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Technical Notes

Complexity of the Fire Sequencing Problem

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ABSTRACT

In this note, we introduce the Fire Sequencing Problem, which arises in military operations. Given m weapons, n fixed targets and required duration of firing of the weapons on the targets, we want to determine the start time of firing on each target so that makespan is minimized while satisfying various operational constraints.

We show that the decision problem of the Fire Sequencing problem is strongly NP-complete and remains strongly NP-complete even if the number of weapons is two. We also briefly discuss the results with respect to the complexities of several well-known scheduling models.

1. INTRODUCTION

In this note, we introduce the Fire Sequencing Problem which arises in military operations research and show that this problem is strongly NP-complete.

We have m weapons (possibly non-identical) and n fixed targets such as common post, communication center, assembly area, fire support means, principal terrain features, and logistic facilities, etc. Each weapon i must fire on a target j for p_{ij} time units. Firing duration time p_{ij}'s are calculated from another decision model that considers tactical mission, ability of each weapon, desired target destruction level, available ammunition amount, characteristics of each target, and so on. If p_{ij} is equal to 0, it implies weapon i does not need to fire on target j. In addition, we have the following *operational constraints*. First, each weapon can fire only one target at a time. Second, once a weapon starts firing on a target, it should continue firing on that target for the entire planned time units. If preemption on firing is allowed, the effectiveness of surprise would be diminished and the military tactics do not permit that kind of operation. Surprise is an important principle of strategy, operational arts and tactics. Finally, considered from a target side, the start time of firing from all planned weapons should be the same. The reason is also to achieve maximum tactical surprise. The Fire Sequencing Problem is to determine the start time of firing on each target so as to minimize makespan while satisfying operational constraints. We investigate the computational complexity of the problem in the next section.

2. COMPUTATIONAL COMPLEXITY OF THE PROBLEM

Let W be the set of weapons and T be the set of targets. Also let $W(i)$ be the set of weapons which are planned to fire on target i, for each $i \in T$. Note that we can start firing on target i and j at the same time if $W(i) \cap W(j) = \phi$. The decision problem associated with the Fire Sequencing Problem (FSP) is defined as follows :

(FSP) : *Instance: Set of weapons W, set of targets T, nonnegative integer p_{ij}, for all $i \in W$, $j \in T$, and a positive integer L.*

Question: Is there a firing sequence whose makespan is less than or equal to L ?

We now prove the (FSP) is strongly NP-complete. To prove a problem Π is strongly NP-complete, we first need to check that the problem Π is in NP, and then select a candidate strongly NP-complete problem, $\hat{\Pi}$, for which we can establish a pseudopolynomial transformation from $\hat{\Pi}$ to Π (or to a special case of Π)[2]. Clearly, (FSP) is in NP. As a candidate strongly NP-complete problem we choose the GRAPH K-COLORABILITY, which is strongly NP-complete[2]. GRAPH K-COLORABILITY is restated as follows:

Instance : *Undirected graph* $G = (V, E)$, *positive integer* $K \leq |V|$.

Question : *Is G K-colorable, i.e. does there exist a function* $f : V \rightarrow \{1, 2, \ldots, K\}$ *such that* $f(u) \neq f(v)$ *whenever* $(u, v) \in E$?

Given an arbitrary instance of GRAPH K-COLORABILITY we will construct (in a polynomial number of steps) an instance of (FSP) as follows : Let $W = \{[i, j] \mid (i, j) \in E\}$ and $T = \{i \mid i \in V\}$. Set $p_{[i,j],i} = p_{[i,j],j} = 1$, for all $[i, j] \in W$ and $i, j \in T$ and $L = K$. From this transformation, $W(i) \cap W(j) = \{[i, j]\}$, for all $i, j \in T$ if and only if $(i,j) \in E$.

Theorem 1. *G is K-colorable if and only if there is a firing sequence for* (FSP) *whose makespan is less than or equal to L.*

The proof is clear from the above construction of an instance of (FSP). If G is K-colorable, there exist mutually disjoint subsets $V_1, V_2, \ldots, V_k$ of V, where $k \leq K$ and $\bigcup_{i=1}^{k} V_i = V$, each of which is an independent set (node packing) in G. Let T_i be the set of targets which corresponds to V_i, for all $1 \leq i \leq k$. Then $T_1, T_2, \ldots, T_k$, subsets of T, are mutually disjoint and $\bigcup_{i=1}^{k} T_i = T$. Clearly, $W(u) \cap W(v) = \phi$, for all $u, v \in T_i$, for all $1 \leq i \leq k$. Therefore, we can start firing on all targets of T_i at the same time, for all $1 \leq i \leq k$, and all planned duration of firing are equal to 1 time unit. That is, there exists a firing sequence whose makespan is k. Conversely, if there exists a firing sequence whose makespan is k ($\leq L$), then clearly there exist mutually exclusive and exhaustive subsets of V, $V_1, V_2, \ldots, V_k$, each of which corresponds to an independent set in G, so that G is K-colorable.

Corollary 2. (FSP) *is NP–complete even if* $p_{ij} = 1$ *for all* $i \in W(j)$ and $j \in T$.

Corollary 2 establishes that (FSP) is strongly NP-complete. Next we consider (FSP) with only two weapons available, which is denoted by (FSP2). Note that if there exists only one weapon, the problem can be solved trivially. We will show that (FSP2) is strongly NP-complete. To show the result, we transform an arbitrary instance of 3-PARTITION[2] to an instance of (FSP2) using a pseudopolynomial transformation. 3- PARTITION may be stated as follows.

Instance : *Set A of 3m elements, a bound* $B \in Z^+$, *and a size* $s(a) \in Z^+$ *for each* $a \in A$ *such that* $B/4 < s(a) < B/2$ *and such that* $\sum_{a \in A} s(a) = mB$.

Question : *Can A be partitioned into m disjoint sets* $A_1, A_2, \ldots, A_m$ *such that for all* $1 \le i \le m$, $\sum_{a \in A_i} s(a) = B$ *?*

3-PARTITION is known to be NP-complete in the strong sense[2]. Given an arbitrary instance of 3-PARTITION, we will construct an instance of (FSP2) as follows : Let $W = \{1, 2\}$ and $T = A \cup \{1, \ldots, m\}$. Set $p_{1i} = 1$ and $p_{2i} = B+1$ for all $i = 1, \ldots, m$, $p_{1a} = s(a)$ and $p_{2a} = 0$ for all $a \in A$, *and* $L = mB + m$. Then it can be easily proved that a given instance of 3-PARTITION gives an affirmative answer if and only if there exists a firing sequence for the transformed problem whose makespan is equal to L. Moreover, the size of the largest numbers in the transformed instance is bounded by a polynomial function of the size of the numbers in 3-PARTITION. Therefore we have :

Theorem 3. (FSP) *with only two weapons is strongly NP-complete.*

3. DISCUSSION

Now we compare the complexity of (FSP) with the complexities of other well-known scheduling models. First, consider the parallel machine scheduling problem with the objective of minimizing makespan without preemptions[3]. The problem is known to be NP-complete even if there are only two machines. However, the problem can be solved in pseudopolynomial time if the number of machines is fixed. It is well-known that the scheduling problem with the objective of minimizing makespan in Flow Shops with unlimited intermediate storage can be solved in polynomial time when there are only two machines[3], though the problem with three machines is strongly NP-complete. We have shown in the previous section (FSP) is already strongly NP-complete even if only two weapons (machines) are available. Moreover, even when the processing times (p_{ij}) are all the same, the problem remains to be strongly NP-complete, but in this case, the other two problems (parallel machines, flow shop) can be solved trivially. These results show that (FSP) is a very difficult scheduling problem.

A problem similar to (FSP) is the simultaneous resource scheduling problem, which was addressed by Dobson and Karmarkar[1]. The objective of this problem is to minimize the total weighted flow time of n tasks on m resources. Each task j requires a subset S_j of the m resources and consumes time t_j. The additional restriction is that the task j must capture all of the resources S_j simultaneously. This problem is known to be NP-complete[1]. Differences between the simultaneous resource scheduling problem and (FSP) lie in 1) the objective function and 2) a task j does not release any of the resources S_j, until it is completely processed in the simultaneous resource scheduling problem.

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