코달 및 순열 그래프의 레이블링 번호 상한에 대한 연구

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요 약

그래프 $G=(V, E)$에, G의 $L_d(2, 1)$ 레이블링은 하나의 함수 $f: V(G) \rightarrow [0, \infty)$로서, $dist_G(x, y)$가 x, y 사이의 최소 거리일 때, 두 개의 버텍스 $x, y(\in V)$가 인접하면 $|f(x) - f(y)| \geq 2d$이며, x, y의 거리가 2이면 $|f(x) - f(y)| \geq d$이다. $L_d(2, 1)$ 레이블링 번호 $\lambda(G, d)$는 최소 번호 m으로서 G는 최대 레이블이 m인 $L_d(2, 1)$ 레이블링 f를 갖는다. 이 문제는 Griggs와 Yeh 그리고 Sakai에 의해 여러 가지 종류의 그래프를 대상으로 연구되어졌다. 본 논문은 코달 그래프 G의 $\lambda(G)$ 및 순열그래프 G'의 $\lambda(G')$에 대하여 논하고자 한다.

The Study on the Upper-bound of Labeling Number for Chordal and Permutation Graphs

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ABSTRACT

Given a graph $G=(V, E)$, $L_d(2, 1)$-labeling of G is a function $f: V(G) \rightarrow [0, \infty)$ such that, if $v1, v2 \in V$ are adjacent, $|f(v_1)-f(v_2)| \geq 2d$, and, if the distance between v_1 and v_2 is two, $|f(v_1)-f(v_2)| \geq d$, where $d_G(v_1, v_2)$ is the shortest distance between v_1 and v_2 in G. The $L(2, 1)$-labeling number $\lambda(G)$ is the smallest number m such that G has an $L(2, 1)$-labeling f with maximum m of $f(v)$ for $v \in V$. This problem has been studied by Griggs, Yeh and Sakai for the various classes of graphs. In this paper, we discuss the upper-bound of $\lambda(G)$ for a chordal graph G and that of $\lambda(G')$ for a permutation graph G'.

1. Introduction

The *channel assignment problem* is the task of assigning channels (non-negative integers) to radio transmitters such that interfering transmitters get channels whose separation is not in a set of disallowed separations. Hale[3] first formulated this problem into a graph coloring problem, i.e., the notion of the T-coloring of a so-called interference graph, where transmitters are represented by the vertices and interference by the edges in the graph, colors assigned to the vertices are channels, and T is the set of disallowed separations. Subsequently, Roberts[7] proposed a variation of the channel assignment problem, where radio channels are efficiently assigned to transmitters at several locations such that close

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transmitters receive different channels, and very close transmitters receive channels at least two apart. Griggs and Yeh[2] and Yeh[11] considered a more general problem such that, given a real number $d>0$, $L_d(2, 1)$-labeling of G is an assignment f of non-negative real numbers to the vertices of G; a function $f: V(G) \rightarrow [0, \infty)$ such that, if x, $y \in V$ are adjacent, $|f(x)-f(y)| \geq 2d$, and, if the distance between x and y is two, $|f(x)-f(y)| \geq d$.

Griggs and Yeh[2] concentrated on the $L_d(2, 1)$-labeling number of G, denoted by $\lambda(G, d)$, which is the smallest number m such that G has an $L_d(2, 1)$-labeling with no label greater than m with $max\{f(v)|v \in V\}=m$. They further simplified this problem as $L_1(2, 1)$-labeling, and showed that for $L_1(2, 1)$-labeling it suffices to consider labelings with non-negative integers. Therefore, we consider only $L_1(2, 1)$-labeling with non-negative integers as the labels in this paper. For simplicity we denote $L_1(2, 1)$ by $L(2, 1)$ and $\lambda(G, d)$ by $\lambda(G)$ from now on.

Using the property that every chordal graph G contains a simplicial vertex, Sakai[9] showed that $\lambda(G) \leq (\Delta(G)+3)^2/4$ if G is a chordal graph, where $\Delta(G)$ is the maximum degree of G. Sakai also questioned whether this bound is sharp. In this paper, by adapting our *high-only scheme* (defined later in this paper) we show that, if G is chordal, $\lambda(G) \leq (1/6)\Delta(G)+(19/6)\Delta(G)+1/24$, which is better than Sakai's if $\Delta(G) \geq 19$. In particular, we show that, if both G and G^2 are chordal graphs, the upper-bound is dramatically reduced to $3\Delta(G)-2$ by using the coloring of G^2. We further extend our scheme to the class of permutation graphs and prove that $\lambda(G) \leq 4\Delta(G)-2$ if G is a permutation graph.

2. Preliminaries

Two vertices of a graph G are *adjacent* if they are contained in an edge. A *path* in G is a sequence of distinct vertices $v_1, v_2, ..., v_k$ such that v_i and v_{i+1} are adjacent for each i, $1 \leq i \leq k-1$. A *cycle* is a path $v_1, v_2, ..., v_k$, $k \geq 3$, such that v_1 and v_k are adjacent. A *clique* of G is a set of pairwise adjacent vertices. An undirected graph G is called a *chordal graph* if every cycle of length strictly greater than three contains a *chord*, which is an edge joining two nonconsecutive vertices of the cycle[1]. A vertex v of $G=(V, E)$ is called *simplicial* if v and its adjacent vertices induce a complete subgraph of G, i.e., a clique (not necessarily maximal) in G. For any graph $G=(V, E)$, $G^2=(V, E')$ is the graph such that $E'=\{(x, y)|x,y \in V$ and $dist_G(x, y) \leq 2$ in $G\}$, where we denote by $dist_G(x, y)$ the distance of x and y in G.

Let σ be some total ordering of the vertices of a graph $G=(V, E)$. We will implicitly identify the vertices with this ordering. For two vertices x and y we will say that $x<y$ if and only if $\sigma(x)<\sigma(y)$ under σ. If two vertices x and y are adjacent and $\sigma(x)<\sigma(y)$, then we say that y is a *higher-neighbour* of x while x is a *lower-neighbour* of y. $H(v)$ denotes the highest-neighbour of v in an ordered graph G. Let $G=(V, E)$ be an ordered graph with $V=\{v_1, v_2, ..., v_n\}$. Then, G_i denotes the induced subgraph of G by the vertices $v_i, v_{i+1}, ..., v_n$.

$N_G[v]=N_G(v) \cup \{v\}$ denotes the *closed-neighbourhood* of v, where $N_G(v)=\{u|(u, v) \in E\}$ is called *open-neighbour* of v. We denote by $\chi(G)$ and $\omega(G)$ the chromatic number and the size of the maximum clique of G, respectively. Let $G=(V, E)$ be a graph, then $G(S)$ denotes the vertices induced subgraph of G by the set $S \subseteq V$. We denote by $deg_G(v)$ and $diam(G)$ the degree of a vertex v and the diameter of G, respectively. C_n denotes chordless cycle of length $n \geq 4$.

The so-called *high-only scheme* for $L(2, 1)$-labeling of a graph $G=(V, E)$ is the following. First, we order the vertices of G by some total ordering σ. Suppose that the vertex set V has been ordered as $V=\{v_1, v_2, ..., v_n\}$. For $L(2, 1)$-labeling of G we relabel the vertices in reverse order, i.e., $v_n, v_{n-1}, ..., v_1$. Let $1 \leq i$

$\le j \le k \le n$. Then, when v_j is the next vertex to be relabeled, we need to consider the labels of v_k only for some k since the vertex v_i for some i has not been relabeled yet.

3. *L*(2, 1)-labelings of Chordal Graphs

A chordal graph $G=(V, E)$ admits a special ordering on its vertex set V, which is called *perfect elimination ordering* (*peo*).

Definition 3.1 A *peo* of a graph $G=(V, E)$ is an ordering $\sigma=[v_1, v_2, ..., v_n]$ of V with the property that, for each i, j, and k, if $\sigma(i)<\sigma(j)<\sigma(k)$ and (v_i, v_j), $(v_i, v_k) \in E$, then $(v_j, v_k) \in E$.

Rose[8] proved that a graph G is chordal if and only if it admits a *peo*. Note that an *end-high path* is a path in G whose end-vertices are higher than all the internal vertices relative to σ.

Lemma 3.2 (Klein [4]) *Given a chordal graph G and a peo σ, an end-high path has adjacent end-vertices.*

Tarjan and Yannakakis[10] developed an algorithm called *maximum cardinality search* (*MCS*) for computing *peo*'s of chordal graphs. The following Lemma shows that *MCS* ordering also satisfies very interesting property called P-property. Note that any *MCS* ordering is a *peo*, but not vice versa.

Lemma 3.3 (Tarjan and Yannakakis [10]) *MCS ordering $\sigma=[v_1, v_2, ..., v_n]$ of chordal graph G satisfies the following property: (P-property) If $\sigma(i)<\sigma(j)<\sigma(k)$, and v_k is adjacent to v_i and not to v_j, then there is a vertex v_m, $\sigma(m)>\sigma(j)$, adjacent to v_j but not to v_i.*

Lubiw[6] also showed that *doubly lexical ordering* of the neighbourhood matrix of a chordal graph G satisfies the P-property. A doubly lexical ordering of a matrix is an ordering of the rows and the columns of the matrix such that the rows and the columns are lexically increasing as vectors.

Lemma 3.4 *For a chordal graph $G=(V, E)$ with MCS-ordering σ, assume that there are three vertices v_i, v_j, $v_k \in V$ such that v_i is adjacent to both v_j and v_k in G_i, where $\sigma(v_i)<\sigma(v_j)<\sigma(v_k)$. If the vertex v_j have $m(\ge 1)$ adjacent vertices $X=\{x_1, x_2, ..., x_m\}$ other than v_i such that $\sigma(x_1)<\sigma(x_2)<\cdots<\sigma(x_m)$ and v_i is not adjacent to any vertex x_p, $1\le p \le m$, then v_k is adjacent to x_m in G_i.*

Proof. Since v_j and v_k are the higher-neighbours of v_i, v_j is adjacent to v_k in G by Lemma 3.2. If $\sigma(x_m) > \sigma(v_j)$, then clearly v_k is adjacent to x_m by Lemma 3.2. If $\sigma(x_m) < \sigma(v_j)$ but v_k is not adjacent to x_m, then, by the P-property from Lemma 3.3, there exists a vertex, say v_w in G_i, such that v_w is adjacent to x_m but not to v_i and $\sigma(v_w)>\sigma(x_m)$. Note that $v_k \neq v_w$ since v_k is adjacent to v_i. Then, by Lemma 3.2, v_w is adjacent to v_j; this is a contradiction to the assumption that x_m is the highest vertex in X. Therefore, x_m is adjacent to v_k in G_i.

Given a graph $G=(V, E)$ with some total order σ, let $TWO(v)=\{u\in V \mid \sigma(u)>\sigma(v) \text{ and } dist_G(u, v)=2\}$.

Lemma 3.5 *Let $G=(V, E)$ be a chordal graph such that $V=\{v_1, v_2, ..., v_n\}$ has ordered by some MCS σ. If any vertex $v\in V$ has θ higher-neighbours in G, $\theta \ge 1$, then $|TWO(v_i)| \le (\Delta(G) + 1/2)^2/6$.*

Proof. Let u_1, u_2, ..., u_θ be the higher-neighbours of v in G such that $\sigma(u_i)<\sigma(u_{i+1})$ for $1\le i\le \theta-1$. Let u_{i1}, u_{i2}, ..., u_{ip}, for some $p\ge 0$ be the vertices adjacent to u_i but not to v in G for each i. We also assume that $\sigma(u_{i1})<\sigma(u_{i2})<\cdots<\sigma(u_{ip})$ for each i. Then, $TWO(v) = \{u_{ij} \mid 1\le i\le \theta$ and $1\le j\le p$ such that $\sigma(u_{ij})>\sigma(v)$ and $dist_G(u_{ij},v)=2\}$ such that, by Lemma 3.4, u_{1p} is adjacent to u_2, u_3, ..., u_θ, u_{2p} is adjacent to u_3, u_4, ..., u_θ, and so on. Also, by Lemma 3.2, clearly the subgraph induced by $v\cup\{u_i \mid 1\le i\le \theta\}$ is a clique. Let $W=\{u_i \mid u_i$ is adjacent to at least one vertex in $TWO(v)$, where

$1 \le i \le \theta\}$ and $|W| = \theta$. Also, let $deg_G(u_i)$, $1 \le i \le \theta$, be the degree if $u_i \in W$; otherwise, 0. Then, $|TWO(v)|$ is as follows: $|TWO(v_i)| = \sum_{i=1}^{\theta} deg_G(u_i) - \{1+2+\cdots+(\theta-1)\} - \theta^2 \le \theta(\Delta(G)+1/2) - 3\theta^2/2$. The function $g(\theta) = \theta(\Delta(G)+1/2) - 3\theta^2/2$ has its maximum value at $\theta = \Delta(G)/3 + 1/6$ such that $g(\theta) = \theta(\Delta(G)/3+1/6) = (\Delta(G)+1/2)^2/6$.

Theorem 3.6 *Let $G=(V, E)$ be a chordal graph. Then, $\lambda(G) \le \Delta(G)^2/6 + 19\Delta(G)/6 + 1/24$.*

Proof. Assume that $V=\{v_1, v_2, \ldots, v_n\}$ is ordered by some *MCS* ordering σ. For $L(2, 1)$-labeling we use the high-only scheme. Suppose that v_i, $1 \le i \le n$, is the next vertex to be labeled. To prove the theorem it is sufficient to show that there exists at most $\Delta(G)^2/6 + 19\Delta(G)/6 + 1/24$ numbers used by the vertices $v_{i+1}, v_{i+2}, \ldots, v_n$; hence, it must be avoided by v_i. Note that, because of the end-high path property of chordal graphs, we need consider only the higher-neighbours of v_i. Suppose that v_i has θ higher-neighbours. Then, by Lemma 3.5, $|TWO(v_i)| \le (\Delta(G)+1/2)^2/6$. Hence, v_i must avoid $3\theta + (\Delta(G)+1/2)^2/6 \le 3\Delta(G) + (\Delta(G)+1/2)^2/6 = \Delta(G)^2/6 + 19\Delta(G)/6 + 1/24$ numbers. Since we can use 0 as a label when we label v_i, there exists at least one number for v_i.

$L(2, 1)$-labeling of G and coloring of G^2 are related as follows: For a given graph G we compute G^2 first and color the vertices of G^2 by the number 0, 2, 3, ..., $2\chi(G^2)-2$ rather than 1, 2, ..., $\chi(G)$. Then it is easy to see that the coloring of G^2 is the same as $L(2, 1)$-labeling of G. However, in order to apply the scheme used in the above, we must know the size of $\chi(G^2)$ first.

A cycle $Q_n=[v_1, v_2, \ldots, v_n]$ is called *chordal cycle* if the induced subgraph of the vertices of Q_n is chordal. A graph S_n of G is *sunflower* if it consists of a chordal cycle $Q_n=[v_1, v_2, \ldots, v_n]$ together with a set of n independent vertices $u_1, u_2, \ldots, u_n$ such that for each i, u_i is adjacent to only v_j, where $j=i-1 (mod\ n)$. A sunflower S_n of G is called a *suspended sunflower* in G if there exists a vertex $\omega \notin S_n$ such that ω is adjacent to at least one pair of vertices u_j and u_k, where $j \ne k-1 (\text{mod } n)$. A *3sun* is a sunflower such that Q_n is a clique and $n=3$. Let G be a *3SF* chordal graph if G is chordal and G does not contain *3sun* as its induced subgraph.

Let G be a chordal graph, then G^2 is not necessary chordal. However, Laskar and Shier characterized those graphs as follows:

Theorem 3.7 (Laskar and Shier[5]) *Let G be a chordal graph. Then, G^2 is chordal if and only if every sunflower S_n, $n \ge 4$, of G is suspended.*

Let $G=(V, E)$ be a graph and $C=[v_1, v_2, \ldots, v_n]$ be a simple cycle of G, where $n \ge 3$. An edge $(v_i, v_j) \in C$, $1 \le i, j \le n$, is called a dangling edge if there exists no $v_k \in C$, $1 \le k \le n$, such that v_k is adjacent to both v_i and v_j. We denote by C_n the chordless cycle of length $n \ge 4$. It is clear that chordal graph G contains no C_n, $n \ge 4$. The following two Lemmas are well known and easy to verify.

Lemma 3.8 *If G is a chordal graph, then there exist no dangling edges In G.*

Lemma 3.9 *If $G=(V, E)$ is chordal, then every vertex-induced subgraph of G is chordal.*

Lemma 3.10 *Let $G=(V, E)$ be a chordal graph. If $S=\{v_1, v_2, \ldots, v_k\}$ induces a maximal clique in G^2, then $G(S)$ is connected and $diam(G(S)) \le 2$.*

Proof. (i) For the contradiction, suppose that $G(S)$ is not connected. There exists at least one pair of vertices $x, y \in S$ such that $(x, y) \notin E$. However, since $x, y \in S$ there must exist a vertex, say $a \notin S$, such that $(x, a), (a, y) \in E$. Since $a \notin S$, there must exist at least one vertex, say $z \in S$, such that $dist_G(a, z) > 2$ while $dist_G(x, z) \le 2$ and $dist_G(y, z) \le 2$. Hence there exist two vertices b and c such that $(x, b), (b, z), (y, c), (c, z) \in E$ but $(a, b), (a, c), (x, z), (y, z) \notin E$. Suppose that $b \ne c$, then $[x, a, y, c, z, b, x]$ is a simple cycle of six vertices and edges (x, a) and (y, a) are dangling edges. If $b=c$, then $[x, a, y, b, x]$ is a chordless cycle of length four. In both case we have contradictions; hence, $G(S)$ is connected.

(ii) By part *(i)*, S is connected. For contrary, suppose that the diameter of $G(S)$ is greater than 2. There exists at least one pair of vertices $x,y \in S$ such that $dist_G(x, y)=3$. Let $p=[x, a, b, y]$ be such a shortest path between x and y in $G(S)$, where $a,b \in S$. Since $x, y \in S$ there exists at least one vertex, say $z \notin S$, such that z is adjacent ot both x and y in G. Note that z must be adjacent to both a and b in G for otherwise G contains C_4 or C_5. Now, since $z \notin S$, there must exist at least one vertex, say w, such that $dist_G(x, y) \leq 2$ and $dist_G(y, w) \leq 2$ while $dist_G(z, w) \geq 2$. This implies that there exist two vertices c and d such that $(c, x),(c, w),(d, y),(d, w) \in E$ but $(c, z),(d, z),(x, w),(y, w) \notin E$. Note that $c \neq d$ for otherwise G contains $C_4=[x, c, w, z, x]$. Then, it is easy to see that the edge (x, z) is a dangling edge of the simple cycle $[x, z, y, d, w, c, x]$ in G which is a contradiction to the fact that G is chordal. Therefore, $diam(G(S)) \leq 2$.

Note that the previous Lemma is not true for general graphs. Let a vertex v of a graph $G=(V, E)$ be a dominating vertex, $DV(G)$, if v is adjacent to all the vertices of $V-\{v\}$ in G.

Lemma 3.11 *Let $G=(V, E)$ be a graph with $diam(G) \leq 2$. If G contains a cutpoint v, then v is a dominating vertex in G.*

Proof. Let $C_1, C_2, \ldots, C_k$, $k \geq 2$, be the connected components of $G-\{v\}$. Suppose that there exists a vertex $x \in C_i$ such that $(x, v) \notin E$. Then $dist_G(x, y)>2$ for some vertex y such that $y \in C_j$, $i \neq j$. Therefore, x is a dominating vertex in G.

Lemma 3.12 *Let $G=(V, E)$ be a 3SF chordal graph such that $diam(G) \leq 2$. Then G contains at least one dominating vertex.*

Proof. The proof is by induction on $|V|$. If $|V| \leq 5$, the proof is trivial. Let $|V|=6$. If G contains a $DV(G)$, we are done. If not, we must show that G is exactly a *3sun*. Since G is not a clique, there exists at least two nonadjacent simplicial vertices b and c. Also, there exists a vertex w such that w is adjacent to both b and c in G. Since not all the rest of the vertices, say u, v, and a, are adjacent to w without loss of generality, let $(a, w) \notin E$. We have two cases:

(i) $(a, b) \in E$. Note that, since $(a, b) \in E$, $(a, c) \notin E$. Therefore, there exists a vertex v such that $(v, a),(v, c) \in E$. Then, v must be adjacent to both b and w. There is a one more vertex u to be added. However, it is straightforward to see that u can not be adjacent to both a and c, and if u is adjacent to either b or w, then G is a *3sun*.

(ii) $(a, b) \notin E$. Since $dist_G(a, b)=2$, there must exist a vertex, say u, such that $(u, a),(u, b) \in E$. If $(a, c) \in E$, since u must be adjacent to both c and w, this case is the same as the case *(i)*. If $(a, c) \notin E$, then either there exists a vertex, say v, such that (v, a), $(v, c) \in E$ or $(u, c) \in E$. If $(v, a),(v, c) \in E$, it is easy to see that G is a *3sun*. If $(u, c) \in E$, then, because of the last vertex v, $diam(G) \geq 2$ in all cases.

For the induction hypothesis suppose that, if $|V| \leq k$, then G contains at least one $DV(G)$, where $k \geq 7$. Let G be a graph with $|V|=k+1$. If G is a clique, we are done. If not G contains at least two nonadjacent simplicial vertices x and y. Also, there exists at least one vertex z such that z is adjacent to both x and y in G. Let $G'=G-\{z\}$. If z is a cutpoint in G, then, by Lemma 3.11, $x=DV(G)$. If z is not a cutpoint, then G' is a 3SF chordal graph such that $diam(G') \leq 2$. Hence, by the hypothesis G' contains a $DV(G)$. If $x=DV(G')$ or $y=DV(G')$, then x or y is a dominating vertex in G. If $x \neq DV(G')$ and $y \neq DV(G')$, then $DV(G')$ is adjacent to z in G for otherwise G contains a C_4. Therefore, G contains at least one $DV(G)$.

Lemma 3.13 Let G be a *graph such that $diam(G) \leq 2$ and G contains no dominating vertex. Let x and y be any two maximum degree vertices of G. If $(x, y) \notin E$, then G contains either C_4 or C_5.*

Proof. Let $S=\{v\in V|\ v$ is adjacent to both x and y in $G\}$. The proof is by induction on $|S|$. Let $|S|=1$ and z be the vertex such that z is adjacent to x and y in G. Since x and y are maximum degree vertices there must exist at least two vertices u and v such that $(u, x),(v, y)\in E$ and $(z, u),(z, v)\notin E$. Note that since $|S|=1$ $(u, y),(v, x)\notin E$ and $u\neq v$. If $(u, v)\in E$ then the set $\{u, v, x, y, z\}$ induces a C_5. If $(u, v)\neq E$, then there must exist another vertex w such that w is adjacent to both u and y. then the set $\{x, y, z, u, w\}$ induces a C_4 or C_5 in G. for the induction hypothesis assume that if $|S|=k>1$ then G contains either C_4 or C_5. Let $|S|=k+1$ and $G'=G-\{u\}$, where $u\in S$. we need to show that G contains C_4 or C_5. If u is a cutpoint in G, then u is a dominating vertex which is a contradiction. Hence u is not a cutpoint. Now, suppose that G' contains a dominating vertex, say d, then it means that d is not adjacent to only u in G. Note that $u\neq x$ and $u\neq y$. However, both u and d are adjacent to both x and y in G. Therefore, the set $\{x, y, d, u\}$ induces a C_4 in G. Hence, by the hypothesis, G' contains either C_4 or C_5 and any of this chordless cycle will be remained in $G+\{u\}$. Therefore, G contains either C_4 or C_5.

Lemma 3.14 *If $G=(V, E)$ is a chordal graph with $diam(G)\leq 2$, then $G(M)$ induces a clique in G, where $M=\{v\in V|\ deg_G(v)=\Delta(G)\}$.*

Proof. If G contains no dominating vertex, then the proof follows from Lemma 3.13. If G contains a dominating vertex, then $deg_G(v)=|V|-1$ for any $v\in M$. Hence, the vertices of M induces a clique.

Lemma 3.15 *If $G=(V, E)$ is a nontrivial chordal graph with $diam(G)\leq 2$, then $\Delta(G)\geq 2|V|/3$.*

Proof. If G contains a dominating vertex, then clearly $\Delta(G)=|V|-1\geq 2|V|/3$ for any $|V|\geq 3$. From now on we assume that G contains no dominating vertex. The proof is by induction on $|V|$. If $|V|\leq 6$, then the proof is trivial. For the induction hypothesis assume that if $|V|\leq k$, then $\Delta(G)\geq 2k/3$, $k\geq 7$. Let $|V|=k+1$. We need to show that $\Delta(G)\geq 2k/3+2/3$. Let $M=\{x\in V|\ deg_G(x)=\Delta(G)\}$ and $G'=G-\{x\}$, where $x\in M\}$. Note that x is not a cutpoint in G for otherwise x is a dominating vertex in G. If G' contains a dominating vertex, say d, then clearly $deg_{G'}(d)=k-1$. Then, in G, d can not be adjacent to x for otherwise d is a dominating vertex in G. Therefore, $deg_G(d)=k-1$ which means that $deg_G(x)>deg_G(d)$ for otherwise, by Lemma 3.14, $(x, d)\in E$. Hence, x is a dominating vertex in G which is a contradiction. Therefore, by the hypothesis, $\Delta(G')\geq 2/3k$. We have two cases: (*i*) $|M|=1$. Then $deg_G(x)\geq 2k/3+1$. (*ii*) $|M|\geq 2$. Then, by Lemma 3.14, M is a clique. Hence, $\Delta(G')=\Delta(G)-1\geq 2k/3$.

Corollary 3.16 *If $G=(V, E)$ is a chordal graph with $diam(G)\leq 2$, then $|V|\leq 3\Delta(G)/2$.*

Theorem 3.17 *If $G=(V, E)$ is a chordal graph, then $\Delta(G)+1\leq w(G^2)\leq 3\Delta(G)/2$.*

Proof. Let $deg_G(v)=\Delta(G)$, where $v\in V$. In G^2, $N_{G^2}[v]$ forms a clique. Hence $\Delta(G)+1\leq w(G^2)$. Suppose that G^2 contains a maximal clique $S=\{v_1, v_2, ..., v_k\}$, where $k=3\Delta(G)/2+1$. By the Lemma 3.9 and 3.10, we know that $G(S)$ is connected chordal such that $diam(G(S))\leq 2$. Therefore, by Corollary 3.16, $k\leq 3\Delta(G)/2$. Hence, $w(G^2)\leq 3\Delta(G)/2$.

Note that $\chi(G)=\omega(G)$ if G is chordal. Therefore, by the previous observation and Theorem 3.17, the following theorem is immediate.

Theorem 3.18 *Let $G=(V, E)$ be a chordal graph. If $G^2=(V, E)$ is also chordal, then $\lambda(G)\leq 3\Delta(G)-2$.*

We have restricted our discussion to the case when G and G^2 is chordal. However, since $\omega(G)=\chi(G)$ for any perfect graph the following stronger result is immediate.

Theorem 3.19 *Let $G=(V, E)$ be a chordal graph. If G^2 belongs to any perfect graph, then $\lambda(G)\leq 3\Delta(G)-2$.*

4. $L(2, 1)$-labeling of Permutation Graphs

Let $\pi=[\pi(1), \pi(2), ..., \pi(n)]$ be the permutation of the numbers 1, 2, ..., n. For example, if $\pi=[4, 2, 1, 3, 5]$, then $\pi(1)=4$ and $\pi(2)=2$, etc. Let $\pi^{-1}(i)$ denotes the position in the sequence, where the number i can be found; in our example $\pi^{-1}(3)=4$. If π is a permutation of the numbers 1, 2, ..., n., then the graph $G[\pi]=(V, E)$ is defined as follows: $V=\{1, 2, ..., n\}$ and $(v_i, v_j)\in E$ if and only if $(i-j)(\pi^{-1}(i)-\pi^{-1}(j))<0$. An undirected graph G is called a permutation graph if there exists a permutation π such that $G\cong G[\pi]$.

Lemma 4.1 *Let $[v_i, v_j, v_k]$ be a path in a permutation graph $G=(V, E)$, where $i<j<k$. Then, $(v_i, v_k)\in E$.*
Proof. Since v_j is adjacent to both v_i and v_k, clearly $\pi^{-1}(k)\pi^{-1}<\pi^{-1}(j)<\pi^{-1}(i)$. Hence, v_i is adjacent to v_k in G.

In a graph G, let path $p=[v_i, v_2, ..., v_k]$ be called *monotonic* if $v_1<v_2<\cdots<v_k$ or vice versa, where $k\geq 2$.

Lemma 4.2 *Let G be a permutation graph. Then, every monotonic path induces a clique in G.*
Proof. The proof follows by applying Lemma 4.1 recursively.

Lemma 4.3 Let *a vertex v_i has two lower-neighbours v_j and v_k $(v_j<v_k)$ in a permutation graph G. Let $\{v_{j1}, v_{j2}, ..., v_{jp}\}$ be the neighbours of v_j such that $v_i<v_{j1}<v_{j2}<\cdots<v_{jp}$ and $dist_G(v_i, v_{ja})=2$ for all $1\leq a\leq p$. Similarly, let $\{v_{k1}, v_{k2}, \cdots, v_{kq}\}$ be the neighbours of v_k such that $v_i<v_{k1}<v_{k2}<\cdots<v_{kq}$ and $dist_G(v_i, v_{kb})=2$ for all $1\leq b\leq q$. Then,*
(i) v_j and v_k are adjacent to all the higher-neighbours of v_i;
(ii) v_{ja} (resp. v_{kb}) such that $v_{ja}<H(v_i)$ (resp. $v_{kb}<H(v_i)$) is adjacent to $H(v_i)$ for any $1\leq a\leq p$ (resp. $1\leq b\leq q$);
(iii) If $(v_j, v_k)\in E$, then v_j is adjacent to all v_{kb}, $1\leq b\leq q$; and
(iv) If $(v_j, v_k)\notin E$, then v_k is adjacent to all v_{ja}, $1\leq a\leq p$.

Proof. (i) Let $S=\{u\in V|$ u is a higher-neighbour of $v_i\}$. Then, the path $[v_j, v_i, u]$ is a monotonic path for any $u\in S$. Hence, by Lemma 4.2, v_j is adjacent to all $u\in S$. Similarly, v_k is adjacent to all the higher neighbours of v_i.
(ii) By part (i), clearly $\{v_j, v_i, H(v_i)\}$ is a clique in G; hence, $\pi^{-1}(H(v_i))<\pi^{-1}(v_{ja})$ for all $1\leq a\leq p$. Therefore, $H(v_i)$ is adjacent to all such v_{ja}, $1\leq a\leq p$.
(iii) Since $(v_j, v_k)\in E$, $\{v_j, v_k, v_i\}$ is a clique in G; hence, $\pi^{-1}(v_i)<\pi^{-1}(v_k)<\pi^{-1}(v_j)$. Since $(v_i, v_{kb})\notin E$ and $(v_k, v_{kb})\in E$ for all b, $1\leq b\leq q$, $\pi^{-1}(v_i)<\pi^{-1}(v_{kb})<\pi^{-1}(v_k)$; hence, $\pi^{-1}(v_{kb})<\pi^{-1}(v_j)$ for all such b, $1\leq b\leq q$. Therefore, v_j is adjacent to all v_{kb}, $1\leq b\leq q$.
(iv) Since $(v_j, v_k)\notin E$, $\pi^{-1}(v_i)<\pi^{-1}(v_j)<\pi^{-1}(v_k)$. Since $(v_i, v_{ja})\notin E$ and $(v_j, v_{ja})\in E$ for all a, $1\leq a\leq p$, $\pi^{-1}(v_i)<\pi^{-1}(v_{ja})<\pi^{-1}(v_j)$ for all a, $1\leq a\leq p$; hence, $\pi^{-1}(v_{ja})<\pi^{-1}(v_k)$. Therefore, v_k is adjacent to all v_{ja}, $1\leq a\leq p$.

Lemma 4.4 *Let a vertex v has $S=\{u_1, u_2, ..., u_k\}$ lower-neighbours in a permutation graph G. Let $\{u_{p1}, u_{p2}, ..., u_{pq}\}$ be the vertices adjacent to u_p for some number q for each p, $1\leq p\leq k$, and $u_{ij}>v$, and $dist_G(v, u_{ij})=2$ for all j, $1\leq j\leq p$, for each i, $1\leq i\leq k$. Then, there exists a vertex u_m, $1\leq m\leq k$, such that u_m is adjacent to all u_{ij}.*
Proof. The proof is by induction on $|S|$. If $|S|=1$, the proof is trivial. If $|S|=2$, then the proof is followed by part (iii) and (iv) of Lemma 4.3. For the induction hypothesis, assume that it is true for $|S|\leq k-1$ for some $k>2$. Let $|S|=k$, and consider the graph $G'=G-\{u_k\}$. Then, by the hypothesis, there exists a vertex, say $u_c\in S-\{u_k\}$ such that u_c is adjacent to all u_{ij} in G'. In G, it is clear that either $u_c<u_k$ or $u_c>u_k$ and either $(u_c, u_k)\in E$ or $(u_c, u_k)\notin E$. Hence, by the part (iii) and (iv) of Lemma 4.3, u_c is adjacent to all u_{ij} or u_k is adjacent to all u_{ij} in G.

Lemma 4.5. *Let $G=(V, E)$ be a permutation graph with some permutation π. Let v_i have two higher-neighbours v_j and v_k $(v_j<v_k)$, and $v_p\in V$ be a vertex with $(v_j, v_p)\in E$ and $dist_G(v_i, v_p)=2$. Then, $(v_p, v_k)\in E$ in G.*

Proof. Since v_j is adjacent to both v_i and v_k and $v_i<v_j<v_k$, clearly $\pi^{-1}(v_j), \pi^{-1}(v_k)<\pi^{-1}(v_i)$. Note that v_i is the lowest indexed vertex in G_i; hence, $v_p>v_i$. Now, since v_i is not adjacent to v_p, we have $\pi^{-1}(v_i)<\pi^{-1}(v_p)$. Since v_j is adjacent to v_p, $\pi^{-1}(v_j)<\pi^{-1}(v_p)$, and clearly $v_k>v_j>v_p$. Therefore, v_k is adjacent to v_p in G_i.

Lemma 4.6. *Let $G=(V, E)$ be a permutation graph. Then, $|TWO(v)| \leq 2\Delta(G)-deg_G(v)-2$ for any vertex $v \in V$.*

Proof. Let $N_G(v)=\{v_1, v_2, ..., v_k, u_1, u_2, ..., u_m\}$ such that $v_1<v_2<\cdots<v_k$ are higher-neighbours and $u_1<u_2<\cdots<u_m$ are lower-neighbours of v. Note that $H(v)=v_k$. Let $X=\{x \in V|\ x>v$ and $dist_G(v, x)=2$ and x is adjacent to some v_i, $1 \leq i \leq k\}$ and $Y=\{y \in V|\ y>v$ and $dist_G(v, y)=2$ and y is adjacent to some u_j, $1 \leq j \leq m\}$. Note that $TWO(v)=X \cup Y$. Then, by Lemma 4.5, v_k is adjacent to all the vertices of X. Also, by Lemma 4.4, there exists a vertex, say u_c, such that u_c is adjacent to all the vertices of Y. Finally, by Lemma 4.3, each u_j, $1 \leq j \leq m$, is adjacent to all v_i, $1 \leq i \leq k$. Therefore, $|TWO(v)|=(deg_G(v_k)-(deg_G(v)-k)-1)+(deg_G(u_c)-k-1)=deg_G(v_k)+deg_G(u_c)-deg_G(v)-2 \leq 2\Delta(G)-deg_G(v)-2$.

Theorem 4.7. *Let $G=(V, E)$ be a permutation graph. Then, $\lambda(G) \leq 4\Delta(G)-2$.*

Proof. We use high-only scheme for $L(2, 1)$-labeling of G. Suppose that v_i, $1 \leq i \leq n$, is the next vertex to be labeled. Then, by Lemma 4.6, v_i must avoid at most $3deg_G(v_i)+2\Delta(G)-deg_G(v_i)-2$ numbers. Hence, $2deg_G(v_i)+2\Delta(G)-2 \leq 4\Delta(G)-2$. Since we can use 0 as a label when we label v_i, there exists at least one number for v_i.

5. Conclusions

We claim that $(\Delta^2+6\Delta+9)/4-(4\Delta^2+76\Delta+1)/24>0$ if $\Delta(G) \geq 19$, the first part of which is Sakai's upper-bound and the latter part of which is ours. It can be proved by induction on $\Delta(G)$. Let $\Delta(G)=19$, then it is trivial. Assume that $(n^2+6n+9)/4-(4n^2+76n+1)/24>0$ for all $n \geq 20$. We need to prove that $((n+1)^2+6(n+1)+9)/4-(4(n+1)^2+76(n+1)+1)/24=2n^2-36n+15>0$. It can be decomposed into $(2n^2-40n+53)+(4n-38)>0$, the first part of which is greater than 0 by hypothesis and the latter part of which is greater than 0 since $n \geq 20$. Hence, our upper-bound is better than Sakai's if $\Delta(G) \geq 19$. For a chordal graph G, if $G^2=(V, E)$ is also chordal, then $\lambda(G) \leq 3\Delta(G)-2$. If G^2 belongs to any perfect graph, then $\lambda(G) \leq 3\Delta(G)-2$. For a permutation graph G, $\lambda(G) \leq 4\Delta(G)-2$.

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