equivalent, then we find a necessary and sufficient condition for \( f \) and \( g \), in an elementary way. Secondly, if \( f \) and \( g \) belong to the different types, then \( f \) and \( g \) are not analytically equivalent.

Throughout in this paper, we assume that \( 2 \leq n \leq k \leq l \) and all exponents of \( z_0, z_1 \) and \( z_2 \) are positive integers.

**Theorem 3.2.** Let \( n \) and \( k \) be positive integers with \( 2 \leq n \leq k \) and \( f_1 \in T_1, g_1 \in T_1 \).

A: Assume that \( n < k \). Then, \( f_1 \) and \( g_1 \) can be written analytically as follows:

\[
\begin{align*}
    f_1 &= z_0^n + A_0 z_1^k + z_2^l + \sum_{i=1}^{d-1} A_i z_0^{n_1} z_1^k z_2^{l_1}, \\
    g_1 &= z_0^n + B_0 z_1^k + z_2^l + \sum_{i=1}^{d-1} B_i z_0^{n_1} z_1^k z_2^{l_1}
\end{align*}
\]

where \( d = \gcd(n, k) \) with \( n = n_1 d \) and \( k = k_1 d \) for some positive integers \( n_1, k_1 \), and \( A_i \) and \( B_i \) are complex numbers for \( 0 \leq i \leq d-1 \) which satisfy the following properties:
(a): If \( d < n \), then \( A_0 = B_0 = 1 \);
(b): If \( d = n \geq 3 \), then \( A_{d-1} = B_{d-1} = 0 \), and \( A_0 \) and \( B_0 \) are either 1 or 0, respectively. In this case, if \( A_0 = 0 \), then \( A_1 = 1 \), and if \( B_0 = 0 \), then \( B_1 = 1 \);
(c): If \( d = n = 2 \), then \( A_0 = B_0 = 1 \) and \( A_1 = B_1 = 0 \);
(d): \( f_1 \approx g_1 \) if and only if there exists a complex number \( \omega_i \) with \( \omega_1^s = 1 \) such that \( A_i \omega_i^t = B_i \) for \( 0 \leq i \leq d - 1 \).

B: Assume that \( n = k \). Then \( f_1 \) and \( g_1 \) can be written analytically as follows:

\[
\begin{align*}
    f_1 &= z_0^n + z_1^n + z_2^n + \sum_{i=1}^{s} C_i z_0^{i} z_1^{n-i}, \\
    g_1 &= z_0^n + z_1^n + z_2^n + \sum_{i=1}^{t} D_i z_1^{i} z_2^{n-i}
\end{align*}
\]

for some complex numbers \( C_i \) and \( D_i \) where \( s \leq n - 1, t \leq n - 1 \) and \( C_s \neq 0, D_t \neq 0 \). In this case, let \( n \geq 2s + 3, n \geq 2t + 3 \) and \( n \geq 5 \). Then, \( f_1 \approx g_1 \) if and only if exits a complex number \( \omega_2 \) with \( \omega_2^n = 1 \) such that \( \omega_2^s C_i = D_i \) for each \( i = 1, 2, \ldots, s = t \).

**Theorem 3.3.** Let \( k \) and \( l \) be positive integers with \( 2 \leq k \leq l \) and \( f_2 \in T_2, g_2 \in T_2 \).

A: Assume that \( k < l \). Then \( f_1 \) and \( g_1 \) can be written analytically as follows:

\[
\begin{align*}
    f_2 &= z_0^n + z_1^n + A_0 z_2^n + \sum_{i=1}^{d-1} A_i z_0^{k_i} z_1^{l_i(d-i)}, \\
    g_2 &= z_0^n + z_1^n + B_0 z_2^n + \sum_{i=1}^{d-1} B_i z_1^{k_i} z_2^{l_i(d-i)}
\end{align*}
\]

where \( d = \gcd(k, l) \) with \( k = k_1 d \) and \( l = l_1 d \) for some positive integers \( k_1, l_1 \), and \( A_i \) and \( B_i \) are complex numbers for \( 0 \leq i \leq d - 1 \) which satisfy the following properties:

(a): If \( d < k \), then \( A_0 = B_0 = 1 \);
(b): If \( d = k \geq 3 \), then \( A_{d-1} = B_{d-1} = 0 \), and \( A_0 \) and \( B_0 \) are either 1 or 0, respectively. In this case, if \( A_0 = 0 \), then \( A_1 = 1 \), and if \( B_0 = 0 \), then \( B_1 = 1 \);
(c): If \( d = k = 2 \), then \( A_0 = B_0 = 1 \) and \( A_1 = B_1 = 0 \);
(d): \( f_2 \approx g_2 \) if and only if there exists a complex number \( \rho_1 \) with \( \rho_1^n = 1 \) such that \( A_i \rho_i^t = B_i \) for \( 0 \leq i \leq d - 1 \).

B: Assume that \( k = l \). Then \( f_2 \) and \( g_2 \) can be written analytically as follows:

\[
\begin{align*}
    f_2 &= z_0^n + z_1^n + z_2^n + \sum_{i=1}^{s} C_i z_0^{i} z_2^{k_i-l_i}, \\
    g_2 &= z_0^n + z_1^n + z_2^n + \sum_{i=1}^{t} D_i z_1^{i} z_2^{k_i-l_i}
\end{align*}
\]
for some complex numbers $C_i$ and $D_i$ where $s \leq k - 1, t \leq k - 1$ and $C_s \neq 0, D_t \neq 0$. In this case, let $k \geq 2s + 3, k \geq 2t + 3$ and $k \geq 5$. Then, $f_2 \approx g_2$ if and only if there exists a complex number $\rho_2$ with $\rho_2^k = 1$ such that $C_i\rho_2^i = D_i$ for $i = 1, 2, \ldots, s = t$.

**Theorem 3.4.** Let $n$ and $l$ be positive integers with $2 \leq n \leq l$ and $f_3 \in T_3, g_3 \in T_3$.

A: Assume that $n < l$. Then $f_1$ and $g_1$ can be written analytically as follows:

\begin{align*}
f_3 &= z_0^n + z_1^k + A_0 z_2^l + \sum_{i=1}^{d-1} A_i z_0^{n_i} z_2^{l_i(d-i)}, \\
g_3 &= z_0^n + z_1^k + B_0 z_2^l + \sum_{i=1}^{d-1} B_i z_0^{n_i} z_2^{l_i(d-i)}
\end{align*}

where $d = \gcd(n, l)$ with $n = n_1 d$ and $l = l_1 d$ for some positive integers $n_1, l_1$, and $A_i$ and $B_i$ are complex numbers for $0 \leq i \leq d - 1$ which satisfy the following properties:

(a): If $d < n$, then $A_0 = B_0 = 1$;
(b): If $d = n \geq 3$, then $A_{d-1} = B_{d-1} = 0$, and $A_0$ and $B_0$ are either 1 or 0, respectively. In this case, if $A_0 = 0$, then $A_1 = 1$, and if $B_0 = 0$, then $B_1 = 1$;
(c): If $d = n = 2$, then $A_0 = B_0 = 1$ and $A_1 = B_1 = 0$;
(d): $f_3 \approx g_3$ if and only if there exists a complex number $\eta_1$ with $\eta_1^n = 1$ such that $A_i \eta_1^i = B_i$ for $0 \leq i \leq d - 1$.

B: Assume that $n = l$. Then $f_3$ and $g_3$ can be written analytically as follows:

\begin{align*}
f_3 &= z_0^n + z_1^n + z_2^n + \sum_{i=1}^{s} C_i z_0^{n_i} z_2^{-n_i}, \\
g_3 &= z_0^n + z_1^n + z_2^n + \sum_{i=1}^{s} D_i z_1^{n_i} z_2^{-n_i}
\end{align*}

for some complex numbers $C_i$ and $D_i$ where $s \leq n - 1, t \leq n - 1$ and $C_s \neq 0, D_t \neq 0$. In this case, let $n \geq 2s + 3, n \geq 2t + 3$ and $n \geq 5$. Then, $f_3 \approx g_3$ if and only if exits a complex number $\eta_2$ with $\eta_2^n = 1$ such that $C_i \eta_2^i = D_i$ for each $i = 1, 2, \ldots, s = t$.

**Remark 3.5.** Theorem 3.2, Theorem 3.3 and Theorem 3.4 imply the following facts:

(i): If $f_1 \in T_1$ and $n = 2$, then $f_1 \in T_0$;
(ii): If $f_2 \in T_2$ and $n = k = 2$, then $f_2 \in T_0$;
(iii): If $f_3 \in T_3$ and $n = 2$, then $f_3 \in T_0$. 


THEOREM 3.6. Suppose that \(2 < n < k < l\) and that the weighted homogeneous polynomials \(f_i\) and \(g_j\) belong to the type \(T_i\) and \(T_j\), respectively, where \(0 \leq i \leq j \leq 4\). If \(i \neq j\), then \(f_i \not\approx g_j\) except for \(i = 2\) and \(j = 4\).

Assume that \(f_4\) belongs to the type \(T_4\). That is, \(f_4\) can be written as

\[
f_4 = z_0^n + z_1^k + z_2^l + \sum_{\alpha, \beta, \gamma} D_{\alpha, \beta, \gamma} z_0^\alpha z_1^\beta z_2^\gamma.
\]

DEFINITION 3.7. Let \(f = z_0^n + z_1^k + z_2^l + \sum_{\alpha, \beta, \gamma} D_{\alpha, \beta, \gamma} z_0^\alpha z_1^\beta z_2^\gamma\) be given. Define \(\min(f) = \min\{\alpha + \beta + \gamma\}\) for all nonzero monomial \(z_0^\alpha z_1^\beta z_2^\gamma\) in \(f\) and \(S(f) = \{(\alpha, \beta, \gamma) : \alpha + \beta + \gamma = \min(f)\}\).

LEMMA 3.8. If \(k < l\), then there exists a unique element \((\alpha_0, \beta_0, \gamma_0) \in S(f)\) such that \(\alpha_0 \leq \alpha\) for any \((\alpha, \beta, \gamma) \in S(f)\) as in Definition 3.7.

Proof. Note that if \(\alpha_0 = \alpha\) and \(\alpha_0 + \beta_0 + \gamma_0 = \alpha + \beta + \gamma\), then

\[
\frac{\beta_0 - \beta}{k} = \frac{\gamma - \gamma_0}{l}.
\]

Thus \(\beta_0 = \beta\) and \(\gamma_0 = \gamma\) if \(k < l\). Therefore if we choose an element \((\alpha_0, \beta_0, \gamma_0) \in S\) such that \(\alpha_0 \leq \alpha\) for any \((\alpha, \beta, \gamma) \in S\), then the element \((\alpha_0, \beta_0, \gamma_0)\) is unique. \(\square\)

THEOREM 3.9. Suppose that \(2 < n < k < l\) and that \(f_4\) and \(g_4\) belong to the type of \(T_4\). Then \(f_4\) and \(g_4\) can be written as follows:

\[
f_4 = z_0^n + z_1^k + z_2^l + \sum_{(\alpha, \beta, \gamma) \in I_4} D_{\alpha, \beta, \gamma} z_0^\alpha z_1^\beta z_2^\gamma,
\]

\[
g_4 = z_0^n + z_1^k + z_2^l + \sum_{(\alpha', \beta', \gamma') \in I'_4} D'_{\alpha', \beta', \gamma'} z_0'^\alpha z_1'^\beta z_2'^\gamma.
\]

for some nonzero complex numbers \(D_{\alpha, \beta, \gamma}\) and \(D'_{\alpha', \beta', \gamma'}\). For \(f_4\), if we can choose \((\alpha_0, \beta_0, \gamma_0) \in I_4\) with \(\alpha_0 + \beta_0 + \gamma_0 \leq n + k - 2\) which satisfies Definition 3.7 and Lemma 3.8, then the followings hold:

(i): \(f_4 \not\approx f_2\) where \(f_2 \in T_2\);

(ii): If \((\alpha_0, \beta_0, \gamma_0) \notin I'_4\), then \(f_4 \not\approx g_4\).

Furthermore, if \(f_4 \approx g_4\), then \((\alpha_0, \beta_0, \gamma_0) \in I'_4\) and there exist complex numbers \(a, b\) and \(c\) such that \(D_{\alpha_0, \beta_0, \gamma_0} a_0^\alpha b_0^\beta c_0^\gamma = D'_{\alpha_0, \beta_0, \gamma_0}\).
We prove those results by using Theorem 2.5, Theorem 2.9 and Theorem 2.10. From the fact by Theorem 2.5, which two surface singularities at the origin defined by weighted homogeneous polynomials \( f \) and \( g \) are analytically equivalent if and only if \( f \circ \varphi = g \) for some biholomorphisms \( \varphi : (\mathbb{C}^3, O) \to (\mathbb{C}^3, O) \) at the origin, we may apply the fact to prove those results.

We use a notation to be convenient:

\( z_0^\alpha z_1^\beta z_2^\gamma \in P(z_0, z_1, z_2) \) if the monomial \( z_0^\alpha z_1^\beta z_2^\gamma \) belongs to the polynomial or power series \( P(z_0, z_1, z_2) \). That is, the monomial \( z_0^\alpha z_1^\beta z_2^\gamma \) has nonzero coefficient in \( P(z_0, z_1, z_2) \).

Before proving Theorem 3.2, Theorem 3.3 and Theorem 3.4, we remarked the followings: Let \( (X_f, 0) = \{(z_0, \cdots, z_n) : f(z_0, \cdots, z_n) = 0\} \). Teissier(1977) showed that the analytic type of the hypersurface \( X_{f+h} \) defined by \( f(z_0, \cdots, z_n) + h(w_0, \cdots, w_m) = 0 \) depends not only on the analytic types of \( (X_f, 0) \) and of \( (X_h, 0) \), but also in general on the choice of the equations for \( f \) and \( h \). However, the following theorem says that in case \( h \) is quasihomogeneous, then the analytic type of \( X_{f+h} \) indeed depends only on the analytic types of \( (X_f, 0) \) and of \( (X_h, 0) \). In fact, a "subtraction" theorem holds !

**Theorem 3.10.** Let \( f(z_0, \cdots, z_n) \) and \( g(z_0, \cdots, z_n) \) be holomorphic functions with isolated singularity at origin in \( \mathbb{C}^{n+1} \) and \( h(w_0, \cdots, w_m) \) be a quasihomogeneous holomorphic function with an isolated singularity at the origin. Then \( (X_f, 0) \) is analytically equivalent to \( (X_g, 0) \) if and only if \( (X_{f+h}, 0) \) is analytically equivalent to \( (X_{g+h}, 0) \).

**Proof.** See [6].

**Proof of Theorem 3.2.** If \( \gcd(n, k) = d < n \), then \( f_1 \) and \( g_1 \) can be written analytically as \( (1) \) which satisfy (a). Suppose that \( \gcd(n, k) = n \). Then,

\[
f_1 = z_0^n + z_1^k + z_2^l + \sum_{i=1}^{n-1} D_i z_0^{k_i} z_1^{(n-i)}
\]

for some complex numbers \( D_i, 1 \leq i \leq n - 1 \). If \( D_{n-1} = 0 \), then \( f_1 \) can be written analytically as \( (1) \). If \( D_{n-1} \neq 0 \), then, by the biholomorphic change of coordinates \( \varphi \) with

\[
\varphi(z_0, z_1, z_2) = \left( z_0 - \frac{D_{n-1}}{n} z_1^{k_1}, z_1, z_2 \right)
\]
at the origin, \((f_1 \circ \varphi)(z_0, z_1, z_2)\) can be written analytically as (1) which satisfies (b) and (c). Since \(f_1 \circ \varphi \approx f_1\), \(f_1\) can be written analytically as (1) which satisfies (b) and (c). By a similar method, \(g_1\) can be written analytically as (1) which satisfies (a), (b) and (c). Let \(h(z_2) = z_2^l\). Then \(h\) is a quasihomogeneous holomorphic function. Let \(f = f_1 - h\) and \(g = g_1 - h\). By Theorem 3.10,

\[
f_1 = f + h \approx g + h = g_1 \text{ if and only if } f \approx g.
\]

This is the case of plane curve singularities. The result of Theorem 2.9 implies A. The proof of B is similar if we set \(h(z_0) = z_0^n\).

This completes the proof of Theorem 3.2.

Proofs of Theorem 3.3 and Theorem 3.4. By a similar argument of the proof of Theorem 3.2, we can prove Theorem 3.3 and Theorem 3.4 without any difficulty.

We prove Theorem 3.6 by Proposition 3.11, Proposition 3.12 and Proposition 3.13.

PROPOSITION 3.11. Assume that \(2 < n < k < l\) and that \(f_0 \in T_0\) and \(f_j \in T_j\) for \(1 \leq j \leq 4\) in a sense of Definition 3.1. Then \(f_0 \not\approx f_j\).

Proof. We prove that \(f_0 \not\approx f_1\). Suppose that \(f_0 \approx f_1\). Then \(f_0\) and \(f_1\) can be written analytically as

\[
\begin{align*}
f_0 &= z_0^n + z_1^k + z_2^l, \\
f_1 &= z_0^n + z_1^k + z_2^l + \sum_{\alpha, \beta} A_{\alpha, \beta} z_0^\alpha z_1^\beta,
\end{align*}
\]

which are weighted homogeneous polynomials with weights \((n, k, l)\). Let

\[
\begin{align*}
f(z_0, z_1) &= z_0^n + z_1^k + \sum_{\alpha, \beta} A_{\alpha, \beta} z_0^\alpha z_1^\beta, \\
g(z_0, z_1) &= z_0^n + z_1^k, \\
h(z_2) &= z_2^l.
\end{align*}
\]

Then \(f\) and \(g\) are weighted homogeneous polynomials with weights \((n, k)\) and \(h\) is a quasihomogeneous holomorphic function. By Theorem 3.10, \(f_0 = g + h \approx f + h = f_1\) if and only if \(f \approx g\). But, \(f \not\approx g\) by Theorem 2.9. This leads to a contradiction. Thus \(f_0 \not\approx f_1\).

The other proofs are similar to the above.

PROPOSITION 3.12. Suppose that \(2 < n < k < l\) and that \(f_1 \in T_1\) and \(f_j \in T_j\) for \(0 \leq j \leq 4, j \neq 1\) in a sense of Definition 3.1. Then \(f_1 \not\approx f_j\).
Proof. Suppose that $f_1 \approx f_j$ for some $j$ with $j \neq 1$ and $0 \leq j \leq 4$.

Case I) $\gcd(n, k) = d_1 < n$.

Then $n = n_1 d_1$ and $k = k_1 d_1$ for some positive integers $k_1$ and $n_1$. Note that $d_1 > 1$. By Theorem 3.2, A, $f_1$ can be written analytically as

$$f_1 = z_0^n + z_1^k + z_2^l + \sum_{i=1}^{d} A_i z_0^{n_1 i} z_1^{k_1 (d_1 - i)}$$

for some complex numbers $A_i$ where $d$ is the largest number of $i$ with $A_i \neq 0$ and $1 \leq i \leq d_1 - 1$. Choose a biholomorphism $\varphi : (\mathbb{C}^3, O) \rightarrow (\mathbb{C}^3, O)$ at the origin $O = (0,0,0)$ such that $f_1 \circ \varphi = f_j$. We set $\varphi(z_0, z_1, z_2) = (H, L, M)$ as follows:

$$H = a_1 z_0 + b_1 z_1 + c_1 z_2 + H_2 + \cdots + H_s + \cdots, \quad H_s = \sum_{p+q+r=s} A_{p,q,r} z_0^p z_1^q z_2^r,$$
$$L = a_2 z_0 + b_2 z_1 + c_2 z_2 + L_2 + \cdots + L_s + \cdots, \quad L_s = \sum_{p+q+r=s} B_{p,q,r} z_0^p z_1^q z_2^r,$$
$$M = a_3 z_0 + b_3 z_1 + c_3 z_2 + M_2 + \cdots + M_s + \cdots, \quad M_s = \sum_{p+q+r=s} C_{p,q,r} z_0^p z_1^q z_2^r.$$

Since $\varphi$ is a biholomorphism at the origin, we have

$$| J_\varphi(O) | = \det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \neq 0.$$  \hspace{1cm} (8)

Consider the expansion of $f_1 \circ \varphi = f_j$,

$$H^n + L^k + M^l + \sum_{i=1}^{d} A_i H_i^{n_i} L_i^{k_1 (d_1 - i)} = f_j.$$  \hspace{1cm} (9)

Note that

$$n < n_1 d + k_1 (d_1 - d) < n_1 (d - 1) + k_1 (d_1 - d + 1) < \cdots < n_1 + k_1 (d_1 - 1) < k.$$

Then, by comparison of degrees in (9), we have

$$b_1 = c_1 = c_2 = 0$$

and

$$H_2 = \cdots = H_{(k_1 - n_1) (d_1 - d)} = 0.$$

Therefore

$$| J_\varphi(O) | = a_1 b_2 c_3 \neq 0.$$
In the expansion
\[ A_d H^{n_1 d} L^{k_1 (d_1 - d)} \]
\[ = A_d \left( (a_1 z_0 + H_{(k_1-n_1)(d_1-d)+1} + \cdots) \right)^{n_1 d} (a_2 z_0 + b_2 z_1 + L_2 + \cdots)^{k_1 (d_1 - d)}, \]
every monomial with degree \( n_1 d + k_1 (d_1 - d) \) is contained in the expansion of \( A_d \left( (a_1 z_0)\right)^{n_1 d} (a_2 z_0 + b_2 z_1)^{k_1 (d_1 - d)} \). In particular, the monomial \( z_0^{n_1 d} z_1^{k_1 (d_1 - d)} \) has nonzero coefficient \( A_d a_1^{n_1 d} b_2^{k_1 (d_1 - d)} \) in the expansion of \( A_d H^{n_1 d} L^{k_1 (d_1 - d)}. \)

It is clear that the monomial \( z_0^{n_1 d} z_1^{k_1 (d_1 - d)} \) does not belong to \( L^k \) and \( M^l \) by the inequalities
\[ n_1 d + k_1 (d_1 - d) < k < l. \]

We claim that the monomial \( z_0^{n_1 d} z_1^{k_1 (d_1 - d)} \) does not belong to the expansion of \( H^n \). If \( z_0^{n_1 d} z_1^{k_1 (d_1 - d)} \) belongs to \( H^n \), then the monomial is contained in the expansion of \( (a_1 z_0)^{\eta} (H_{(k_1-n_1)(d_1-d)+1} + \cdots)^{n-n} \) for some \( \eta \) where \( 0 \leq \eta \leq n_1 d \). Since \( n \geq n_1 d + 2 \) and \( n_1 d + k_1 (d_1 - d) > n \), the inequalities
\[ \eta + (n-\eta) \left\{ (k_1 - n_1)(d_1 - d) + 1 \right\} \]
\[ \geq n_1 d + n_1 (d_1 - d) \left\{ (d_1 - d)(k_1 - n_1) + 1 \right\} \]
\[ > n_1 d + k_1 (d_1 - d) \]
hold for all \( \eta \) where \( 0 \leq \eta \leq n_1 d \). That is to say, every monomial in the expansion of \( (a_1 z_0)^{\eta} (H_{(k_1-n_1)(d_1-d)+1} + \cdots)^{n-n} \) has degree greater than \( n_1 d + k_1 (d_1 - d) \) if \( 0 \leq \eta \leq n_1 d \). This leads to a contradiction.

Consequently, the monomial \( z_0^{n_1 d} z_1^{k_1 (d_1 - d)} \) has nonzero coefficient \( A_d a_1^{n_1 d} b_2^{k_1 (d_1 - d)} \) in the left expansion of (9) and it must belong to \( f_j \) for \( 0 \leq j \leq 4, j \neq 1 \). This also leads to a contradiction. Thus \( f_1 \not\approx f_j \) if \( j \neq 1 \).

Case II) \( \gcd(n, k_1) = n < k. \)

Then \( k = nk_1 \) for some positive integer \( k_1 \). By Theorem 3.2, A, \( f_1 \) has analytically two different representations as follows:
\[ f_{i1} = z_0^\alpha + z_1^k + z_2^\beta + \sum_{i=1}^{\alpha} A_{1,i} z_0^i z_1^{k_1(n-i)} \]
\[ f_{i2} = z_0^\alpha + z_1^k + \sum_{r=1}^{\beta} A_{2,r} z_0^r z_1^{k_1(n-r)} \]
for some complex numbers \( A_{1,i}, A_{2,r} \) with \( 1 \leq i \leq \alpha \leq n - 2, 1 \leq r \leq \beta \leq n - 2, A_{2,1} = 1 \), where \( \alpha \) and \( \beta \) are the largest numbers of \( i \) and \( r \) such that \( A_{1,i} \neq 0 \) and \( A_{2,r} \neq 0 \), respectively. Therefore, if \( f_1 \approx f_j \), then either \( f_{i1} \approx f_j \) or \( f_{i2} \approx f_j \).

Suppose that \( f_{i1} \approx f_j \). Choose a biholomorphism \( \varphi : (C^3, O) \rightarrow (C^3, O) \) at the origin as (7) such that \( f_{i1} \circ \varphi = f_j \). By a similar method
in the proof of Case I, the monomial \( z_0^a z_1^{k_1(n-\alpha)} \) has nonzero coefficient
\( A_{1,a} a_1^{k_1(n-\alpha)} \) in the expansion of \( f_{11} \circ \varphi \). Thus the monomial \( z_0^a z_1^{k_1(n-\alpha)} \)
must belong to \( f_j \). This leads to a contradiction. Thus \( f_{11} \not\equiv f_j \).
Similarly, we have \( f_{12} \not\equiv f_j \).
Consequently, \( f_i \not\equiv f_j \).
This completes the proof.

\[ \square \]

**Proposition 3.13.** Suppose that \( 2 < n < k < l \) and that \( f_3 \in T_3 \)
and \( f_j \in T_j \) for \( j = 0, 2, 4 \) in a sense of Definition 3.1. Then \( f_3 \not\equiv f_j \).

**Proof.** Suppose that \( f_3 \approx f_j \) for some \( j = 0, 2, 4 \).

**Case I** \( \gcd(n, l) = d_3 < n \).

Then \( n = n_3 d_3 \) and \( l = l_3 d_3 \) for some positive integers \( n_3 \) and \( l_3 \).
Note that \( d_3 > 1 \). By Theorem 3.4, A, \( f_3 \) can be written analytically as
follows:

\[ f_3 = z_0^n + z_1^k + z_2^l + \sum_{i=1}^d E_i z_0^{n_3 i} z_2^{l_3(d_3-i)} \]

for some complex numbers \( E_i \), where \( d \) is the largest number of \( i \) with \( E_i \neq 0 \) and \( 1 \leq i \leq d_3 - 1 \). Take a biholomorphism \( \varphi : (\mathbb{C}^3, O) \to (\mathbb{C}^3, O) \)
at the origin \( O = (0, 0, 0) \) such that \( f_3 \circ \varphi = f_j \) as (7).
Note that
\[ n < n_3 d + l_3(d_3 - d) < n_3(d_3 - 1) + l_3(d_3 - d + 1) < \cdots < n_3 + l_3(d_3 - d) < l. \]

By a similar method in the proof of Proposition 3.12, we have \( b_1 = c_1 = 0 \)
in (7). Therefore, we may write \( H \) as

\[ H = a_1 z_0 + H_{\min\{k-n+1, n_3 d + l_3(d_3-d)-n+1\}} + \cdots. \]

Note that

\[ (n - n_3 d)(n_3 d + l_3(d_3 - d) - n + 1) > l_3(d_3 - d), \]
\[ n(n_3 d + l_3(d_3 - d) - n + 1) > l. \]

Using the above inequalities (13), we have

\[ n \cdot \min\{k-n+1, n_3 d + l_3(d_3 - d) - n + 1\} > k, \]
\[ n_3 d \cdot \min\{k-n+1, n_3 d + l_3(d_3 - d) - n + 1\} > l. \]

So, we have \( c_2 = 0 \) and \( L = a_2 z_0 + b_2 z_1 + L_2 + \cdots \) in (7). Since \( \varphi \) is a
biholomorphism at the origin,

\[ |J_\varphi(O)| = a_1 b_2 c_3 \neq 0. \]

We consider two subcases (1a) and (1b) of Case I.
(Ia) $n < n_3d + l_3(d_3 - d) \leq k$.

Then we may write $H$ as

$$H = a_1z_0 + H_{n_3d+l_3(d_3-d)-n+1} + \cdots.$$ 

In the expansion of

$$E_dH^{n_3d}M^{l_3(d_3-d)} = E_d(a_1z_0 + H_{n_3d+l_3(d_3-d)-n+1} + \cdots)^{n_3d}(a_3z_0 + b_3z_1 + c_3z_2 + M_2 + \cdots)^{l_3(d_3-d)}$$

in $f_3 \circ \varphi$, every monomial with degree $n_3d + l_3(d_3 - d)$ is contained in the expansion of $E_d(a_1z_0)^{n_3d}(a_3z_0 + b_3z_1 + c_3z_2)^{l_3(d_3-d)}$ only. In particular, the monomial $z_0^{n_3d}z_2^{l_3(d_3-d)}$ has nonzero coefficient $E_d a_1^{n_3d} c_3^{l_3(d_3-d)}$ in the expansion of $E_d H^{n_3d} M^{l_3(d_3-d)}$.

We claim that the monomial $z_0^{n_3d}z_2^{l_3(d_3-d)}$ does not belong to $H^n$. If $z_0^{n_3d}z_2^{l_3(d_3-d)} \in H^n$, then the monomial is contained in the expansion of $(a_1z_0) ^\eta (H_{n_3d+l_3(d_3-d)-n+1} + \cdots)^{n-\eta}$ for some $\eta$ where $0 \leq \eta \leq n_3d$. By the inequalities

$$\eta + (n - \eta) \geq n_3d + (n_3d - n_3d)(n_3d + l_3(d_3 - d) - n + 1)$$

$$\geq n_3d + l_3(d_3 - d),$$

(15)

every monomial in the expansion of $(a_1z_0) ^\eta (H_{n_3d+l_3(d_3-d)-n+1} + \cdots)^{n-\eta}$ has degree greater than $n_3d + l_3(d_3 - d)$ if $0 \leq \eta \leq n_3d$. It is impossible if $z_0^{n_3d}z_2^{l_3(d_3-d)}$ belongs to $H^n$.

Since $L = a_2z_0 + b_2z_1 + L_2 + \cdots$ and $n_3d + l_3(d_3 - d) \leq k < l$, the monomial $z_0^{n_3d}z_2^{l_3(d_3-d)}$ does not belong to $L^k$ and $M^l$. That is to say, the monomial $z_0^{n_3d}z_2^{l_3(d_3-d)}$ has nonzero coefficient $E_d a_1^{n_3d} c_3^{l_3(d_3-d)}$ in the expansion of $f_3 \circ \varphi$ and it must belong to $f_j$ for some $j = 0, 2, 4$. This leads to a contradiction.

(Ib) $k < n_3d + l_3(d_3 - d)$.

In this case, $H$ can be written as

$$H = a_1z_0 + H_{k-n+1} + \cdots.$$ 

In the expansion of $f_3 \circ \varphi$, every monomial with degree $k$ is contained in the expansions of $(a_2z_0 + b_2z_1)^k$ in $L^k$ and $(a_1z_0)^{n-1}H_{k-n+1}$ in $H^n$ only. Since $n \geq n_3d + 2 \geq 3$, the monomial $z_0^k$ does not belong to $(a_1z_0)^{n-1}H_{k-n+1}$. This says that $a_2 = 0$. Note that the following assertion.
Assertion. If \( s < \min\{n - 1, n_3d + l_3(d_3 - d) - k + 1\} = e \), then \( L_s = 0 \) and \( H_{k-n+s} = 0 \). That is to say, \( H \) and \( L \) can be written as follows:

\[
\begin{align*}
H &= a_1z_0 + H_{k-n+e} + \cdots, \\
L &= b_2z_1 + L_e + \cdots.
\end{align*}
\]

Proof of Assertion. It is obvious that \( H_{k-n+1} = 0 \). In the expansion of \( f_3 \circ \varphi \), every monomial with degree \( k+1 \) is contained in the expansions of \( z_0^{n-1}H_{k-n+2} \) in \( H^n \) and \( z_1^{k-1}L_2 \) in \( L^k \) only. Thus \( H_{k-n+2} = L_2 = 0 \) if \( 2 < \min\{n-1, n_3d+l_3(d_3-d)-k+1\} \). If \( \min\{n-1, n_3d+l_3(d_3-d)-k+1\} = 3 \), we are done. If not, i.e., \( \min\{n-1, n_3d+l_3(d_3-d)-k+1\} > 3 \), then every monomial with degree \( k+2 \) is contained in the expansions of \( z_0^{n-1}H_{k-n+3} \) in \( H^n \) and \( z_1^{k-1}L_3 \) in \( L^k \) only in \( f_3 \circ \varphi \). Thus we have \( H_{k-n+3} = L_3 = 0 \) if \( \min\{n-1, n_3d+l_3(d_3-d)-k+1\} > 3 \). Continuing this process, we have the above assertion.

We claim that the monomial \( z_0^{n_3d}z_2^{l_3(d_3-d)} \) does not belong to \( H^n \) and \( L^k \) by using the assertion. Consider two subcases of the case (Ib).

(Ib-I) \( n_3d + l_3(d_3 - d) - k + 1 \leq n - 1 \).

In this case, \( H \) and \( L \) can be written as follows:

\[
\begin{align*}
H &= a_1z_0 + H_{n_3d+l_3(d_3-d)-n+1} + \cdots, \\
L &= b_2z_1 + L_{n_3d+l_3(d_3-d)-k+1} + \cdots.
\end{align*}
\]

If the monomial \( z_0^{n_3d}z_2^{l_3(d_3-d)} \) belongs to \( H^n \), then the monomial \( z_0^{n_3d}z_2^{l_3(d_3-d)} \) belongs to \( (a_1z_0)^\eta(H_{n_3d+l_3(d_3-d)-n+1} + \cdots)^{n-\eta} \) for some \( \eta \) where \( 0 \leq \eta \leq n_3d \). The inequality (15) implies that every monomial in the expansion of \( (a_1z_0)^\eta(H_{n_3d+l_3(d_3-d)-n+1} + \cdots)^{n-\eta} \) has degree greater than \( n_3d + l_3(d_3 - d) \) if \( 0 \leq \eta \leq n_3d \). This leads to a contradiction.

Similarly, we have \( z_0^{n_3d}z_2^{l_3(d_3-d)} \notin L^k \) by using the inequality

\[
k(n_3d + l_3(d_3 - d) - k + 1) > n_3d + l_3(d_3 - d).
\]

(Ib-II) \( n - 1 < n_3d + l_3(d_3 - d) - k + 1 \).

In this case, we claim that \( H \) and \( L \) can be written as follows:

\[
\begin{align*}
H &= z_0P_1 + z_1Q_1 + H_{n_3d+l_3(d_3-d)-n+1} + \cdots, \\
L &= z_0P_2 + z_1Q_2 + L_{n_3d+l_3(d_3-d)-k+1} + \cdots.
\end{align*}
\]
for some polynomials $P_1, Q_1$ with degrees less than $n_3d + l_3(d_3 - d) - n + 1$ and $P_2, Q_2$ with degrees less than $n_3d + l_3(d_3 - d) - k + 1$. Note that

$$(19) \quad H = a_1z_0 + H_{k-1} + \cdots, \quad L = b_2z_1 + L_{n-1} + \cdots.$$ 

Since $n - 1 + k - 1 < n_3d + l_3(d_3 - d)$, every monomial with degree $k + n - 2$ is contained in the expansions of $(a_1z_0)^{n-1}H_{k-1}$ in $H^n$ and $(b_2z_1)^{k-1}L_{n-1}$ in $L^k$ in the expansion of $f_3 \circ \varphi$. Since the monomial $z_0^{n-1}z_2^{k-1}$ with coefficient $A_{0,0,k-1}a_1^{n-1}$ is not contained in $(b_2z_1)^{k-1}L_{n-1}$ and the monomial $z_1^{k-1}z_2^{n-1}$ with coefficient $B_{0,0,n-1}b_2^{k-1}$ is not contained in $(a_1z_0)^{n-1}H_{k-1}$, we have $A_{0,0,k-1} = B_{0,0,n-1} = 0$. That is to say, $z_2^{k-1} \notin H_{k-1}$ and $z_2^{n-1} \notin L_{n-1}$. If $n_3d + l_3(d_3 - d) = k + n - 1$, we are done. If not, i.e., $n_3d + l_3(d_3 - d) > k + n - 1$, continue this process. In $H^n$, we claim that the monomial $z_0^{n-1}z_2^{k}$ with degree $k + n - 1$ is contained in $(a_1z_0)^{n-1}H_k$ only. In the expansion

$$H^n = (a_1z_0 + H_{k-1} + H_k + \cdots)^n = \sum_{\eta=0}^{n} nC_{\eta}(a_1z_0 + H_{k-1})^{n-\eta}(H_k + \cdots)^{\eta},$$

we have $z_0^{n-1}z_2^{k} \notin (a_1z_0 + H_{k-1})^{n-\eta}(H_k + \cdots)^{\eta}$ if $\eta = 0$ or $\eta > 1$. Thus

$$z_0^{n-1}z_2^{k} \in (a_1z_0 + H_{k-1})^{n-1}(H_k + \cdots).$$

if $z_0^{n-1}z_2^{k} \in H^n$. In particular, $z_0^{n-1}z_2^{k} \in (a_1z_0)^{n-1}H_k$ only. Since the monomial $z_0^{n-1}z_2^{k}$ does not belong to $L^k$, $M^l$ and $H^{n_3d}M^{l_3(d_3 - d)}$ if $n_3d + l_3(d_3 - d) > k + n - 1$, the monomial $z_0^{n-1}z_2^{k}$ has coefficient $nA_{0,0,k}a_1^{n-1}$ in the expansion of $f_3 \circ \varphi$. Thus $A_{0,0,k} = 0$.

Similarly, we can show that the monomial $z_1^{k-1}z_2^{n}$ is contained in the expansion of $L^k$ only and has coefficient $kB_{0,0,a}b_2^{k-1}$ in the expansion of $f_3 \circ \varphi$. Thus $B_{0,0,n} = 0$. That is, $z_2^{k} \notin H_k$ and $z_2^{n} \notin L_n$. Continue this process. Then we have the following facts:

$$(20) \quad A_{0,0,k-1} = A_{0,0,k} = \cdots = A_{0,0,n_3d+l_3(d_3-d)-n} = 0, \quad B_{0,0,n-1} = B_{0,0,n} = \cdots = B_{0,0,n_3d+l_3(d_3-d)-k} = 0.$$ 

This says that (18) holds. Using the facts (20), we prove that $z_0^{n_3d}z_2^{l_3(d_3-d)}$ does not belong to $H^n$ and $L^k$. If $z_0^{n_3d}z_2^{l_3(d_3-d)} \in H^n$, then

$$z_0^{n_3d}z_2^{l_3(d_3-d)} \in (z_0P_1 + z_1Q_1)^{n-\eta}(H_{n_3d+l_3(d_3-d)-n+1} + \cdots)^{\eta}$$

for some $\eta$ where $0 \leq \eta \leq n$. If either $\eta = 0$ or $\eta > 1$, then

$$z_0^{n_3d}z_2^{l_3(d_3-d)} \notin (z_0P_1 + z_1Q_1)^{n-\eta}(H_{n_3d+l_3(d_3-d)-n+1} + \cdots)^{\eta},$$
since \( z_0^{n_3 d} z_2^{l_3(d_3-d)} \notin (z_0 P_1 + z_1 Q_1)^n \) and
\[
n - \eta + \eta(n_3 d + l_3(d_3 - d) - n + 1) > n_3 d + l_3(d_3 - d),
\]
if \( \eta > 1 \). Thus, if \( z_0^{n_3 d} z_2^{l_3(d_3-d)} \in H^n \), then
\[
z_0^{n_3 d} z_2^{l_3(d_3-d)} \in (z_0 P_1 + z_1 Q_1)^{n-1}(H_{n_3 d+l_3(d_3-d)-n+1} + \cdots).
\]
But it is impossible, since \( n_3 d < n - 1 \).

Similarly, we can prove that if \( z_0^{n_3 d} z_2^{l_3(d_3-d)} \in L^k \), then
\[
z_0^{n_3 d} z_2^{l_3(d_3-d)} \in (z_0 P_2 + z_1 Q_2)^{k-1}(L_{n_3 d+l_3(d_3-d)-k+1} + \cdots).
\]
But it is also impossible, since \( n_3 d < k - 1 \). Thus \( z_0^{n_3 d} z_2^{l_3(d_3-d)} \) does not belong to the expansions of \( H^n \) and \( L^k \) in this case.

Consequently, \( z_0^{n_3 d} z_2^{l_3(d_3-d)} \) does not belong to \( H^n \) and \( L^k \) at any case. Furthermore, the monomial \( z_0^{n_3 d} z_2^{l_3(d_3-d)} \) does not belong to \( M^l \), since \( n_3 d + l_3(d_3 - d) < l \).

These show that the monomial \( z_0^{n_3 d} z_2^{l_3(d_3-d)} \) has nonzero coefficient \( E_{q_1} a_1^{n_3 d} c_3^{l_3(d_3-d)} \) in the expansion of \( f_3 \circ \varphi \). But the monomial \( z_0^{n_3 d} z_2^{l_3(d_3-d)} \) does not belong to \( f_j \) for \( j = 0, 2, 4 \). This leads to a contradiction.

Case II) \( \gcd(n, l) = n < l \).

Then \( l = n l_3 \) for some positive integer \( l_3 \). By Theorem 3.4, A, \( f_3 \) has analytically two different representations as follows:
\[
f_{31} = z_0^{n_3 d} + z_1^{l_3} + z_2^{l_3} + \sum_{i=1}^{\alpha} E_{1,i} z_0^{i} z_2^{l_3(n-i)},
\]
\[
f_{32} = z_0^{n_3 d} + z_1^{l_3} + \sum_{r=1}^{\beta} E_{2,r} z_0^{r} z_2^{l_3(n-r)}
\]
for some complex numbers \( E_{1,i} \) and \( E_{2,r} \), where \( \alpha \) is the largest number of \( i \) with \( E_{1,i} \neq 0 \) and \( \beta \) is the largest number of \( r \) with \( E_{2,r} \neq 0 \) for \( 1 \leq i, r \leq n - 2 \). Therefore, if \( f_3 \approx f_j \), then either \( f_{31} \approx f_j \) or \( f_{32} \approx f_j \).

Suppose that \( f_{31} \approx f_j \). Then there is a biholomorphism \( \varphi : (\mathbb{C}^3, O) \rightarrow (\mathbb{C}^3, O) \) at the origin as (7) such that \( f_{31} \circ \varphi = f_j \). Note that
\[
n < \alpha + l_3(n-\alpha) < \alpha - 1 + l_3(n - \alpha + 1) < \cdots < 1 + l_3(n - 1) < l.
\]
As in the proof of Case I, the monomial \( z_0^{\alpha} z_2^{l_3(n-\alpha)} \) has nonzero coefficient \( E_{1,\alpha} a_1^{\alpha} c_3^{l_3(n-\alpha)} \) in the expansion of \( f_{31} \circ \varphi \). But the monomial \( z_0^{\alpha} z_2^{l_3(n-\alpha)} \) does not belong to \( f_j \) for \( j = 0, 2, 4 \). This leads to a contradiction.

Similarly, suppose that \( f_{32} \approx f_j \) and \( \varphi : (\mathbb{C}^3, O) \rightarrow (\mathbb{C}^3, O) \) are chosen as (7) so that \( f_{32} \circ \varphi = f_j \). Then the monomial \( z_0^{\beta} z_2^{l_3(n-\beta)} \) has nonzero
coefficient \( E_{2,\beta}z_1^\beta c_3^{(n-\beta)} \) in the expansion of \( f_{32} \circ \varphi \). But the monomial \( z_0^\beta z_2^\gamma c_3^{(n-\beta)} \) does not belong to \( f_j \). This also leads to a contradiction.

Consequently, we show that neither \( f_{31} \approx f_j \) nor \( f_{32} \approx f_j \). Thus \( f_3 \not\approx f_j \).

By Case I and Case II, \( f_3 \not\approx f_j \) at any case.

This completes the proof. \( \square \)

**Proof of Theorem 3.6.** Proposition 3.11, Proposition 3.12 and Proposition 3.13 imply Theorem 3.6. \( \square \)

**Proof of Theorem 3.9.** Suppose that \( f_4 \approx f_2 \). Choose a biholomorphic \( \varphi : (\mathbb{C}^3, O) \rightarrow (\mathbb{C}^3, O) \) at the origin \( O = (0,0,0) \) as (7) so that \( f_4 \circ \varphi = f_2 \).

Claim that the monomial \( z_0^{\alpha_0}z_1^\beta z_2^\gamma \) has nonzero coefficient \( D_{\alpha_0,\beta_0,\gamma_0}^\alpha a_1 b_2^\beta c_3^\gamma \) for some complex numbers \( a_1, b_2 \) and \( c_3 \) in the expansion of \( f_4 \circ \varphi \).

To prove the claim, it is enough to consider two cases: \( \alpha_0 + \beta_0 + \gamma_0 \leq k \) and \( k < \alpha_0 + \beta_0 + \gamma_0 \leq k + n - 2 \).

Case I \( \alpha_0 + \beta_0 + \gamma_0 \leq k \)

In this case, we may write \( H \) as follows:

\[
H = a_1 z_0 + H_{\alpha_0+\beta_0+\gamma_0-n+1} + \cdots
\]

by the similar method as in the proof of Proposition 3.13. Consider the expansion of \( f_4 \circ \varphi \) as follows:

\[
(21) \quad H^n + L^k + M^l + \sum_{(a,\beta,\gamma)\in I_4} D_{\alpha,\beta,\gamma}^\alpha H^a L^\beta M^\gamma
\]

\[
= (a_1 z_0 + H_{\alpha_0+\beta_0+\gamma_0-n+1} + \cdots)^n + (a_2 z_0 + b_2 z_1 + c_2 z_1 + L^2 + \cdots)^k
\]

\[
+ (a_3 z_0 + b_3 z_1 + c_3 z_2 + M_2 + \cdots)^l + \sum_{(a,\beta,\gamma)\in I_4} D_{\alpha,\beta,\gamma}^\alpha (a_1 z_0
\]

\[
+ H_{\alpha_0+\beta_0+\gamma_0-n+1} + \cdots)^n (a_2 z_1 + b_2 z_1 + c_2 z_2 + L_2 + \cdots)^k (a_3 z_0 +
\]

\[
b_3 z_1 + c_3 z_2 + M_2 + \cdots)^l.
\]

Note that the monomial \( z_2^k \) does not belong to the expansions of \( H^n, M^l \) and \( D_{\alpha_0,\beta_0,\gamma_0} H^a L^\beta M^\gamma \) by the inequalities \( n(\alpha_0 + \beta_0 + \gamma_0 - n + 1) > k \), \( \alpha_0(\alpha_0 + \beta_0 + \gamma_0 - n + 1) + \beta_0 + \gamma_0 > k \) and \( l > k \). These show that \( z_2^k \in L^k \) only in the expansion of (21) and the monomial \( z_2^k \) has coefficient \( c_2^k \). Since the monomial \( z_2^k \) does not belong to \( f_2(z, z_1, z_2) \), we have \( c_2 = 0 \). Thus \( | J_\varphi(0) | = a_1 b_2 c_3 \neq 0 \).

We claim that \( z_0^{\alpha_0} z_1^{\beta_0} z_2^{\gamma_0} \) has nonzero coefficient \( D_{\alpha_0,\beta_0,\gamma_0}^\alpha a_1 b_2^\beta c_3^\gamma \) in the expansion of (21). By the inequalities \( \alpha_0 + (n-\alpha_0)(\alpha_0 + \beta_0 + \gamma_0 - n + 1) > \)

\[
\alpha_0(\alpha_0 + \beta_0 + \gamma_0 - n + 1) + \beta_0 + \gamma_0 > k \) and \( \alpha_0 + \beta_0 + \gamma_0 \leq k \) we have \( c_2 = 0 \). Thus \( | J_\varphi(0) | = a_1 b_2 c_3 \neq 0 \).

We claim that \( z_0^{\alpha_0} z_1^{\beta_0} z_2^{\gamma_0} \) has nonzero coefficient \( D_{\alpha_0,\beta_0,\gamma_0}^\alpha a_1 b_2^\beta c_3^\gamma \) in the expansion of (21). By the inequalities \( \alpha_0 + (n-\alpha_0)(\alpha_0 + \beta_0 + \gamma_0 - n + 1) > \)

\[
\alpha_0 + (n-\alpha_0)(\alpha_0 + \beta_0 + \gamma_0 - n + 1) + \beta_0 + \gamma_0 > k \) and \( \alpha_0 + \beta_0 + \gamma_0 \leq k \) we have \( c_2 = 0 \). Thus \( | J_\varphi(0) | = a_1 b_2 c_3 \neq 0 \).
$\alpha_0 + \beta_0 + \gamma_0$ and $l > \alpha_0 + \beta_0 + \gamma_0$, the monomial $z_0^{\alpha_0} z_1^{\beta_0} z_2^{\gamma_0}$ does not belong to the expansions of $H^n$ and $M^l$. If $k > \alpha_0 + \beta_0 + \gamma_0$, then $z_0^{\alpha_0} z_1^{\beta_0} z_2^{\gamma_0} \notin L^k$ is clear. If $k = \alpha_0 + \beta_0 + \gamma_0$, then $z_0^{\alpha_0} z_1^{\beta_0} z_2^{\gamma_0} \notin L^k$, since every monomial with degree $k = \alpha_0 + \beta_0 + \gamma_0$ is contained in the expansion of $(a_2 z_0 + b_2 z_1)^k$ in $L^k$, and the monomial $z_0^{\alpha_0} z_1^{\beta_0} z_2^{\gamma_0}$ does not belong to the expansion of $(a_2 z_0 + b_2 z_1)^k$.

Consequently, the monomial $z_0^{\alpha_0} z_1^{\beta_0} z_2^{\gamma_0}$ belongs to $D_{\alpha_0, \beta_0, \gamma_0} H^{\alpha_0} L^{\beta_0} M^{\gamma_0}$ only in the expansion of (21). Thus the monomial $z_0^{\alpha_0} z_1^{\beta_0} z_2^{\gamma_0}$ has nonzero coefficient $D_{\alpha_0, \beta_0, \gamma_0} a_1^{\alpha_0} b_2^{\beta_0} c_3^{\gamma_0}$ in the expansion of $f_4 \circ \varphi$.

Case II) $k < \alpha_0 + \beta_0 + \gamma_0 \leq k + n - 2$

In this case, we may write $H$ as follows:

$$H = a_1 z_0 + H_{k-n+1} + \cdots.$$ 

In the expansion of $f_4 \circ \varphi$, every monomial with degree $k$ is contained in the expansions of $n(a_1 z_0)^{n-1} H_{k-n+1}$ in $H^n$ and $(a_2 z_0 + b_2 z_1 + c_2 z_2)^k$ in $L^k$ only. Note that $n \geq \alpha_0 + 2 \geq 3$. Since the monomials $z_0 z_1^{k-1}$ and $z_2^k$ do not belong to the expansion of $n(a_1 z_0)^{n-1} H_{k-n+1}$ and $f_2(z_0, z_1, z_2)$, we have $a_2 = c_2 = 0$. This follows

$$L = b_2 z_1 + L_2 + \cdots.$$ 

By the Assertion in the proof of Proposition 3.13, $H$ and $L$ can be written as follows:

$$H = a_1 z_0 + H_{\alpha_0 + \beta_0 + \gamma_0 - n + 1} + \cdots,$$

$$L = b_2 z_1 + L_{\alpha_0 + \beta_0 + \gamma_0 - k + 1} + \cdots.$$ 

Thus the monomial $z_0^{\alpha_0} z_1^{\beta_0} z_2^{\gamma_0}$ does not belong to the expansions of $H^n$ and $L^k$ by the inequalities $\alpha_0 + (n-\alpha_0)(\alpha_0 + \beta_0 + \gamma_0 - n + 1) > \alpha_0 + \beta_0 + \gamma_0$ and $\beta_0 + (k-\beta_0)(\alpha_0 + \beta_0 + \gamma_0 - k + 1) > \alpha_0 + \beta_0 + \gamma_0$, since $n \geq \alpha_0 + 2$ and $k \geq \beta_0 + 2$. It is clear that the monomial $z_0^{\alpha_0} z_1^{\beta_0} z_2^{\gamma_0}$ does not belong to $M^l$, since $l > \alpha_0 + \beta_0 + \gamma_0$.

Consequently, the monomial $z_0^{\alpha_0} z_1^{\beta_0} z_2^{\gamma_0}$ is contained in the expansion of $D_{\alpha_0, \beta_0, \gamma_0} H^{\alpha_0} L^{\beta_0} M^{\gamma_0}$ only in the expansion of $f_4 \circ \varphi$. Thus the monomial $z_0^{\alpha_0} z_1^{\beta_0} z_2^{\gamma_0}$ has nonzero coefficient $D_{\alpha_0, \beta_0, \gamma_0} a_1^{\alpha_0} b_2^{\beta_0} c_3^{\gamma_0}$ and must belong to $f_2(z_0, z_1, z_2)$. This says that $f_4 \neq f_2$ if $(\alpha, \beta, \gamma) \neq (\alpha', \beta', \gamma')$. In particular, if $f_4 \approx g_4$, then $(\alpha, \beta, \gamma)$ belongs to $I_1'$ and there exist complex numbers $a_1, b_2$ and $c_3$ such that $D_{\alpha_0, \beta_0, \gamma_0} a_1^{\alpha_0} b_2^{\beta_0} c_3^{\gamma_0} = D_{\alpha_0, \beta_0, \gamma_0}$. This completes the proof of Theorem 3.9. \qed
References


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