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CENTRAL LIMIT THEOREMS FOR BELLMAN-HARRIS PROCESSES

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ABSTRACT. In this paper we consider functionals of the empirical age distribution of supercritical Bellman-Harris processes. Let $f : R^+ \to R$ be a measurable function that integrates to zero with respect to the stable age distribution in a supercritical Bellman-Harris process with no extinction. We present sufficient conditions for the asymptotic normality of the mean of f with respect to the empirical age distribution at time t.

1. Introduction

Let $\{Z(t); t \geq 0\}$ be a one-dimensional supercritical Bellman-Harris process evolving from one particle of age 0 at time 0 with lifetime distribution G and offspring law $\{p_k\}$. For any family history ω, let $\{a_j(t,\omega); j = 1, \cdots, Z(t,\omega)\}$ be the age-chart at time t and for any $f; R^+ \to R$ define

$$Z_f(t,\omega) = \sum_{j=1}^{Z(t,\omega)} f(a_j(t,\omega)) \quad \text{and} \quad m_f(t) = E(Z_f(t,\omega)).$$

Let $\alpha \geq 0$ be the Malthusian parameter for m and G which is the unique solution to the equation $m\int_0^\infty e^{-\alpha t} dG(t) = 1$, where $m = \sum_{j=0}^\infty jp_j$. If $e^{-\alpha t} f(t)(1 - G(t))$ is directly Riemann integrable (in short, d.R.i) then it is well-known (see Jagers (1975)) that

$$\lim_{t\to\infty} e^{-\alpha t} m_f(t) = \frac{1}{\beta} \int_0^\infty e^{-\alpha u}(1 - G(u)) f(u)\, du \equiv m_f^\alpha,$$

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where $\beta = m\int_0^\infty e^{-\alpha u}G(du)$ and also (see Athreya and Kaplan (1976)) that

$$\frac{Z_f(t)}{Z(t)} \xrightarrow{\text{a.s.}} m_f^\alpha \quad \text{on the set of nonextinction.}$$

In this paper we develop limit theorems for this class of stochastic processes $\{Z_f(t); t \geq 0\}$ when $m_f^\alpha = 0$. In section 2 we describe the basic setup, terminologies and notations and state the results. Section 3 analyzes the first and second moments of $Z_f(t)$ while section 4 gives the proof of Theorem 1.

In a sequel (Kang (1999)) we extend the results to multitype cases.

2. Statement of Results

For any family tree ω let $Z(t,\omega)$ be the number of particles living at time t, and let $Z(t,a,\omega)$ be the number of particles living at time t whose age $\leq a$. We shall consider the following assumptions which are not valid at all times.

(A 1) $p_0 = 0$.
(A 2) $m_2 \equiv \sum_{j=0}^\infty j^2 p_j < \infty$.
(A 3) $G(0+) = 0$, G is non-lattice.
(A 4) $\int_0^\infty uG(du) < \infty$.

The assumption (A 1) is primarily of convenience of exposition. Otherwise, one has to keep qualifying "on the set of nonextinction". With (A 1) and (A 2) (in fact, the condition $\sum_{j=1}^\infty (j\log j)p_j < \infty$ is enough) we know (see Athreya and Ney (1972)) that there exists a random variable W such that

$$\lim_{t\to\infty} e^{-\alpha t}Z(t) = W \quad \text{a.s.} \quad \text{and} \quad P(W>0) = 1.$$

The assumption (A 3) is standard and guarantees that $Z(t)$ is finite for any finite t. We add superscript a to random variables and their moments to indicate the case when P is supported by those ω's which start with one particle of age $a \geq 0$.

We impose the following assumptions on a measurable function $f : R^+ \to R$.

(F 1) f is continuous a.e. (w.r.t. Lebesgue measure) on the support of G.

(F 2) $e^{-\alpha t}(1-G(t))f(t)$ is d.R.i. and $\int_0^\infty e^{-\alpha t}(1-G(t))f(t)\,dt = 0$.

(F 3) $e^{-\alpha t} f^2(t) \to 0$ as $t \to \infty$.
(F 4) $e^{-\alpha t}(m_f^2 * G)(t)$ is d.R.i.
(F 5) There exists $s_0 > 0$ such that for $s \geq s_0$,

$$\sup_{a \geq 0} |f(a+s)(1 - G^a(s))| < \infty, \quad \sup_{a \geq 0} |f^2(a+s)(1 - G^a(s))G^a(s)| < \infty,$$

where $G^a(t) = (G(t+a) - G(a))/(1 - G(a))$.

REMARK 1. 1. (F 3) with (A 4) implies (F 3)$'$: $e^{-\alpha t} f^2(t)(1 - G(t))$ is d.R.i.
2. (F 4) is not directly in terms of f and G and is difficult to verify in general. However, (F 4) can be verified easily for Markov branching processes (see Corollary 1 below) and when G is gamma (see Corollary 2 below).

Now we are ready to state the results.

THEOREM 1. *Let $m > 1$. Assume* (A 1) - (A 4). *Let f satisfy* (F 1) - (F 5). *Then*

$$\frac{Z_f(t)}{\sqrt{Z(t)}} \xrightarrow{\text{d}} N(0, \sigma_f^2) \quad \text{as} \quad t \to \infty,$$

where $0 < \sigma_f^2 \equiv n_1^{-1} \lim_{t \to \infty} e^{-\alpha t} D_f(t) < \infty$, $D_f(t) = E(Z_f(t))^2$ *and* $n_1 = \int_0^\infty e^{-\alpha u}(1 - G(u))\, du / (m \int_0^\infty u e^{-\alpha u} G(du))$.

REMARK 2. In the next section, we'll show that $\lim_{t \to \infty} e^{-\alpha t} D_f(t)$ exists and is finite and positive.

If the lifetime has exponential distribution, then the renewal function can be found explicitly and we have the following

COROLLARY 1. *Consider a Markov branching process with offspring mean $m > 1$ and exponential life time with mean $1/b$. Assume* (A 1) *and* (A 2). *Let $f : R^+ \to R$ be bounded and continuous a.e. If* $\displaystyle\int_0^\infty e^{-mbt} f(t)\, dt = 0$, *then*

$$\frac{Z_f(t)}{\sqrt{Z(t)}} \xrightarrow{\text{d}} N(0, \sigma_f^2) \quad \text{as} \quad t \to \infty,$$

where

$$\sigma_f^2 = mb \int_0^\infty e^{-mbt} f^2(t)\, dt + m^2 b^3 (m_2 - m) \int_0^\infty e^{-\alpha t} \left(\int_0^t f(u) e^{-mbu}\, du \right)^2 dt.$$

COROLLARY 2. *Let $\{Z(t); t \geq 0\}$ be a Bellman-Harris process with gamma lifetime distribution with density function $g(x) = \Gamma(k)^{-1} b^k x^{k-1} e^{-bx}$ and offspring distribution $\{p_k\}$, where $b > 0$, $k \geq 2$ integer such that $\cos(2\pi/k) < \frac{1}{2}(1 + m^{1/k})$. Assume that* (A 1) - (A 2) *hold. Let $f : R^+ \longrightarrow R$ be bounded, continuous and differentiable a.e. If f satisfies* (F 2) *and $||f'||_\infty$ is finite, then*

$$\frac{Z_f(t)}{\sqrt{Z(t)}} \xrightarrow{\text{d}} N(0, \sigma_f^2) \quad \text{as} \quad t \to \infty,$$

where σ_f^2 is as in Theorem 1.

One can strengthen Theorem 1 to claim asymptotic independence of $W(t) = e^{-\alpha t} Z(t)$ and $\frac{Z_f(t)}{\sqrt{Z(t)}}$ as in Athreya (1969). Thus we have

THEOREM 1′. *Under the assumptions and notations of* Theorem 1,

$$\lim_{t \to \infty} P(e^{-\alpha t} Z(t) \leq x, \frac{Z_f(t)}{\sqrt{Z(t)}} \leq y) = P(W \leq x) \Phi\left(\frac{y}{\sigma_f}\right).$$

COROLLARY 1′. *Under the assumptions and notations of* Theorem 1,

$$\lim_{t \to \infty} P(e^{-\frac{\alpha t}{2}} Z_f(t) \leq y) = \int_0^\infty \Phi\left(\frac{y}{\sigma_f w}\right) dP(W \leq w).$$

3. The First and Second Moments

Put $\mu_\alpha(t) \equiv m \int_0^t e^{-\alpha u} G(du)$. By the definition of α, $\mu_\alpha(\cdot)$ is a probability distribution. Let $U_\alpha(t) \equiv \sum_{n=0}^\infty \mu_\alpha^{*n}(t)$, where μ_α^{*n} denote the n-fold convolution of μ_α with itself and $\mu_\alpha^{*0}(t) = 1$ for $t \geq 0$, be the associated renewal function.

PROPOSITION 1. *Let $m > 1$.*
(a) *Assume that $e^{-\alpha t} f(t)(1 - G(t))$ is bounded on finite intervals. Then*

$$e^{-\alpha t} m_f(t) = \int_0^t e^{-\alpha(t-u)} f(t-u)(1 - G(t-u)) U_\alpha(du).$$

(b) *Assume* $e^{-\alpha t}f^2(t)(1-G(t))$ *and* $e^{-\alpha t}(m_f^2 * G)(t)$ *are bounded on finite intervals, then*

$$\begin{aligned} e^{-\alpha t}D_f(t) &= \int_0^t [e^{-\alpha(t-u)}f^2(t-u)(1-G(t-u)) \\ &\qquad + (m_2 - m)e^{-\alpha(t-u)}(m_f^2 * G)(t-u)]U_\alpha(du). \end{aligned}$$

Proof. (a) Recall that by the additive property of branching process we have

$$Z_f(t) = I(\lambda_0 > t)f(t) + \sum_{j=1}^{\xi} Z_{f,j}(t-\lambda_0) \tag{1}$$

where λ_0 is the lifetime of the ancestor and ξ is the number of offsprings produced by it and $\{Z_{f,j}(u); u \geq 0\}$ are independent copies of $\{Z_f(u); u \geq 0\}$. Since we assume the independence of ξ and λ_0, taking expectation we get

$$m_f(t) = f(t)(1-G(t)) + m\int_0^t m_f(t-u)G(du).$$

We multiply both sides by $e^{-\alpha t}$ to get the following renewal equation

$$e^{-\alpha t}m_f(t) = e^{-\alpha t}f(t)(1-G(t)) + \int_0^t e^{-\alpha(t-u)}m_f(t-u)\mu_\alpha(du).$$

Since $e^{-\alpha t}f(t)(1-G(t))$ is bounded on finite intervals, we have (see Asmussen (1987), p. 113)

$$e^{-\alpha t}m_f(t) = \int_0^t e^{-\alpha(t-u)}f(t-u)(1-G(t-u))U_\alpha(du).$$

(b) Squaring both sides of (1) we have

$$Z_f^2(t) = I(\lambda_0 > t)f^2(t) + \sum_{i\neq j}^{\xi} Z_{f,i}(t-\lambda_0)Z_{f,j}(t-\lambda_0) + \sum_{j=1}^{\xi} Z_{f,j}^2(t-\lambda_o).$$

By the independence of ξ and λ_0 we get

$$\begin{aligned} D_f(t) &= (1-G(t))f^2(t) + (m_2 - m)\int_0^t m_f^2(t-u)G(du) \\ &\qquad + m\int_0^t D_f(t-u)G(du). \end{aligned}$$

Multiplying both sides by $e^{-\alpha t}$ we arrive at the following renewal equation,

$$\begin{aligned} e^{-\alpha t}D_f(t) &= e^{-\alpha t}f^2(t)(1-G(t)) + (m_2-m)e^{-\alpha t}(m_f^2 * G)(t) \\ &\quad + \int_0^t e^{-\alpha(t-u)}D_f(t-u)\mu_\alpha(du). \end{aligned} \tag{2}$$

Since $e^{-\alpha t}f^2(t)(1-G(t))$ and $e^{-\alpha t}(m_f^2 * G)(t)$ are bounded on finite intervals,

$$\begin{aligned} e^{-\alpha t}D_f(t) &= \int_0^t (e^{-\alpha(t-u)}f^2(t-u)(1-G(t-u)) \\ &\quad +(m_2-m)e^{-\alpha(t-u)}(m_f^2 * G)(t-u))U_\alpha(du). \end{aligned}$$ □

The following is an immediate consequence of the Key Renewal Theorem (see Asmussen (1987)).

PROPOSITION 2. *Let f satisfy (F 3) and (F 4). Then $D_f^\alpha = \lim_{t\to\infty} e^{-\alpha t}D_f(t)$ exists and is given by*

$$D_f^\alpha = \frac{1}{\beta}\int_0^\infty (e^{-\alpha u}f^2(u)(1-G(u)) + (m_2-m)e^{-\alpha u}(m_f^2 * G)(u))\,du$$

which is finite where $\beta = m\int_0^\infty ue^{-\alpha u}G(du)$.

Now define $M(s): f \longmapsto M(s)f$ by $(M(s)f)(t) = m_f^t(s)$.

PROPOSITION 3. *Let $m > 1$. Then*
(a) $m_{M(s)f}(t) = m_f(t+s)$.
(b) *Further, assume that (F 3) and (F 4) hold, then*

$$\lim_{s\to\infty} e^{-\alpha s}\lim_{t\to\infty} e^{-\alpha t}D_{M(s)f}(t) = 0.$$

Proof (a) Let $\{a_j; j=1,\cdots,Z(t)\}$ be the age-chart at time t.

$$\begin{aligned} m_{M(s)f}(t) &= E(Z_{M(s)f}(t)) \\ &= E\left(\sum_{j=1}^{Z(t)}(M(s)f)(a_j)\right) \\ &= E(Z_f(t+s)) \quad = \quad m_f(t+s). \end{aligned}$$

(b) From equation (2) above with $M(s)f$ in the place of f

$$e^{-\alpha t} D_{M(s)f}(t)$$
$$= e^{-\alpha t}\{(M(s)f)\}^2(t)(1-G(t)) + (m_2 - m)e^{-\alpha t}(m^2_{M(s)f} * G)(t)$$
$$+ \int_0^t e^{-\alpha(t-u)} D_{M(s)f}(t-u)\mu_\alpha(du).$$

First we show that $e^{-\alpha t}\{M(s)f\}^2(t)(1-G(t))$ and $e^{-\alpha t}(m^2_{M(s)f} * G)(t)$ are d.R.i. for fixed s. Beginning with an ancestor of age t at time 0, we have the following identity

$$Z^t_f(s) = I(\lambda^t > s)f(t+s) + \sum_{j=1}^{\xi} Z_{f,j}(s-\lambda^t) \tag{3}$$

where λ^t and ξ are the lifetime and the number of children of the ancestor with initial age t respectively and $\{Z_{f,j}(s); s \geq 0\}$ is the $Z_f(\cdot)$ process initiated by the jth child of the ancestor. Conditioned on λ^t, $\{Z_{f,j}(s-\lambda^t); j = 1, \cdots, \xi\}$ are i.i.d. and further if $\lambda^t = u$, then the conditional distribution of $Z_{f,j}(s-\lambda^t)$ is the same as $Z_f(s-u)$. So we have

$$\{M(s)f\}(t) = f(t+s)(1-G^t(s)) + m\int_0^s m_f(s-u)G^t(du) \tag{4}$$

and so,

(5)
$$\{M(s)f\}^2(t) \leq 2f^2(t+s)(1-G^t(s))^2 + 2m^2\left[\int_0^s m_f(s-u)G^t(du)\right]^2.$$

Using Cauchy-Schwarz inequality,

$$\begin{aligned}\left(\int_0^s m_f(s-u)G^t(du)\right)^2 &\leq \left(\int_0^s m^2_f(s-u)G^t(du)\right)G^t(s)\\ &\leq \frac{(m^2_f * G)(t+s)}{1-G(t)}.\end{aligned}$$

Combining this with inequality (5), we have

$$\begin{aligned}&e^{-\alpha t}(M(s)f)^2(t)(1-G(t))\\ &\leq 2e^{\alpha s}(e^{-\alpha(t+s)}f^2(t+s)(1-G(t+s)) + m^2 e^{-\alpha(t+s)}(m^2_f * G)(t+s)).\end{aligned}$$

(F 3) and (F 4) along with this inequality imply that $e^{-\alpha t}\{M(s)f\}^2(t)$ $(1-G(t))$ is d.R.i. for fixed $s \geq 0$. On the other hand,

$$\begin{aligned} e^{-\alpha t}\int_0^t m^2_{M(s)f}(t-u)G(du) &= e^{-\alpha t}\int_0^t m_f^2(t+s-u)G(du) \\ &\leq e^{-\alpha t}\int_0^{t+s} m_f^2(t+s-u)G(du) \\ &= e^{\alpha s}e^{-\alpha(t+s)}(m_f^2 * G)(t+s). \end{aligned}$$

So $e^{-\alpha t}(m^2_{M(s)f} * G)(t)$ is d.R.i. by (F 4). Hence we can apply the Key renewal Theorem to get

$$\begin{aligned} &\lim_{t\to\infty} e^{-\alpha t}D_{M(s)f}(t) \\ &= \frac{1}{\beta}\int_0^\infty (e^{-\alpha u}\{M(s)f\}^2(u)(1-G(u)) \\ &\qquad +(m_2-m)e^{-\alpha u}(m^2M(s)f * G)(u))\,du \\ &\leq \frac{2e^{\alpha s}}{\beta}\int_s^\infty e^{-\alpha u}(f^2(u)(1-G(u)) + m^2(m_f^2 * G)(u) \\ &\qquad +(m_2-m)(m_f^2 * G)(u))\,du \end{aligned}$$

and since (F 3)′ and (F 4) hold, we conclude that

$$\lim_{s\to\infty} e^{-\alpha s}\lim_{t\to\infty} e^{-\alpha t}D_{M(s)f}(t) = 0.$$

□

4. Proofs

Proof of Theorem 1. Referring to the additive property of branching processes we can write (suppressing ω and (t,ω))

$$Z_f(t+s) = \sum_{j=1}^{Z(t)} Z_f^{a_j}(s), \tag{6}$$

where $\{Z_f^{a_j}(s); s \geq 0\}$ is the process $\{Z_f(s); s \geq 0\}$ initiated by the ancestor of age a_j at time t. It is obvious that conditioned on the age chart at time t, $\{Z_f^{a_j}(s); j = 1, \cdots, Z(t)\}$ are independently distributed. Furthermore, if $a_j = a$ then the conditional distribution of $Z_f^{a_j}(s)$ is the same as $Z_f^a(s)$. Starting from equation (6) we have the following identity

$$Z_f(t+s) = \sum_{j=1}^{Z(t)} (Z_f^{a_j}(s) - m_f^{a_j}(s)) + Z_{M(s)f}(t). \tag{7}$$

Dividing equation (7) by $\sqrt{Z(t+s)}$ we get

$$\begin{aligned}
\frac{Z_f(t+s)}{\sqrt{Z(t+s)}} &= \sqrt{\frac{Z(t)}{Z(t+s)}} \frac{1}{\sqrt{Z(t)}} \sum_{j=1}^{Z(t)} (Z_f^{a_j}(s) - m_f^{a_j}(s)) + \frac{Z_{M(s)f}(t)}{\sqrt{Z(t+s)}} \\
&= \sqrt{\frac{W(t)}{W(t+s)}} \frac{1}{\sqrt{Z(t)}} \sum_{j=1}^{Z(t)} (Z_f^{a_j}(s) - m_f^{a_j}(s))e^{-\frac{\alpha s}{2}} + \frac{Z_{M(s)f}(t)}{\sqrt{Z(t+s)}} \\
&\equiv \sqrt{\frac{W(t)}{W(t+s)}} A_1(t,s) + A_2(t,s), \quad \text{say,}
\end{aligned}$$

where $W(t) = e^{-\alpha t} Z(t)$. Here is the basic idea of the proof; we first choose s large enough to make $A_2(t,s)$ small in probability and then with this large but fixed s, we show using the Lindberg-Feller theorem that as $t \to \infty$, $A_1(t,s)$ converges to the desired normal distribution. We carry this out in a series of lemmas below where we assume that (F 1)-(F 5) hold.

LEMMA 1. *For any $\eta > 0$, $\delta > 0$, there exists $s_0(\eta, \delta)$ such that*

$$\lim_{t\to\infty} P(|A_2(t,s)| > \eta) < \delta, \quad \textit{for all} \quad s \geq s_0(\eta,\delta).$$

Proof. Recall that there exists $W = \lim_{t\to\infty} W(t)$ a.s. and $P(W > 0) = 1$ if $p_0 = 0$. Choose x such that $P(W \leq x) \leq \delta/3$ and let $\varepsilon > 0$ be such that $\varepsilon < x/2$. Since $W(t)$ converges to W a.s., it does so in probability and hence we can choose $s_0' = s_0'(\delta)$ such that

$$P(|W(t+s_0') - W| \geq \varepsilon) < \delta/3, \qquad \text{for all} \qquad t \geq 0.$$

So for $s \geq s_0'$ and for all $t \geq 0$,

$$\begin{aligned}
P(|A_2(t,s)| > \eta) &\leq P(|A_2(t,s)| > \eta, |W(t+s) - W| < \varepsilon, W > x) \\
&\quad + P(|W(t+s) - W| \geq \varepsilon) + P(W \leq x) \\
&\leq P(|A_2(t,s)| > \eta, |W(t+s) - W| < \varepsilon, W > x) + \frac{2\delta}{3}. \qquad (8)
\end{aligned}$$

Now,

$$
\begin{aligned}
&P(|A_2(t,s)| > \eta, |W(t+s) - W| < \varepsilon, W > x) \\
(9) \qquad &\le P(|\frac{Z_{M(s)f}(t)}{\sqrt{Z(t+s)}}| > \eta, Z(t+s) > (x-\varepsilon)e^{\alpha(t+s)}) \\
&\le P(|Z_{M(s)f}(t)| > \eta\sqrt{x-\varepsilon}\, e^{\frac{\alpha}{2}(t+s)}) \\
&\le \frac{e^{-\alpha(t+s)}}{\eta^2(x-\varepsilon)} E(Z^2_{M(s)f}(t)),
\end{aligned}
$$

by Markov' inequality. We can choose s_0'' by Proposition 3(b) such that $s \ge s_0''$ imply

$$
(10) \qquad e^{-\alpha s} \lim_{t\to\infty} E(Z^2_{M(s)f}(t))e^{-\alpha t} \le \frac{\delta}{3}(x-\varepsilon)\eta^2.
$$

Let $s_0 = \max(s_0', s_0'')$, then from inequality (8), (9), and (10) we have for all $s \ge s_0$, $\lim_{t\to\infty} P(A_2(t,s) > \eta) < \delta$. □

LEMMA 2. *Fix $s_0 > 0$ and let $\mathcal{F}_t$ be the σ-algebra containing all the informations up to time t. Then $\lim_{t\to\infty} Var(A_1(t,s_0)|\mathcal{F}_t) = \sigma_f^2(s_0)$* a.s., *where*

$$
\begin{aligned}
\sigma_f^2(s_0) &= e^{-\alpha s_0}\int_0^\infty \sum_{i=1}^5 V_i(a,s_0)A(da), \\
V_1(a,s_0) &= f^2(a+s_0)G^a(s_0)(1-G^a(s_0)), \\
V_2(a,s_0) &= m(D_f * G^a)(s_0), \\
V_3(a,s_0) &= (m_2 - m)(m_f^2 * G^a)(s_0), \\
V_4(a,s_0) &= -m^2(m_f * G^a)^2(s_0), \\
V_5(a,s_0) &= -2mf(a+s_0)(1-G^a(s_0))(m_f * G^a)(s_0), \\
A(a) &= \frac{\int_0^a e^{-\alpha u}(1-G(u))\,du}{\int_0^\infty e^{-\alpha u}(1-G(u))\,du} \quad \textit{the stable age distribution.}
\end{aligned}
$$

Proof. Write $Y_t^{a_j}(s_0) = [Z_f^{a_j}(s_0) - m_f^{a_j}(s_0)]e^{-\frac{\alpha s_0}{2}}$, then

$$
A_1(t,s_0) = \frac{1}{\sqrt{Z(t)}}\sum_{j=1}^{Z(t)} Y_t^{a_j}(s_0).
$$

Since $\{Y_t^{a_j}(s_0); j = 1, \cdots, Z(t)\}$ are mutually independent conditioned on $\mathcal{F}_t$ and also independent of $Z(t)$,

$$Var(A_1(t, s_0)|\mathcal{F}_t) = \frac{1}{Z(t)} \sum_{j=1}^{Z(t)} Var(Y_t^{a_j}(s_0)|\mathcal{F}_t).$$

Recall equations (3) and (4)

$$\begin{aligned} Z_f^{a_j}(s_0) &= I(\lambda^{a_j} > s_0) f(a_j + s_0) + \sum_{i=1}^{\xi} Z_{f,i}(s_0 - \lambda^{a_j}), \\ m_f^{a_j}(s_0) &= (1 - G^{a_j}(s_0)) f(a_j + s_0) + m \int_0^{s_0} m_f(s_0 - u) G^{a_j}(du). \end{aligned}$$

So $E([Y_t^{a_j}(s_0)]^2|\mathcal{F}_t) = e^{-\alpha s_0} \sum_{i=1}^5 V_i(a_j, s_0)$ and

$$\begin{aligned} Var(A_1(t, s_0)|\mathcal{F}_t) &= e^{-\alpha s_0} \sum_{i=1}^{5} \frac{1}{Z(t)} \sum_{j=1}^{Z(t)} V_i(a_j, s_0) \\ &= e^{-\alpha s_0} \sum_{i=1}^{5} \int_0^{\infty} V_i(a, s_0) A(da, t), \end{aligned}$$

where $A(a, t) \equiv \frac{Z(a,t)}{Z(t)} \equiv \frac{1}{Z(t)} \sum_{j=1}^{Z(t)} I_{[0,a]}(a_j)$ is the empirical age distribution at time t. Note that since $\sup_{0 \le s \le s_0} m_f(s)$ and $\sup_{0 \le s \le s_0} D_f(s)$ are finite $V_2(\cdot, s_0)$, $V_3(\cdot, s_0)$ and $V_4(\cdot, s_0)$ are bounded. The boundedness of $V_1(\cdot, s_0)$ and $V_5(\cdot, s_0)$ is direct from (F 5). Also $V_i(\cdot, s_0), i = 1, \cdots, 5$ are continuous a.e. on the support of G. So the proof is complete by the convergence of the empirical age distribution to the stable age distribution (see Athreya and Kaplan (1976)), i.e.,

$$\begin{aligned} Var(A_1(t, s_0)|\mathcal{F}_t) &= e^{-\alpha s_0} \sum_{i=1}^{5} \int_0^{\infty} V_i(a, s_0) A(da, t) \\ &\xrightarrow{\text{a.s.}} e^{-\alpha s_0} \sum_{i=1}^{5} \int_0^{\infty} V_i(a, s_0) A(da) \quad \text{as} \quad t \to \infty. \end{aligned}$$ □

Lemma 3. *For a fixed $s_0 > 0$ and $\eta > 0$*

$$\sup_{0 \le a \le t} E(|Y_t^a(s_0)|^2; |Y_t^a(s_0)| > \eta e^{\frac{\alpha}{2} t}) \to 0 \quad \text{as} \quad t \to \infty.$$

Proof. First we see from equation (4) that

$$\sup_{0\le a\le t}|m_f^a(s_0)| \le \sup_{0\le a\le t}|(1-G^a(s_0))f(a+s_0)| + m\sup_{0\le s\le s_0}|m_f(s)|,$$

which is finite by (F 5) and that

$$\begin{aligned} &E([Y_t^a(s_0)]^2; |Y_t^a(s_0)| > \eta e^{\frac{\alpha}{2}t}) \\ &\quad\le 2E(e^{-\alpha s_0}|Z_f^a(s_0)|^2; |Z_f^a(s_0)-m_f^a(s_0)| > \eta e^{\frac{\alpha}{2}(t+s_0)}) \\ &\qquad +2E(e^{-\alpha s_0}|m_f^a(s_0)|^2; |Z_f^a(s_0)-m_f^a(s_0)| > \eta e^{\frac{\alpha}{2}(t+s_0)}). \end{aligned}$$

So it is enough to show that

$$(i)\qquad \sup_{0\le a\le t} P(|Z_f^a(s_0)| > \eta e^{\frac{\alpha}{2}(t+s_0)}) \to 0 \quad \text{as} \quad t\to\infty,$$

$$(ii)\qquad \sup_{0\le a\le t} E(|Z_f^a(s_0)|^2; |Z_f^a(s_0)| > \eta e^{\frac{\alpha}{2}(t+s_0)}) \to 0 \quad \text{as} \quad t\to\infty.$$

Note that

$$(11)\qquad \left|\sum_{i=1}^{\xi} Z_{f,i}(s_0-\lambda^a)\right| \overset{s}{\le} \sum_{i=1}^{\xi} Z_i(s_0-\lambda^a)\overline{f}(s_0) = X\overline{f}(s_0), \quad \text{say,}$$

where $\overline{f}(s_0) = \sup_{0\le s\le s_0}|f(s_0)|$ and $\overset{s}{\le}$ denote the stochastic order;$X \overset{s}{\le} Y$ implies $P(X\ge x) \le P(Y\ge x)$ for all x. Combining the equation (3) with the inequality (11) we get

$$(12)\qquad |Z_f^a(s_0)| \overset{s}{\le} |f(a+s_0)|I(\lambda^a > s_0) + X\overline{f}(s_0).$$

So

$$(13)\qquad \sup_{0\le a\le t}|Z_f^a(s_0)| \overset{s}{\le} \overline{f}(t+s_0) + X\overline{f}(s_0).$$

Now we observe from (12) that

$$\begin{aligned} \sup_{0\le a\le t} P(|Z_f^a(s_0)| > \eta e^{\frac{\alpha}{2}(t+s_0)}) &\le P(\sup_{0\le a\le t}|Z_f^a(s_0)| > \eta e^{\frac{\alpha}{2}(t+s_0)}) \\ &\le P(\overline{f}(t+s_0) > (1/2)\eta e^{\frac{\alpha}{2}(t+s_0)}) + P(X\overline{f}(s_0) > (1/2)\eta e^{\frac{\alpha}{2}(t+s_0)}). \end{aligned}$$

Since (F 3) implies that $e^{-\frac{\alpha t}{2}}\overline{f}(t) \to 0$ as $t\to\infty$ the first term is zero for large t, and since X is finite a.s., the second term goes to zero as $t\to\infty$ so (i) is proved.

Turning to (ii) we note first from inequality (12) that

$$(14)\qquad |Z_f^a(s_0)|^2 \overset{s}{\le} f(a+s_0)^2 I(\lambda^a > s_0) + X^2\overline{f}^2(s_0).$$

Note that $Z_f^a(s_0) = f(a+s_0)$ on $\{\lambda^a > s_0\}$ and so,

$$\begin{aligned}
&\sup_{0\le a\le t} E[f^2(a+s_0)I(\lambda^a > s_0)I(|Z_f^a(s_0)| > \eta e^{\frac{\alpha}{2}(t+s_0)})] \\
&\quad = \sup_{0\le a\le t} f^2(a+s_0)E[I(\lambda^a > s_0)I(|f(a+s_0)| > \eta e^{\frac{\alpha}{2}(t+s_0)})] \\
&\quad = \sup_{0\le a\le t} f^2(a+s_0)(1-G^a(s_0))I(|f(a+s_0)| > \eta e^{\frac{\alpha}{2}(t+s_0)}).
\end{aligned}$$

Since $I(|f(t+s_0)| > \eta e^{\frac{\alpha}{2}(t+s_0)}) = 0$ for t large enough

$$\lim_{t\to\infty}\sup_{0\le a\le t} E[f^2(a+s_0)I(\lambda^a > s_0)I(|Z_f^a(s_0)| > \eta e^{\frac{\alpha}{2}})] = 0. \tag{15}$$

On the other hand, by (13) we have

$$\begin{aligned}
&\sup_{0\le a\le t} E[X^2\overline{f}^2(s_0)I(|Z_f^a(s_0)| > \eta e^{\frac{\alpha}{2}(t+s_0)})] \\
&\le \overline{f}^2(s_0)E[X^2\{I(\overline{f}(t+s_0) > \frac{1}{2}\eta e^{\frac{\alpha}{2}(t+s_0)}) + I(X\overline{f}(s_0) > \frac{1}{2}\eta e^{\frac{\alpha}{2}(t+s_0)})\}].
\end{aligned}$$

Since $e^{-\frac{\alpha}{2}(t+s_0)}\overline{f}(t+s_0) \to 0$ as $t \to \infty$ and $E(X^2) < \infty$ we conclude that

$$\lim_{t\to\infty}\sup_{0\le a\le t} E[X^2\overline{f}(s_0)I(|Z_f^a(s_0)| > \eta e^{\frac{\alpha}{2}(t+s_0)})] = 0 \tag{16}$$

by the Lebesgue dominated convergence theorem. Now, (14) and (15), along with (16) together prove (*ii*). □

The following lemma concerns the conditional Lindeberg-Feller condition.

LEMMA 4. *Fix $s_0 > 0$, $\eta > 0$, then*

$$\sum_{j=1}^{Z(t)} E\left(\frac{\{Y_t^{a_j}(s_0)\}^2}{Z(t)}; \left|\frac{Y_t^{a_j}(s_0)}{\sqrt{Z(t)}}\right| > \eta|\mathcal{F}_t\right) \xrightarrow{\text{pr}} 0 \quad \text{as} \quad t\to\infty.$$

Proof. Write $S_t(s_0,\eta) \equiv \dfrac{1}{Z(t)}\displaystyle\sum_{j=1}^{Z(t)} E(\{Y_t^{a_j}(s_0)\}^2; |Y_t^{a_j}(s_0)| > \eta\sqrt{Z(t)}\ |\mathcal{F}_t)$.

Given $\delta_1 > 0$, $\delta_2 > 0$, there exist $t_0 > 0$ and a set A such that

(17) (*i*) $P(A) > 1-\delta_1$,

(18) (*ii*) $t > t_0$ and $\omega \in A$ imply together $Z(t,\omega) \ge \delta_2 e^{\alpha t}$.

So for any $\varepsilon > 0$, from (17) we get

$$\begin{aligned} P(S_t(s_0,\eta) > \varepsilon) &= P(S_t(s_0,\eta) > \varepsilon; A) + P(S_t(s_0,\eta) > \varepsilon; A^c) \\ &\le P(S_t(s_0,\eta) > \varepsilon; A) + \delta_1 \end{aligned}$$

and for $t \ge t_0$, we have the following from (18)

$$\begin{aligned} &P(S_t(s_0,\eta) > \varepsilon; A) \\ &\le P(\frac{1}{Z(t)}\sum_{j=1}^{Z(t)} E(\{Y_t^{a_j}(s_0)\}^2; |Y_t^{a_j}(s_0)| > \eta\sqrt{\delta_2}e^{\frac{\alpha}{2}t}|\mathcal{F}_t) > \varepsilon) \\ &\le P(\{\sup_{0\le a\le t} E(\{Y_t^a(s_0)\}^2; |Y_t^a(s_0)| > \eta\sqrt{\delta_2}e^{\frac{\alpha}{2}t})\} > \varepsilon) \end{aligned}$$

which is zero for t large enough by Lemma 3. So we conclude that $\lim_{t\to\infty} P(S_t(s_0,\eta) > \varepsilon) < \delta_1$. Letting $\delta_1 \downarrow 0$ we get the result. □

LEMMA 5. *For fixed* s_0, $A_1(t,s_0) \xrightarrow{\text{d}} N(0,\sigma_f^2(s_0))$ *as* $t \to \infty$.

Proof. Conditioned on $\mathcal{F}_t$, $\dfrac{Y_t^{a_j}(s_0)}{\sqrt{Z(t)}}$'s are independent. Hence

$$\begin{aligned} E(\exp(i\theta A_1(t,s_0))|\mathcal{F}_t) &= \prod_{j=1}^{Z(t)} E\left(\exp\left(i\theta\frac{Y_t^{a_j}(s_0)}{\sqrt{Z(t)}}\right)|\mathcal{F}_t\right) \\ &\equiv \prod_{j=1}^{Z(t)} \phi_t^{a_j}(s_0,\theta), \quad \text{say.} \end{aligned}$$

As in the proof of the usual Lindeberg-Feller central limit theorem (see Durrett p. 98) it is possible to show that

$$\prod_{j=1}^{Z(t)} \phi_t^{a_j}(s_0,\theta) \xrightarrow{\text{pr}} \exp\left(-\frac{\theta^2}{2}\sigma_f^2(s_0)\right) \quad \text{as} \quad t\to\infty \tag{19}$$

with the aid of Lemma 2 and 4. Therefore, the dominated convergence theorem completes the proof. That is, since (19) holds and since

$\left|\prod_{j=1}^{Z(t)} \phi_t^{a_j}(s_0,\theta) - \exp(-\frac{\theta^2}{2}\sigma_f^2(s_0))\right| \le 2$, we have

$$\begin{aligned} E(\exp(i\theta A_1(t,s_0)) &= E(E(\exp(i\theta A_1(t,s_0)))|\mathcal{F}_t)) \\ &= E(E(\exp(i\theta A_1(t,s_0))) - \exp\left(-\frac{\theta^2}{2}\sigma_f^2(s_0))|\mathcal{F}_t)\right) \\ &\quad + \exp(-\frac{\theta^2}{2}\sigma_f^2(s_0)) \\ &\longrightarrow \exp(-\frac{\theta^2}{2}\sigma_f^2(s_0)) \quad \text{as} \quad t \to \infty. \end{aligned}$$

So $A_1(t,s_0)$ has the desired limit distribution. □

LEMMA 6. $\sigma_f^2(s) \to \sigma_f^2 = n_1^{-1} D_f^\alpha$ as $s \to \infty$.

Proof. Let $c_1 = (\int_0^\infty e^{-\alpha u}(1-G(u))\,du)^{-1}$, so that $A(a) = c_1 \int_0^a e^{-\alpha u}(1-G(u))\,du$. Then

$$\begin{aligned} e^{-\alpha s}\int_0^\infty V_1(a,s)A(da) &= c_1 e^{-\alpha s}\int_0^\infty f^2(a+s)G^a(s)e^{-\alpha a}(1-G(a+s))da \\ &\le c_1\int_0^\infty e^{-\alpha(a+s)}f^2(a+s)(1-G(a+s))da \\ &= c_1\int_s^\infty e^{-\alpha a}f^2(a)(1-G(a))da. \end{aligned}$$

The last term goes to 0 as $s \to \infty$ since $e^{-\alpha a}f^2(a)(1-G(a))$ is integrable ((F 3)′).

$$\begin{aligned} & e^{-\alpha s}\int_0^\infty V_2(a,s)A(da) \\ &= c_1 m e^{-\alpha s}\int_0^\infty\int_0^s D_f(s-u)G^a(du)e^{-\alpha a}(1-G(a))da \\ &= c_1 m\int_0^\infty\int_a^{a+s} e^{-(a+s-u)}D_f(a+s-u)e^{-\alpha u}G(du)da \\ &\to c_1 m\int_0^\infty\int_a^\infty D_f^\alpha e^{-\alpha u}G(du)da \\ &\quad \text{by Lebesgue dominated convergence theorem} \\ &= c_1 m D_f^\alpha\int_0^\infty\int_0^u da e^{-\alpha u}G(du) \\ &= c_1 m D_f^\alpha\int_0^\infty u e^{-\alpha u}G(du) \;=\; n_1^{-1}D_f^\alpha. \end{aligned}$$

$$\begin{aligned}
&e^{-\alpha s}\int_0^\infty V_3(a,s)A(da)\\
&= c_1 e^{-\alpha s}\int_0^\infty (m_2-m)(m_f^2 * G^a)(s)e^{-\alpha a}(1-G(a))da\\
&\le c_1(m_2-m)\int_0^\infty e^{-\alpha(a+s)}(m_f^2 * G)(a+s)da\\
&= c_1(m_2-m)\int_s^\infty e^{-\alpha a}(m_f^2 * G)(a)da\\
&\to 0 \quad \text{as} \quad s\to\infty \quad \text{by} \quad (\text{F } 4).
\end{aligned}$$

Since $(m_f * G^a)^2(s) \le (m_f^2 * G^a)(s)$, we have from above that $e^{-\alpha s}\int_0^\infty V_4(a,s)A(da) \to 0$ as $s\to\infty$. Finally,

$$\begin{aligned}
&\left|e^{-\alpha s}\int_0^\infty V_5(a,s)A(da)\right|^2\\
&= \; 4m^2c_1^2\left|\int_0^\infty e^{-\alpha(a+s)}f(a+s)(1-G(a+s))(m_f * G^a)(s)da\right|^2\\
&\le \; 4m^2c_1^2\int_s^\infty e^{-\alpha a}f^2(a)(1-G(a))da\int_s^\infty e^{-\alpha a}(m_f^2 * G)(a)da\\
&\to \; 0 \quad \text{as} \quad s\to\infty
\end{aligned}$$

by the fact that $e^{-\alpha a}f^2(a)(1-G(a))$ is integrable and by the assumption (F 4). Hence

$$\sigma_f^2(s) = \sum_{i=1}^5 e^{-\alpha s}\int_0^\infty V_i(a,s)A(da) \to n_1^{-1}D_f^\alpha \quad \text{as} \quad s\to\infty. \qquad \square$$

Now we complete the proof of Theorem 1 by assembling all the lemmas together. Let $\varepsilon > 0$ be arbitrary and y fixed. Choose $\eta_\varepsilon > 0$ such that

$$\left|\Phi\left(\frac{y+\eta_\varepsilon}{\sigma_f}\right) - \Phi\left(\frac{y-\eta_\varepsilon}{\sigma_f}\right)\right| < \frac{\varepsilon}{3}. \tag{20}$$

Since $\lim_{s\to\infty}\sigma_f^2(s) = \sigma_f^2$, there exists $s_1(\varepsilon)$ such that $s \ge s_1(\varepsilon)$ implies

$$\left|\Phi\left(\frac{y+r\eta_\varepsilon}{\sigma_f}\right) - \Phi\left(\frac{y+r\eta_\varepsilon}{\sigma_f(s)}\right)\right| < \frac{\varepsilon}{3} \quad \text{for} \quad r=\pm 1. \tag{21}$$

Let $\delta = \varepsilon/3$ and let $s^* = \max\{s_0(\eta_\varepsilon, \delta), s_1(\varepsilon)\}$ where $s_0(\eta_\varepsilon, \delta)$ is defined in Lemma 1. Then

$$\begin{aligned}
&\limsup_{t\to\infty} P(A_1(t,s^*) + A_2(t,s^*) \le y) \\
(22) \quad &\le \limsup_{t\to\infty} P(A_1(t,s^*) \le y + \eta_\varepsilon) + \limsup_{t\to\infty} P(|A_2(t,s^*)| \ge \eta_\varepsilon) \\
&\le \Phi\left(\frac{y+\eta_\varepsilon}{\sigma_f(s^*)}\right) + \frac{\varepsilon}{3}
\end{aligned}$$

and

$$\begin{aligned}
&\liminf_{t\to\infty} P(A_1(t,s^*) + A_2(t,s^*) \le y) \\
(23) \quad &\ge \liminf_{t\to\infty} P(A_1(t;s^*) \le y - \eta_\varepsilon) - \liminf_{t\to\infty} P(|A_2(t,s^*)| \ge \eta_\varepsilon) \\
&\ge \Phi\left(\frac{y-\eta_\varepsilon}{\sigma_f(s^*)}\right) - \frac{\varepsilon}{3}.
\end{aligned}$$

(20), (21), (22) and (23) and the fact $\lim_{t\to\infty} \dfrac{Z(t)e^{-\alpha t}}{Z(t+s^*)e^{-\alpha(t+s^*)}} = 1$ a.s. imply together that

$$\Phi\left(\frac{y}{\sigma_f}\right) - \varepsilon \le \lim_{t\to\infty} P\left(\frac{Z_f(t+s^*)}{\sqrt{Z(t+s^*)}} \le y\right) \le \Phi\left(\frac{y}{\sigma_f}\right) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the proof is completed. □

Proof of Corollary 1. Clearly boundedness of f implies (F 3) and (F 5). It remains to check that the condition (F 4) is satisfied. In exponential case, the Malthusian parameter α can be found easily, i.e., $\alpha = b(m-1)$. So we have $\mu_\alpha(dt) = bme^{-bmt}\,dt$, $U_\alpha(dt) = bm\,dt$. Hence

$$\begin{aligned}
e^{-\alpha t} m_f(t) &= \int_0^t e^{-\alpha(t-u)} f(t-u)(1-G(t-u)) bm\,du \\
&= bm \int_0^t e^{-\alpha u} f(u) e^{-bu}\,du \\
&= -bm \int_t^\infty e^{-bmu} f(u)\,du
\end{aligned}$$

where the third equation comes from the assumption (F 2). Since f is bounded $|m_f(t)| \le ||f||e^{\alpha t}e^{-bmt} = ||f||e^{-bt}$. So $e^{-\alpha t}(m_f^2 * G)(t) =$

$O(e^{-bmt})$ and it is d.R.i. Furthermore, $\beta = m\int_0^\infty te^{-\alpha t}G(dt) = \dfrac{1}{bm}$, $n_1 = 1$,

$$\begin{aligned}
\sigma_f^2 &= \lim_{t\to\infty} e^{-\alpha t}D_f(t) \\
&= \frac{1}{\beta}\int_0^\infty (e^{-\alpha t}f^2(t)e^{-bt} + (m_2 - m)e^{-\alpha t}(m_f^2 * G)(t))\,dt \\
&= mb\left(\int_0^\infty e^{-bmt}f^2(t)\,dt\right. \\
&\qquad \left.+(m_2 - m)\int_0^\infty e^{-\alpha t}m_f^2(t)\,dt\int_0^\infty e^{-\alpha t}be^{-bt}\,dt\right) \\
&= mb\left(\int_0^\infty e^{-mbt}f^2(t)\,dt\right. \\
&\qquad \left.+mb^2(m_2 - m)\int_0^\infty e^{\alpha t}\left(\int_0^t e^{-mbu}f(u)\,du\right)^2 dt\right). \quad \square
\end{aligned}$$

Before proving Corollary 2 we provide some interesting results on convergence rates in renewal theorems with gamma interarrival distribution.

LEMMA 7. *Let G be a gamma distribution with density function $g(x) = \Gamma(k)^{-1}b^kx^{k-1}e^{-bx}$ and let U be the renewal function with interarrival distribution G. If r is d.R.i., differentiable a.e., and if $||\frac{r'}{g}||_\infty$ is finite, then*

$$\left|(r * U)(t) - \frac{1}{\mu}\int_0^\infty r(u)\,du\right| = O(e^{-c_1t})$$

where $||\cdot||_\infty$ is the supremum norm and $\mu = \int_0^\infty uG(du) = k/b$, and $c_1 = b(1-\cos\frac{2\pi}{k})$.

The following is an immediate result with $r = I_{[0,h]}$.

COROLLARY 3. *With the notations in* Lemma 7 *for any $h > 0$,*

$$U(t) - U(t-h) = \frac{h}{\mu} + O(e^{-c_1t}).$$

Proof of Lemma 7. Let $\{Y_n\}_{n=1}^\infty$ be i.i.d. with common distribution G. Define a renewal process $S_0 = 0$, $S_n = \sum_{j=1}^n Y_j$, $n \geq 1$. Let B_t denote the forward recurrence time, i.e., the waiting time until the next renewal after t. For an almost surely continuous function $h : R \to R$, define $H(t) = E(h(B_t))$. Then we get the following renewal equation

$$H(t) = (1 - G(t))E(h(Y_1)|Y_1 > t) + (H * G)(t). \tag{24}$$

Now, let r be a function which is bounded on bounded sets and consider the following renewal equation

$$(r * U)(t) = r(t) + [(r * U) * G](t).$$

If $r(t) = (1 - G(t))E(h(Y_1)|Y_1 > t)$, we conclude that $H(t) = (r * U)(t)$ by the uniqueness of bounded solution to the renewal equation (24). Furthermore, if r is d.R.i.,

$$\begin{aligned} \left|(r * U)(t) - \frac{1}{\mu}\int_0^\infty r(u)\,du\right| &= |E(h(B_t)) - E(h(B_\infty))| \\ &\le ||h||_\infty ||P(B_t \in \cdot) - P(B_\infty \in \cdot)||, \end{aligned} \tag{25}$$

where $||\cdot||$ denote the total variation norm. Now, for each $n \ge 1$, we may write

$$Y_n = Y_{n,1} + \cdots + Y_{n,k},$$

where $\{Y_{n,j}; j = 1, \cdots, k\}_{n=1}^\infty$ are i.i.d. with $P(Y_{1,1} > x) = e^{-bx}$. Define a Markov process $X(t)$ on state space $S = \{0, 1, \cdots, k-1\}$ by

$$X(t) = j \quad \text{if} \quad S_{m-1} + \sum_{i=1}^{j} Y_{m,i} \le t < S_{m-1} + \sum_{i=1}^{j+1} Y_{m,i}, \quad \text{for some } m \ge 1,$$

where $\sum_{i=1}^0 Y_{m,i}$ is defined by 0. Clearly, the process is irreducible and so positive recurrent and for each $t \ge 0$,

$$\mathbf{P}^t = e^{\Lambda t}$$

where $\{\mathbf{P}^t\}_{t\ge 0}$ is the transition semigroup of $X(t)$; $P^t(i,j) = P(X(t) = j|X(0) = i)$, and Λ is its intensity matrix, i.e.,

$$\Lambda = \begin{pmatrix} -b & b & 0 & \cdots & 0 \\ 0 & -b & b & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & 0 & 0 & \cdots & -b \end{pmatrix}.$$

The eigenvalues of Λ are $d_j = b(\exp\{\frac{2\pi j}{k} i\} - 1)$, $j = 1, \cdots, k$. Hence, we have

$$||P^t(\cdot) - \pi(\cdot)|| = O(e^{-c_1 t}) \quad \text{as} \quad t \to \infty,$$

where $\pi = (\pi_1, \cdots, \pi_k)$ is the stationary probability measure of $X(t)$ and $c_1 = -Re\, d_1$. Since $P(B_t \in \cdot) = \sum_{j=0}^{k-1} P(X(t) = j)P(T_{k-j} \in \cdot)$ and $P(B_\infty \in \cdot) = \sum_{j=0}^{k-1} \pi_j P(T_{k-j} \in \cdot)$, where T_l is a gamma random variable with parameters (b, l), we conclude that

$$||P(B_t \in \cdot) - P(B_\infty \in \cdot)|| = O(e^{-c_1 t}). \tag{26}$$

Now consider the equation

$$r(t) = (1 - G(t))E(h(Y_1)|Y_1 > t) = \int_t^\infty h(u)G(du).$$

Since $r'(t) = h(t)g(t)$ a.e. and $||\frac{r'}{g}||_\infty$ is finite, so is $||h||_\infty$. Combining this fact with (25) and (26) the proof is completed. □

Proof of Corollary 2. First it is easy to see that $\alpha = b(m^{\frac{1}{k}} - 1)$ and that $\mu_\alpha(t)$ is a gamma distribution with density function $g_\alpha(x) = \Gamma(k)^{-1} m b^k x^{k-1} e^{-bm^{\frac{1}{k}}x}$. Since f is bounded (F 3) and (F 5) are trivially satisfied. So it is enough to show that $e^{-\alpha t}(m_f^2 * G)(t)$ is d.R.i. Let $f_\alpha(t) = e^{-\alpha t} f(t)(1 - G(t))$. Since $\frac{1 - G(t)}{g_\alpha(t)} = O(e^{\alpha t})$, and $g(t)/g_\alpha(t) = e^{\alpha t}/m$, $||\frac{f'_\alpha}{g_\alpha}||_\infty < \infty$. So $m_f(t) = e^{\alpha t}(f_\alpha * U_\alpha)(t) = O(e^{(\alpha - c_1)t})$ by Proposition 1(a) and Lemma 7, where $c_1 = m^{\frac{1}{k}} b(1 - \cos\frac{2\pi}{k})$. Hence $e^{-\alpha t}(m_f^2 * G)(t) = O(e^{(\alpha - 2c_1)t})$, which is d.R.i. if $\alpha - 2c_1 < 0$, or equivalently if $\cos(2\pi/k) < (1 + m^{-1/k})/2$. □

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