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Rao-Wald Test for Variance Ratios of a General Linear Model

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Abstract

In this paper, we propose a method to test $H: \rho_i = r_i$ for $1 \leq i \leq \ell$ against $K: \rho_i \neq r_i$ for some i in k-variance component random or mixed linear model, where ρ_i denotes the ratio of the i-th variance component to the error variance and $\ell \leq k$. The test, which we call Rao-Wald test, is exact and does not depend upon nuisance parameters. From a numerical study of the power performance of the test of the interaction effect for the case of a two-way random model, Rao-Wald test was seen to be quite comparable to the locally best invariant (LBI) test when the nuisance parameters of the LBI test are assumed known. When the nuisance parameters of the LBI test are replaced by maximum likelihood estimators, Rao-Wald test outperformed the LBI test.

1. Introduction

One objective of Linear models for classificatory data may be inference about the ratio of the variance of a particular set of random effects to the residual variance. For instance, animal and plant breeders often investigate the heritability of some trait, and, under certain assumptions, heritability is expressible as a strictly increasing function of a variance component ratio. However, when a random or mixed model is unbalanced, we could not find unified approach for testing variance ratio. AOV(Analysis of Variance) procedure fails to decompose the total sum of squares into independently distributed sums of squares, and we can not directly apply AOV procedure to test the variance ratios of a general linear model. One of the classical approaches to handle the problem is employ Satterthwaite's argument to obtain an approximate test. However, in some cases the nominal significance level of the approximate test is highly unreliable (see, e.g., Khuri and Little 1987; Kleffe and Seifert 1988). Thus an exact test procedure is desirable.

Wald(1940, 1941) constructed an exact confidence interval for the variance ratio in one way random model, which is closely related to the testing problem. After Wald, many studies have attempted to obtain exact tests for null variance ratio on specific models. For example,

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Spjotvoll(1967, 1968) and Thomsen(1975) derived an exact test for null variance ratios in a two-way crossed classification model. These results were generalized by Seely and El-Bassiouni(1983). They obtained a Wald test for testing null variance ratios by applying reductions in sum of squares in a general linear model and provided the necessary and sufficient conditions for the existence of a Wald test. They also showed that the test statistic derived by Spjotvoll and many others are identical to that of Wald. Recently, attention has been given to testing whether a variance ratio equals a specific value. For this testing problem, the corresponding Wald procedure and various alternatives to Wald's procedure in two variance components models have been devised (e.g., Burdick, Masqsood, and Graybill 1986; LaMotte, McWhorter, and Prasad 1988; Lin and Harville 1991; Huh and Li 1996). Huh and Li (1996) showed that exact test for variance ratio is essentially unique.

Although an exact test is desirable for the problem, the test should have some optimal properties. It is well known, when a design is unbalanced, that global optimum test for variance ratios does not exist. Thus we should employ tests which have certain local optimum properties such as LBI(Locally Best Invariant) test or Neyman-Pearson test. Lin and Harville (1991) showed by numerical study that the Neyman-Pearson test and the LBI test are slightly better than Wald test in two variance-component model. However, these tests have several drawbacks. Firstly, when the model contains more than two variance components, the tests depend on nuisance paramters. Secondly, Neyman-Pearson test statistic and LBI test statistic do not follow known distributions. Hence to obtain the critical value for the test, we need to solve nonlinear equation or to rely on simulation results. Thirdly, the computation of the test statistics requires eigen value computation.

In this note, we derive a test statistic, which we will call Rao-Wald, to test selected subset of variance ratios equal simultaneously to specified values in a general linear model. This result is a slight generalization of Huh and Li's work and resolves all the three problems of the LBI test. The conventional Wald test for testing only one variance ratio equal to a specific value is a special case of this work. We then examine the power performance of the Rao-Wald test relative to the LBI(Locally Best Inavriance) test for the ratio of interaction effect variance component over the error component in the case of two way random model with interaction. Since an exact power of LBI test is not available, we conducted a numerical study to obtain the power of LBI test. Also for the LBI test, two main effect component ratios are nuisance parameters. We considered two approaches for the nuisance parameters. First approach is to assume the nuisance parameters are known, and we call this test as theoretical LBI test. Second approach is to replace the nuisance parameters by the maximum likelihood estimates. We call this test as two-step LBI test.

2. Derivation of Rao-Wald Test

Consider the following general linear model

$$y = X_0\pi + X_1\xi_1 + \cdots + X_k\xi_k + \varepsilon, \tag{1}$$

where y is a vector of n observations, X_i is a $n\times b_i$ design matrix, π is a $b_0\times 1$ vector of unknown constants, ξ_i is a vector of b_i uncorrelated random effects, and ε is a vector of n random errors. Further we assume that ξ_i and ε are statistically independent and are multivariate normal random vectors with $\mathrm{E}(\xi_i) = \mathbf{0}$, $\mathrm{Var}(\xi_i) = \sigma_i^2 \mathrm{I}$, for $i = 1, 2, \dots, k$ and $\mathrm{E}(\varepsilon) = \mathbf{0}$, $\mathrm{Var}(\varepsilon) = \sigma_{k+1}^2 \mathrm{I}$.

We consider testing a subset of $(\rho_1, \dots, \rho_k)$ equal to some values where $\rho_i = \sigma_i^2/\sigma_{k+1}^2$ $i = 1, \dots, k$. Without loss of generality, we can state the problem as $H: \rho_1 = r_1, \dots, \rho_\ell = r_\ell$ against $K: \rho_i \neq r_i$, for at least one i where $1 \le i \le \ell \le k$. To derive a test statistic for the problem, we rewrite model (1) as

$$y = X\xi + W\zeta + \varepsilon \tag{2}$$

where $X = (X_1, \dots, X_\ell)$, $W = (X_{\ell+1}, \dots, X_k)$, $\zeta' = (\pi', \xi'_{\ell+1}, \dots, \xi'_k)$ and $\xi' = (\xi'_1, \dots, \xi'_\ell)$. For the model we assume that each column space of X_i, $i = 1, \dots, \ell$ is not a proper subset of the column space of W. *i.e.*,

$$\mathrm{col}(X_i) \not\subset \mathrm{col}(W) \tag{3}$$

where $\mathrm{col}(A)$ denotes the column space of a matrix A. When $\ell = 1$, the assumption is equal to the case of Seely and El-Bassiouni's work to derive an Wald test for $H: \rho_1 = 0$ against $K: \rho_1 > 0$.

Under the conditions (3), we have

$$\mathrm{rank}\,(X, W) - \mathrm{rank}\,(W) = m > 0. \tag{4}$$

There are three cases under this assumption. First one is that the column spaces of X and W are essentially disjoint, *i.e.*, $\mathrm{col}(X) \cap \mathrm{col}(W) = \{\mathbf{0}\}$. Balanced designs belong to this case. Second one is that the column spaces intersect each other. Third one is that the column space of W is a proper subset of the column space of X. We only consider the second case here, because it is trivial to derive a test statistic for the first case. and the third case is identical to the second one.

Let C be a full column rank matrix of order $(n-d)\times n$ such that $CW=0$, $CC'=I_{n-d}$ and $C'C=I_n-W(W'W)^{-}W'$, where d is the rank of W. Multiplying both sides of equation (2) by C, we have

$$z=Cy=CX\xi+C\varepsilon, \tag{5}$$

Now consider a test based on z. It can be shown (see Rao, section 9, 1971, for example) that the MINQUE of $\sigma=(\sigma_1^2,\cdots,\sigma_\ell^2;\sigma_{k+1}^2)$ for model (5) is a solution to the equation

$$S\hat{\sigma}=u \tag{6}$$

where $S_{(\ell+1)\times(\ell+1)}=\mathrm{tr}(RV_iRV_j)$, $u_{(\ell+1)}=z'RV_iRz$, $V_i=CX_iX_i'C'$ for $i=1,\cdots,\ell$, $V_{\ell+1}=I_{n-d}$ and

$$R=(\sum_{i=1}^{\ell+1}r_iV_i)^{-1}=(I+\sum_{i=1}^{\ell}r_iV_i)^{-1} \tag{7}$$

with r_i denoting the a-priori values of $\rho_i=\sigma_i^2/\sigma_{k+1}^2$, for $i=1,\cdots,\ell$, and $r_{\ell+1}=1$.

To derive a test statistic for testing $H:\rho_i=r_i$ for $i=1,2,\ldots,\ell$ against $K:\rho_i\neq r_i$ for at least one i, we consider the linear combination $\sum_{i=1}^{\ell+1}r_iu_i$ where u_i is the i-th element of u of equation (6). $\sum_{i=1}^{\ell+1}r_iu_i$ can be rewritten follows:

$$\sum_{i=1}^{\ell+1}r_iu_i=z'R(I+\sum_{i=1}^{\ell}r_iV_i)Rz=z'Rz \tag{8}$$

We will decompose the quadratic form $z'Rz$ into two parts so that each of them follows an independent central chi-square distribution under the null hypothesis.

Since $\sum_{i=1}^{\ell}r_iV_i$ of (7) is symmetric, it can be expressed as

$$\sum_{i=1}^{\ell}r_iV_i=QD(r)Q' \tag{9}$$

where $D(r)$ is a diagonal matrix of eigenvalues of $\sum_{i=1}^{\ell}r_iV_i$ and Q is the matrix of corresponding eigenvectors. Let g be the rank of $\sum_{i=1}^{\ell}r_iV_i$, and assume the first g diagonal elements of $D(r)$ be nonzero. Note that $m-g>0$ when some of r_i's are specified to be zeros. Since Q is orthogonal, we have

$$
\begin{aligned}
R &= \left(\mathrm{I} + \sum_{i=1}^{\ell} r_i V_i \right)^{-1} = (\mathrm{I} + QD(r)Q')^{-1} \\
&= Q(\mathrm{I} + D(r))^{-1} Q' = QD_1^{-}(r)\,Q' + QD_2 Q'
\end{aligned} \tag{10}
$$

where

$$
\begin{aligned}
D_1^{-}(r) &= \mathrm{diag}\,(1/(1+\lambda_1(r)), \ldots, 1/(1+\lambda_g(r)), \overbrace{1, \ldots, 1}^{m-g}, \overbrace{0, \ldots, 0}^{n-d-m}) \\
D_2 &= \mathrm{diag}\,(0, \ldots, 0, \underbrace{1, \ldots, 1}_{n-d-m})
\end{aligned}
$$

and $\lambda_i(r)$ are the nonzero eigenvalues of $\sum_{i=1}^{\ell} r_i V_i$.

Combining the results of (8) and (10), we can decompose $\sum_{i=1}^{\ell+1} r_i u_i$ into two parts as follows:

$$
\sum_{i=1}^{\ell+1} r_i u_i = z' R z = z' QD_1^{-}(r)\,Q' z + z' QD_2 Q' z. \tag{11}
$$

Theorem 1. *Under the null hypothesis, the distribution of* $z' QD_1^{-}(r)\,Q' z / \sigma_{k+1}^2$ *is chi-square with* m *degrees of freedom. Also* $z' QD_2 Q' z / \sigma_{k+1}^2$ *is a chi-square random variable with* $n-d-m$ *degrees of freedom and is independent of* $z' QD_1^{-}(r)\,Q' z / \sigma_{k+1}^2$ *for all* ρ_i.

Theorem 1 is straightforward and suggests to consider the following quantity as a test statistic for $H: \rho_i = r_i$ for $i = 1, 2, \ldots, \ell$ against $K: \rho_i \neq r_i$ for at least one i:

$$
F = \frac{n-d-m}{m} \, \frac{z' QD_1^{-}(r) Q' z}{z' QD_2 Q' z} \tag{12}
$$

The distribution of F is $F(m, n-d-m)$ under the null hypothesis and the null hypothesis is rejected if the observed value of F is too large or too small.

Remark 2.1: Under model (5), $\sum_{i=1}^{\ell+1} r_i u_i$ is a linear combination of MINQUE's of ρ_i's. From the decomposition of $\sum_{i=1}^{\ell+1} r_i u_i$, it is clear that $z' QD_1^{-}(r)\,Q' z$ is a linear combination of the MINQUE's of ρ_i's. Hence based on z, $z' QD_1^{-}(r) Q' z$ is locally minimum variance unbiased quadratic estimator at $\rho_i = r_i$ for all i by the result of Rao (1973, pp 303--305).

Now we prove the uniqueness property of the test statistics (12). Theorem 2 is due to

Seely and El-Bassiouni(1983). The theorem indicates that the quadratic forms $z'QD_2Q'z$ in (12) is the residual sums of squares of the model (1).

Theorem 2. *Let* $Q_e = y'\mathrm{H}_e y$ *where* H_e *is any real symmetric matrix. If* $Q_e/\sigma^2_{k+1} \sim \chi^2_\nu$, *then* $\nu \le n-d-m$. *Moreover, if* $\nu = n-d-m$, *then* Q_e *is the residual sum of squares of the model (1).*

Theorem 3. *Assume the conditions (3). Let* A *be any real symmetric matrix. If* $y'Ay/\sigma^2_{k+1} \sim \chi^2_u$ *under the null hypothesis,* $H: \rho_i = r_i$ *for* $i = 1, 2, \ldots, \ell$, *and is independent of the residual sum of squares of the model (1), then* $\mathrm{rank}(A) \le m$. *Moreover, if* $\mathrm{rank}(A) = m$, *then* $A = C'QD_1^-(r)Q'C$.

Proof : Because $\mathrm{Var}(y)$ is a positive definite matrix and $y'Ay/\sigma^2_{k+1}$ is a central chi-square random variable under H, we have

$$A\left(\mathrm{I} + \sum_{i=1}^{\ell} r_i X_i X_i' + \sum_{i=\ell+1}^{k} \rho_i X_i X_i' \right) A = A \text{ for all } \rho_i \ge 0 \text{ and}$$

$$X_0' A X_0 = \mathbf{0}.$$

These imply that $\mathrm{col}(A) \subset \mathrm{col}^{\perp}(X_0, X_{\ell+1}, \ldots, X_k)$ where $\mathrm{col}^{\perp}(B)$ denotes the orthogonal complement of $\mathrm{col}(B)$. Since $C'C$ is the orthogonal projection matrices on column space $\mathrm{col}^{\perp}(X_0, X_{\ell+1}, \ldots, X_k)$, we have $C'CA = A$. Hence

$$\begin{aligned} A &= A\left(\mathrm{I}_n + \sum_{i=1}^{\ell} r_i X_i X_i' \right) A = AC'\left(\mathrm{I}_{n-d} + \sum_{i=1}^{\ell} r_i C X_i X_i' C' \right) CA \\ &= AC'\left(\mathrm{I} + \sum_{i=1}^{\ell} r_i V_i \right) CA = AC'(QD_1(r)Q' + QD_2Q')CA \end{aligned} \tag{13}$$

The independence between $y'Ay$ and the residual sum of squares of the model (1), which is equal to $y'C'QD_2Q'Cy$, implies

$$C'QD_2Q'CA = \mathbf{0}$$

and (13) is equivalent to

$$AC'QD_1(r)Q'CA = A. \tag{14}$$

i.e., $C'QD_1(r)QC$ is a generalized inverse of A. Hence

$$\mathrm{rank}(A) \le \mathrm{rank}(C'QD_1(r)Q'C) = m.$$

Now we assume $\mathrm{rank}(A) = m$. Let

$$D_1^* = \mathrm{diag}\,(\overbrace{1,\ldots,1}^{m},0,\ldots,0).$$

Then $I_{n-d} = QD_1^*Q' + QD_2Q'$, and

$$\begin{aligned} A &= C'CA \\ &= C'(QD_1^*Q' + QD_2Q')CA \\ &= C'QD_1^*Q'CA \qquad (15) \\ &= C'QD_1^-(r)Q'QD_1(r)Q'CA \\ &= C'QD_1^-(r)Q'CC'QD_1(r)Q'CA \end{aligned}$$

Combining the results of (14) and (15), we have

$$(C'QD_1^-(r)Q'C - A)C'QD_1(r)Q'CA = \mathbf{0}.$$

i.e., $\mathrm{col}(C'QD_1^-(r)Q'C - A) \subset \mathrm{col}^{\perp}(C'QD_1(r)Q'CA)$. Noting that $\mathrm{col}(A) \subset \mathrm{col}(C'QD_1^*Q'C)$, but $\mathrm{rank}\,(A) = m = \mathrm{rank}\,(C'QD_1^*Q'C)$ by assumption, we have

$$\begin{aligned} \mathrm{col}(A) &= \mathrm{col}(C'QD_1^*Q'C) = \mathrm{col}(C'QD_1^-(r)Q'C) \\ &= \mathrm{col}(C'QD_1(r)Q'C) = \mathrm{col}(C'QD_1^-(r)Q'CA) \end{aligned}$$

Thus

$$\mathrm{col}(C'QD_1^-(r)Q'C - A) \subset \mathrm{col}^{\perp}(C'QD_1^*Q'C),$$

and

$$\mathrm{col}(C'QD_1^-(r)Q'C - A) \subset \mathrm{col}(C'QD_1^*Q'C).$$

Hence it must be the case that

$$A = C'QD_1^-(r)Q'C.$$

Theorem 3 extends Seely and El-Bassouni(1983), and Huh and Li(1996)'s work. The theorem is useful not only for testing non-zero null-hypothesis but also is good for simultaneous testing problem.

Remark 2.2: When $\ell = 1$, the problem is to test a specific component in a general linear model. Seely and El-Bassiouni derived an exact F-test by the reduction sums of squares technique for testing problem with r_1 being specified to zero. Their test can be considered as a special case of our test.

To obtain the power of the test, we note that the first m nonzero elements of the diagonal matrix $D_1^-(r_1)$ are $1/(1+r_1\lambda_1), \ldots, 1/(1+r_1\lambda_m)$, where the λ_i's are the nonzero eigenvalues of $CX_1X_1'C'$. Thus $z'QD_1^-(r_1)Q'z/\sigma_{k+1}^2$ is distributed as $\sum_{i=1}^{m}\frac{1+\rho_1\lambda_i}{1+r_1\lambda_i}\chi_i^2$ where χ_i^2's are independent chi-square random variables with 1 degree of freedom. Since the numerator of F in (12) is increasing with ρ_1 while the denominator is independent of ρ_1, we reject the null hypothesis $H:\rho_1\le r_1$ against $K:\rho_1> r_1$ if the observed value of F is too large. Hence the power of the test is given by

$$\pi(\rho_1)=\Pr\left[\sum_{i=1}^{m}\frac{1+\rho_1\lambda_i}{1+r_1\lambda_i}\chi_i^2>\frac{m}{n-d-m}c\chi^2\right] \tag{16}$$

where χ^2 is a chi-square random variable with $n-d-m$ degrees of freedom and c is an appropriate constant so that the test is of size α. It can be observed that the power of the test depends on the design matrix through the eigenvalues of V_1. To compute the power of the test, we need to compute the probability of a linear combination of central chi-square random variables. Farebrother(1984) gives an algorithm for this problem.

Remark 2.3: When r_1 is nonzero, usual Wald procedure which is described in Harville and Fenech (1985), and Budick and Graybill(1992) require the compuation of eignvalues and eigenvectors of nontrivial matrix in contrast to Rao-Wald procedure. As we noted earlier, $z'QD_2Q'z$ in (12) is the residual sums of squares of model (1). The test statistics in (12) can be obtained easily by subtracting the residual sum of squares from $z'Rz$.

Remark 2.4: When $k=1$, test statistic (12) is based on the MINQUE of model (1). This point was argued by Huh and Li(1996). Since MINQUE is the locally minimum variance quadratic unbiased estimator, Wald test or Rao-Wald test have certain optimal properties in two variance-component model. When $k>1$, Rao-Wald test does not based on the MINQUE of model (1). It is based on the MINQUE of model (5). However, it should be noted that any test of the null hypothesis H that does not depend on nuisance parameters is only through the value of z.

Lin and Haville compared power performance of Wald test with LBI test and Neyman-Pearson test. Under their model setting LBI test and Neyman-pearson test does not contain nuisance parameter. Thus later two tests may be proposed test for the situation. However, as we mentioned before two tests have several drawbacks in more than two variance component models.

3. Numerical study of the power performance of Rao-Wald test

There have been many numerical studies of power comparisons between Wald test and other alternative tests including the LBI test. However, all the comparisons were done under the models with only one random effect. In this note we compare the power performance of the Rao-Wald test against the LBI test in two-way random effects with interaction model. Our proposed test is of the form: $H: \rho \leq r$ against $K: \rho > r$. Without loss of generality, however, we consider testing $H: \rho = 0$ against $K: \rho > 0$ where ρ is the ratio of interaction variance component to the error variance component.

For the numerical study, we consider four n_{ij} patterns as given in Table 1. We categorize (a) and (b) as "small sample designs", and (c) and (d) as "large sample designs". Also we may classify (a) and (c) as "almost balanced" designs, and (b) and (d) as "highly unbalanced" designs. For the parameter values, we consider the following combinations.

- Main effect variance ratios: $\{0.5, 1.0, 5.0\}$
- Interaction effect variance ratio: $\{0, 0.01, 0.02, \ldots, 0.30\}$

We can obtain exact power of Rao-Wald test from (16). However, we cannot obtain the critical region and power of LBI test analytically. Hence we used simulations to obtain the critical region and power of the LBI test. Our numerical study consists of 36 combinations of experiments (4 designs, and 9 combinations of main effect variance components for each design). For each experiment, we ran 31 simulations for each interaction effect variance

Table 1: four patterns for numerical study

nearly balanced						highly unbalanced					
(a)	3	2	3		8	(b)	6	2	1		9
	3	2	4		8		5	1	1		7
	3	3	2		8		6	1	1		8
	2	4	3		9		1	1	7		9
	10	11	12		33		18	15	10		33
(c)	5	4	5	6	20	(d)	8	2	8	1	19
	4	4	6	5	19		9	1	2	7	19
	5	5	4	4	18		2	8	1	2	13
	4	6	5	4	19		9	8	1	7	25
	18	19	20	19	76		28	19	12	17	76

component ratio. For each simulation, we used 100,000 repetitions to obtain the power of theoretical LBI and two-step LBI tests. Numerical study showed that the pattern of the power performance of theorectical LBI and two-step LBI tests were consistent over the 4 designs with respect to the two nuisance parameters. Among the 4 designs, we present the results for design (b) in Table 2 when $\rho_1=0.5, \rho_2=5.0$ and $\rho_1=5.0, \rho_2=0.5$. The pattern of the power performance were similar for the other three designs considered. Table 3 gives the results for all 4 designs when $\rho_1=\rho_2=1.0$.

We can observe several points from the tables.

1. Power of all three tests increases as sample size inreases.
2. Power of all three tests increases as the design gets more balanced
3. Pattern of the power performance of theoretical and two-step LBI tests remains almost same for different values of ρ_1 and $\rho_2=5.0$.
4. Power performance of two-step LBI test is quite unreliable for all the four designs considered and for all the choices of ρ (refer to Figure 1). It can be seen that the power of two-step LBI test is about 50% of the theoretical LBI test. Especially the power drops to less than 40% of the theorectical LBI test for $\rho > 0.15$ for the design (d) which is specified as "large sample size" and "highly unbalanced".

Table 2: Power of Rao-Wald, theoretical LBI and two-step LBI test for design b when $\rho_1=0.5, \rho_2=5.0$ and $\rho_1=5.0, \rho_2=0.5$

		$\rho_1=0.5, \rho_2=5.0$		$\rho_1=5.0, \rho_2=0.5$	
ρ_3	Rao-Wald	theortical LBI	two-step LBI	theortical LBI	two-step LBI
0.03	0.05933	0.06131	0.05430	0.06097	0.05338
0.06	0.06941	0.07356	0.05605	0.07302	0.05457
0.09	0.08018	0.08927	0.05972	0.08632	0.05996
0.12	0.09157	0.10380	0.06266	0.09930	0.06459
0.15	0.10351	0.11898	0.06686	0.11334	0.06726
0.18	0.11593	0.13296	0.07003	0.12685	0.07220
0.21	0.12876	0.15016	0.07426	0.14182	0.07547
0.24	0.14194	0.16393	0.07683	0.15523	0.07846
0.27	0.15540	0.17777	0.07642	0.16728	0.08222
0.30	0.16910	0.19655	0.08117	0.18675	0.08547

Table 3: Power of theoretical LBI, Rao-Wald and two-step LBI test for all 4 designs when $\rho_1 = \rho_2 = 1$

	Design (a)			Design (b)		
ρ_3	theortical LBI	Rao-Wald	two-step LBI	theortical LBI	Rao-Wald	two-step LBI
0.03	0.06507	0.06483	0.05736	0.06279	0.05933	0.05335
0.06	0.08331	0.08135	0.06432	0.07697	0.06941	0.05822
0.09	0.10209	0.09934	0.07278	0.09043	0.08018	0.06115
0.12	0.12054	0.11855	0.08122	0.10306	0.09157	0.06418
0.15	0.14164	0.13876	0.08760	0.12025	0.10351	0.06671
0.18	0.16345	0.15972	0.09486	0.13484	0.11593	0.07051
0.21	0.18652	0.18125	0.10388	0.15037	0.12876	0.07543
0.24	0.20788	0.20314	0.11137	0.16218	0.14194	0.08038
0.27	0.22974	0.22522	0.12152	0.17996	0.15540	0.08338
0.30	0.25288	0.24736	0.12949	0.19739	0.16910	0.08728

	Design (a)			Design (b)		
ρ_3	theortical LBI	Rao-Wald	two-step LBI	theortical LBI	Rao-Wald	two-step LBI
0.03	0.09038	0.08923	0.07505	0.08747	0.07810	0.06215
0.06	0.14076	0.13828	0.11139	0.13083	0.11221	0.06948
0.09	0.19716	0.19400	0.14333	0.17600	0.15080	0.08058
0.12	0.25545	0.25325	0.18454	0.22295	0.19230	0.09148
0.15	0.31554	0.31333	0.22261	0.27042	0.23530	0.10932
0.18	0.37607	0.37224	0.26073	0.31404	0.27867	0.11877
0.21	0.43099	0.42857	0.29058	0.35469	0.32153	0.14248
0.24	0.48270	0.48147	0.33012	0.39195	0.36323	0.14969
0.27	0.53119	0.53049	0.36318	0.42950	0.40336	0.16260
0.30	0.57580	0.57545	0.39584	0.46566	0.44161	0.18010

5. For design (b) and (d) which are classified as "highly unbalanced", the power performance of Rao-Wald test drops. The relative performance of Rao-Wald test, however, stays above 82% of the theoretical LBI test for all the values of ρ considered in this work.
6. Power of Rao-Wald test is generally quite close to that of theoretical LBI test for all the four designs and for all the choices of ρ that is to be tested.
7. For designs (a) and (c) which are classified as "almost balanced", the power of Rao-Wald test is almost equal to the power of the theorectical LBI test.

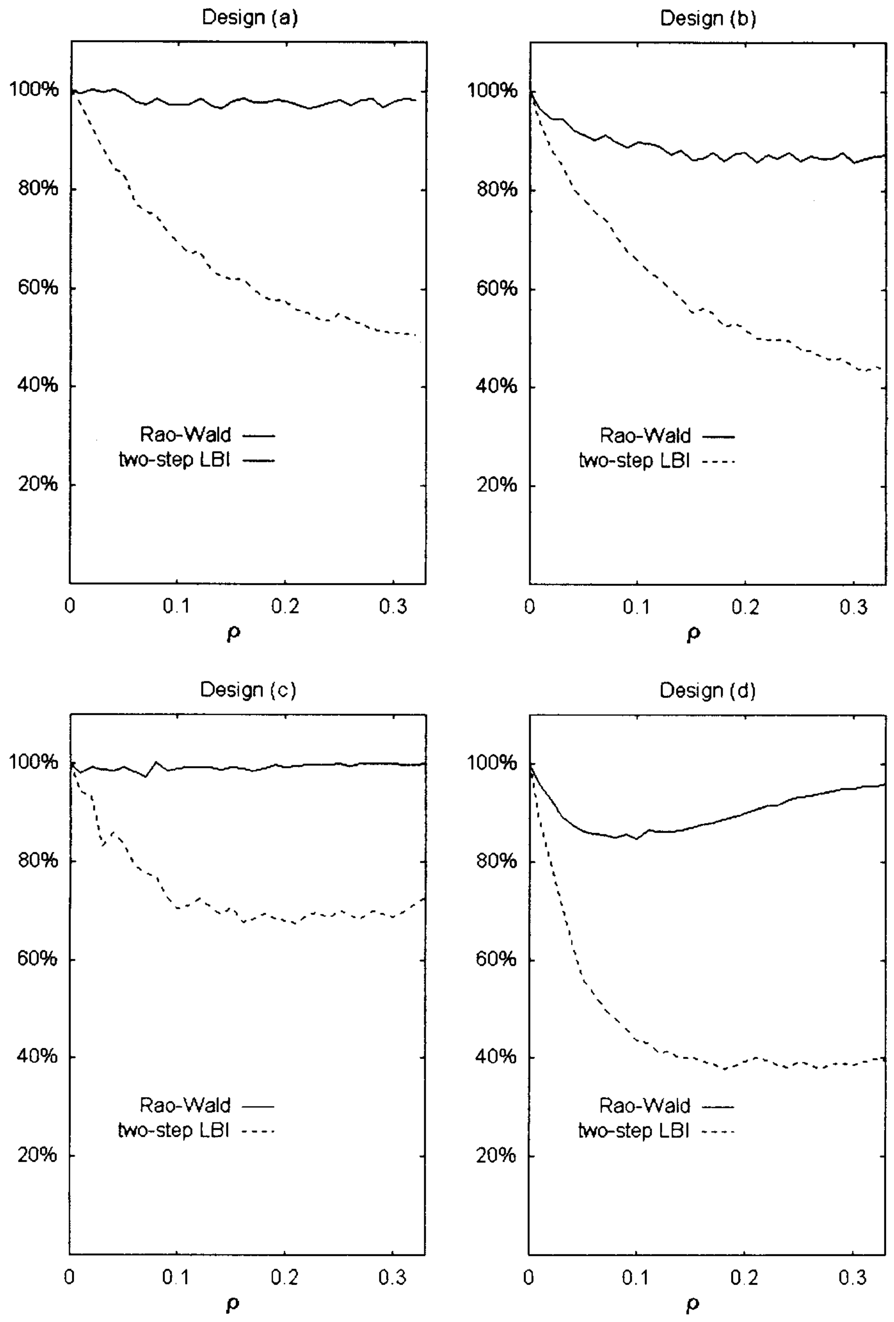

Figure 1: $100\times$ ratio of the power of Rao-Wald and two-step LBI tests over theoretical LBI test for different values of ρ when $\rho_1=\rho_2=1$

4. Conclusion

Rao-Wald test procedure suggested in this work is applicable to simultaneous test of any subset of variance ratios of a general linear model under some mild conditions. The computation of the test statistic is straightforward since the terms $z'QD_2Q'z$ in (12) is the residual sum of squares, and the terms $z'QD_1^-(r)Q'z$ can be obtained easily by subtracting the residual sums of squares from $z'Rz$. The computation of matrix R is straightforward (Huh, 1981). Also Rao-Wald test statistic follows F-distribution under the null hypothesis. Since Rao-Wald test does not depend upon the nuisance parameters, and the power performance of the test is seen to be efficient for all the designs and parameter values considered in this work, we propose that the test statistic suggested in this work can safely be applied to a general setting.

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