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Comparison of Best Invariant Estimators with Best Unbiased Estimators in Location-scale Families[1]

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Abstract

In order to estimate a parameter (α, β^r), $r \in \boldsymbol{N}$, in a distribution belonging to a location-scale family we usually use best invariant estimator (BIE) and best unbiased estimator (BUE). But in some conditions Ryu (1996) showed that BIE is better than BUE. In this paper, we calculate risks of BIE and BUE in a normal and an exponential distribution respectively and calculate a percentage risk improvement (PRI). We find the sample size n which make no significant differences between BIE and BUE in a normal distribution. And we show that BIE is always significantly better than BUE in an exponential distribution. Also, simulation in a normal distribution is given to convince us of our result.

1. Introduction

If any probability density function (pdf) g has the property that

$$g(\boldsymbol{x};\alpha,\beta) = \frac{1}{\beta^n} f\left(\frac{x_1-\alpha}{\beta}, \frac{x_2-\alpha}{\beta}, \cdots, \frac{x_n-\alpha}{\beta}\right),$$

where f is known and $\boldsymbol{\theta} = (\alpha, \beta)$ is unknown, $\alpha \in \boldsymbol{R}$, $\beta > 0$, $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ then this pdf is said to be belonged to a location-scale family. In order to estimate (α, β^r), $r \in \boldsymbol{N}$, BIE and BUE are usually used.

For a long time, some people had found the way to get BIE of location parameter α and scale parameter β in a location-scale family. Pitman (1939) found BIE of α in the conditions that loss was $L(\alpha, \delta_1) = (\alpha - \delta_1)^2$ and $\beta = 1$. And Pitman (1939) got BIE of β in the conditions that loss was $L(\beta, \delta_2) = \left(1 - \frac{\beta}{\delta_2}\right)^2$ and $\alpha = 0$. These results were respectively

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marginal BIE of α and β. Lehmann (1983) presented more general marginal estimator. He respectively calculated BIE of α and β^r, where $r \in \boldsymbol{N}$.

Prabakaran and Chandrasekar (1994) found simultaneous BIE of (α, β^r) in Q_A loss,

$$L(\alpha,\beta^r) = a_{11}\left(\frac{\alpha-\delta_1}{\beta}\right)^2 + 2a_{12}\left(\frac{\alpha-\delta_1}{\beta}\right)\left(1-\frac{\delta_2}{\beta}\right) + a_{22}\left(1-\frac{\delta_2}{\beta}\right)^2,$$

where $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22}\end{pmatrix}$ was positive definite. Let $(\delta_{01}, \delta_{02})$ be invariant estimator of (α, β^r). They defined g as $g(b(\boldsymbol{X}+a\boldsymbol{1})) = bg(\boldsymbol{X})$ and $\boldsymbol{Z}=(Z_1, Z_2, \cdots, Z_{n-1}) = \left(\frac{X_1-X_n}{g(\boldsymbol{X})}, \frac{X_2-X_n}{g(\boldsymbol{X})}, \cdots, \frac{X_{n-1}-X_n}{g(\boldsymbol{X})}\right)$. Then Prabakaran and Chandrasekar's BIE was

$$\delta^*(\boldsymbol{X}) = (\delta_1^*(\boldsymbol{X}), \delta_2^*(\boldsymbol{X})) = \left(\delta_{01} - gw_1^*, \frac{\delta_{02}}{w_2^*}\right),$$

where

$$\begin{aligned} w_1^* = & \ [a_{11}a_{22}E(\delta_{01}^2|z)E(\delta_{01}g|z) - a_{12}^2E(\delta_{02}g|z)E(\delta_{01}\delta_{02}|z) \\ & - a_{12}a_{22}\{E(\delta_{02}^2|z)E(g|z) - E(\delta_{02}g|z)E(\delta_{02}|z)\}]/ \\ & \{a_{11}a_{22}E(g^2|z)E(\delta_{02}^2|z) - a_{12}^2E(\delta_{02}g|z)\} \end{aligned}$$

and

$$\begin{aligned} 1/w_2^* = & \ [-a_{11}a_{12}(E(g^2|z)E(\delta_{01}\delta_{02}|z) - E(\delta_{02}g|z)E(\delta_{01}g|z)) \\ & + a_{11}a_{22}E(g^2|z)E(\delta_{02}|z) - a_{12}^2E(\delta_{02}g|z)E(g|z)]/ \\ & \{a_{11}a_{22}E(g^2|z)E(\delta_{02}^2|z) - a_{12}^2E(\delta_{02}g|z)\} \end{aligned}.$$

Some people concerned to show the relations of BIE and BUE. Takada (1981) showed that BIE of α was expressed by a linear combination of BUE of α and β. And Ryu (1996) showed that simultaneous BUE was simultaneous invariant estimator in some conditions that there was complete and sufficient statistic. So, we can see that simultaneous BIE of (α, β^r) is better than simultaneous BUE of (α, β^r) if there is complete and sufficient statistic. And by the definition of invariant estimator and invariant loss, risks of invariant estimators are independent of (α, β^r) and are effected by sample size n and invariant loss. If there is complete and sufficient statistic risk of BIE is always little than or equal to risk of BUE. Although BIE is better than BUE for every n, there is no significant differences for some n.

In this paper, we want to get risks of some special location-scale probabilities and find n in which risks of BIE and BUE have no significant differences. Section 2 provide simultaneous BIE and simultaneous BUE of a normal and an exponential distribution in Q_A loss and calculate the percentage risk improvement (PRI). In section 3, we find n at which risks of BIE and BUE have no significant differences. We make PRI graphs for a normal and an exponential distribution. Also for some selected n and a_{12} tables are given. In order to convince us of our result simulation is performed and the results are presented.

2. Risks of Best Invariant Estimator and Best Unbiased Estimator

In order to compare two estimator we use PRI. The definition of PRI for BIE and BUE is below :

$$\mathrm{PRI} = \left(1 - \frac{\text{Risk of BIE}}{\text{Risk of BUE}}\right) \times 100 .$$

That is, if PRI is positive then we can think that risk of BIE is little than risk of BUE. And PRI indicates degree of goodness. For example, if PRI is 40% then risk of BIE is only 60% of risk of BUE. So, we can think that BIE is better than BUE such as 40%.

This PRI is related to α, β, loss function and sample size n. If we use Q_A loss and BUE is invariant then PRI is a function of $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$ and sample size n because risk of invariant estimator is independent of parameters α, β. But we usually think that location and scale parameters are equally important. So, in loss function, we usually set $a_{11} = a_{22}$. And if $B = kA = k\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$, then the risk when Q_B loss used is k multiple of the risk when Q_A loss used. That is, we think that $B = kA$ and A give us same information for every estimators. From these reasons, we want to suppose $a_{11} = a_{22} = 1$ in calculation of risks and PRI.

In normal probability, $f_{(\alpha,\beta)}(\boldsymbol{x}) = \left(\frac{1}{2\pi\beta^2}\right)^{\frac{n}{2}} \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i - \alpha}{\beta}\right)^2\right)$, where x_i is the i-th order coordinate of $\boldsymbol{x}$, we can usually want to estimate (α, β^2). BIE is given by Prabakaran and Chandrasekar (1994). BIE for general Q_A loss is

$$(\overline{X} - k_1 S, k_2 S^2),$$

where

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$$
$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{X})^2$$

$$k_1 = a_{12}a_{22}\sqrt{\frac{n-1}{2}}\ \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n+2}{2}\right)-\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+3}{2}\right)}{a_{11}a_{22}\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n+3}{2}\right)-a_{12}^2\Gamma^2\left(\frac{n+2}{2}\right)}$$

$$k_2 = \frac{(n-1)}{2}\ \frac{a_{11}a_{22}\Gamma^2\left(\frac{n+1}{2}\right)-a_{12}^2\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+2}{2}\right)}{a_{11}a_{22}\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n+3}{2}\right)-a_{12}^2\Gamma^2\left(\frac{n+2}{2}\right)}.$$

Rao-Blackwell theorem is usually proved with squared error loss. But Rao-Blackwell theorem is true with Q_A loss. The proof with Q_A loss is very similar to the proof with squared error loss. So, we can see the way to find the marginal BUE with squared error loss can be used to get simultaneous BUE in Q_A loss. Thus, BUE for general Q_A loss is

$$(\overline{X}, S^2).$$

Since we know BIE and BUE, we can calculate risks of BIE and BUE in Q_A loss, where $A=\begin{pmatrix} 1 & a_{12} \\ a_{12} & 1 \end{pmatrix}$. The reason of assumption is explained above.

$$\begin{aligned}
&\text{Risk of BIE} \\
&= R((\alpha,\beta),(\overline{X}-k_1S, k_2S^2)) \\
&= R((0,1),(\overline{X}-k_1S, k_2S^2)) \\
&= E[(\overline{X}-k_1S)^2+2a_{12}(\overline{X}-k_1S)(k_2S^2-1)+(k_2S^2-1)^2 \mid z].
\end{aligned}$$

Since z is an ancillary statistic and $(\overline{X}, S^2)$ is minimal sufficient statistic, by Basu's theorem $(\overline{X}, S^2)$ is independent of z.

$$\begin{aligned}
&E[(\overline{X}-k_1S)^2+2a_{12}(\overline{X}-k_1S)(k_2S^2-1)+(k_2S-1)^2 \mid z] \\
&= E[(\overline{X}-k_1S)^2]+2a_{12}E[(\overline{X}-k_1S)(k_2S^2-1)]+E[(k_2S-1)^2] \\
&= \frac{(3n+1)(n-1)^2\Gamma\left(\frac{n-1}{2}\right)^2-2n(3n-1)a_{12}^2\Gamma\left(\frac{n}{2}\right)^2}{(n-1)^2n(n+1)\Gamma\left(\frac{n-1}{2}\right)^2-2n^3a_{12}^2\Gamma\left(\frac{n}{2}\right)^2}.
\end{aligned}$$

The last calculation is simple but very tedious. So, we use computer software, "Mathematica" version 2.2.3 for windows.

We can easily see that BUE is invariant. So, risk of BUE is independent of (α,β). And we use Q_A loss, where $A=\begin{pmatrix} 1 & a_{12} \\ a_{12} & 1 \end{pmatrix}$.

$$\begin{aligned}
&\text{Risk of BUE} \\
&= R((\alpha,\beta),(\overline{X}, S^2)) \\
&= R((0,1),(\overline{X}, S^2)) \\
&= E[(\overline{X})^2+2a_{12}(\overline{X})(S^2-1)+(S^2-1)^2 \mid z] \\
&= E[(\overline{X})^2]+2a_{12}E[(\overline{X})(S^2-1)]+E[(S^2-1)^2] \\
&= \frac{3n-1}{n(n-1)}.
\end{aligned}$$

Above calculation indicates that risk of BUE is independent of a_{12}.

We can get PRI with risk of BIE and BUE. The calculation of PRI is done by "Mathematica" version 2.2.3 for windows. PRI in normal case is

$$PRI=\frac{200n\left(2(n-1)^2\Gamma(\frac{n-1}{2})^2-(3n-1)a_{12}^2\Gamma(\frac{n}{2})^2\right)}{(3n-1)\left((n-1)^2(n+1)\Gamma(\frac{n-1}{2})^2-2n^2a_{12}^2\Gamma(\frac{n}{2})^2\right)}.$$

In exponential probability, $f_{(\alpha,\beta)}(\boldsymbol{x})=\frac{1}{\beta}\exp\left(-\frac{1}{\beta}\Pi_{i=1}^n(x_i-\alpha)\right)$, where $x_i\geq\alpha$, $\alpha\in R$, $\beta>0$ and x_i is the i-th order coordinate of $\boldsymbol{x}$, we can usually want to estimate (α,β). This pdf is sometimes called as two parameter exponential probability. We represent the r-th order statistic in $\boldsymbol{X}$ as $X_{(r)}$. BIE is

$$\left(\frac{n+1}{n}X_{(1)}-\frac{1}{n}\overline{X},\overline{X}-X_{(1)}\right),$$

which is given by Prabakaran and Chandrasekar (1994). And BUE is

$$\left(\frac{1}{n-1}(nX_{(1)}-\overline{X}),\frac{n}{n-1}(\overline{X}-X_{(1)})\right),$$

which is given by Epstein and Sobel (1954).

Risks of BIE and BUE in Q_A is calculated below. In this calculation, we suppose $A=\begin{pmatrix}1 & a_{12}\\ a_{12} & 1\end{pmatrix}$.

$$\begin{aligned}
&\text{Risk of BIE}\\
&=R\left((\alpha,\beta),(\frac{n+1}{n}X_{(1)}-\frac{1}{n}\overline{X},\overline{X}-X_{(1)})\right)\\
&=R\left((0,1),(\frac{n+1}{n}X_{(1)}-\frac{1}{n}\overline{X},\overline{X}-X_{(1)})\right)\\
&=E\Big[(\frac{n+1}{n}X_{(1)}-\frac{1}{n}\overline{X})^2\\
&\qquad+2a_{12}(\frac{n+1}{n}X_{(1)}-\frac{1}{n}\overline{X})(\{\overline{X}-X_{(1)}\}-1)\\
&\qquad+(\{\overline{X}-X_{(1)}\}-1)^2\mid \boldsymbol{z}\Big]\\
&=\frac{n^2+1-2na_{12}}{n^4}
\end{aligned}$$

Since $(\overline{X},X_{(1)})$ is minimal sufficient statistic in an exponential distribution, $(\overline{X},X_{(1)})$ is independent of $\boldsymbol{z}$. In the above calculation we use the fact that $(\overline{X},X_{(1)})$ is independent of $\boldsymbol{z}$ and calculate with computer software like normal case. Because BUE is invariant, we can think risk of BUE is independent of (α,β). So, we set $(\alpha,\beta)=(0,1)$.

Risk of BUE

$$= R\Big((\alpha,\beta),\Big(\frac{1}{n-1}(nX_{(1)}-\overline{X}),\frac{n}{n-1}(\overline{X}-X_{(1)})\Big)\Big)$$
$$= R\Big((0,1),\Big(\frac{1}{n-1}(nX_{(1)}-\overline{X}),\frac{n}{n-1}(\overline{X}-X_{(1)})\Big)\Big)$$
$$= E\Big[(\frac{1}{n-1}(nX_{(1)}-\overline{X}))^2 + 2a_{12}(\frac{1}{n-1}(nX_{(1)}-\overline{X}))(\frac{n}{n-1}(\overline{X}-X_{(1)})-1) + (\frac{n}{n-1}(\overline{X}-X_{(1)})-1)^2 \mid z\Big]$$
$$= \frac{4}{(n-1)^2} .$$

This result is calculated by "Mathematica" version 2.2.3 for windows. And we can calculate PRI in an exponential distribution.

$$\text{PRI in exponental distribution} = 25\frac{3n^4+2n^3-2n^2+2n-1+2n(n-1)^2a_{12}}{n^4} .$$

3. Percentage Risk Improvement in Some Conditions

In the above section, we suppose $a_{11}=a_{22}=1$ and explain the reason of assumption. Then an invariant risk is only related to a_{12} and n. So, we can draw 3 dimensional graphs of PRI. Two axes are for a_{12} and n. The other axis is for PRI at a_{12} and n. n is an integer which is greater than 2 or equal to 2. And since A is positive definite, $a_{11}a_{22}-a_{12}^2=1-a_{12}^2>0$. That is, a_{12} is any real number which is greater than -1 and less than 1.

In normal case, we draw Figure 1 with computer software Mathematica. For fixed n, PRI has "U" shape with relation to a_{12}, that is, PRI increases as a_{12} approaches 1 or -1. For fixed a_{12}, PRI has "L" shape with relation to n. This result for fixed n is very obvious and indicates that PRI has limit 0 as n increases. So, we can find that an appropriate n can be chosen for no significant differences. In order for PRI to be less than 5%, n is usually greater than 30. But when a_{12} approaches 1 or -1, then n must be greater than 60. Table 1 shows us PRI values for some fixed n and a_{12}. This Table 1 will be compared with Table 2 which we get in the result of simulation.

In exponential case, we draw Figure 2 with computer software Mathematica. For fixed n, PRI has "L" shape with relation to a_{12}. For fixed a_{12}, PRI has "L" shape with relation to n. But PRI has the limit 75% as n increases for every a_{12}. Thus there is no n at which BIE and BUE have significant differences. BIE is always significantly better than BUE. And the limit 75% of PRI indicates that risk of BIE is only 25% of risk of BUE.

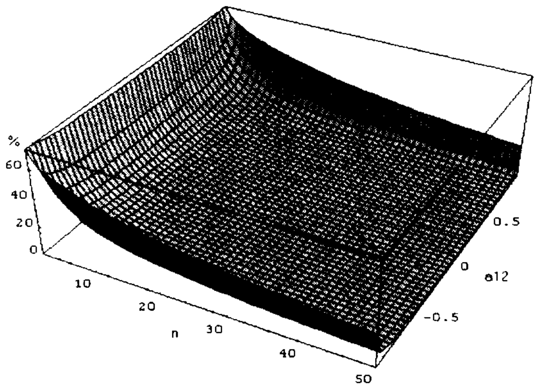

Figure 1 : Normal case

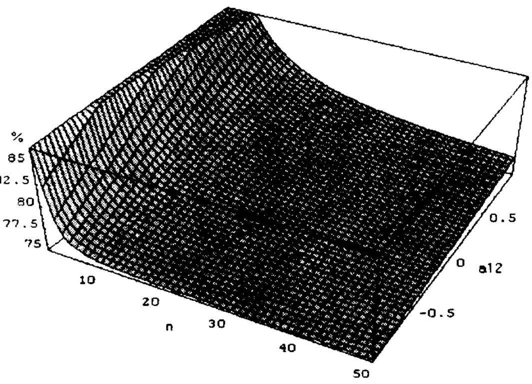

Figure 2 : Exponential case

In order to convince us of our result in normal case. First, we select n and a_{12}. For selected n and a_{12}, 1000 data are observed from normal probability with $\alpha=0$ and $\beta=1$. And we calculate loss for BIE and BUE. Second first step is repeated for 2000 times, and L_i is defined the i-th loss. Define R= $\frac{1}{2000}\sum_{i=1}^{2000} L_i$ for each estimators and use R as risk of each estimators. So, we can calculate PRI for selected n and a_{12}. Third, for other selected n and a_{12}, first and second step are repeated. As the result we make Figure 3 and Table 2. Simulation shows us that our result is true because Figure 3 and Table 2 are very similar to respectively Figure 1 and Table 1. Because PRI in normal case is dependent on a_{12}^2, PRI has same values at $a_{12}=t$ and $a_{12}=-t$ in Table 1.

n \ a_{12}	-10/11	-8/11	-6/11	-4/11	-2/11	0	2/11	4/11	6/11	8/11	10/11
2	61.17	56.05	54.46	53.75	53.43	53.33	53.43	53.75	54.46	56.05	61.17
6	31.87	23.57	21.51	20.66	20.28	20.17	20.28	20.66	21.51	23.57	31.87
10	22.08	15.13	13.55	12.91	12.62	12.54	12.62	12.91	13.55	15.13	22.08
14	16.93	11.16	9.90	9.39	9.17	9.11	9.17	9.39	9.90	11.16	16.93
18	13.73	8.84	7.80	7.39	7.20	7.15	7.20	7.39	7.80	8.84	13.73
22	11.55	7.32	6.44	6.09	5.93	5.89	5.93	6.09	6.44	7.32	11.55
26	9.97	6.25	5.48	5.17	5.04	5.00	5.04	5.17	5.48	6.25	9.97
30	8.77	5.45	4.77	4.50	4.38	4.35	4.38	4.50	4.77	5.45	8.77
34	7.83	4.83	4.22	3.98	3.88	3.85	3.88	3.98	4.22	4.83	7.83
38	7.07	4.34	3.79	3.57	3.48	3.45	3.48	3.57	3.79	4.34	7.07
42	6.45	3.94	3.44	3.24	3.15	3.13	3.15	3.24	3.44	3.94	6.45
46	5.92	3.61	3.14	2.96	2.88	2.86	2.88	2.96	3.14	3.61	5.92
50	5.48	3.33	2.90	2.73	2.65	2.63	2.65	2.73	2.90	3.33	5.48
54	5.10	3.09	2.68	2.53	2.46	2.44	2.46	2.53	2.68	3.09	5.10
58	4.76	2.88	2.50	2.36	2.29	2.27	2.29	2.36	2.50	2.88	4.76
62	4.47	2.70	2.34	2.21	2.15	2.13	2.15	2.21	2.34	2.70	4.47
66	4.21	2.54	2.20	2.07	2.02	2.00	2.02	2.07	2.20	2.54	4.21
70	3.98	2.39	2.08	1.96	1.90	1.89	1.90	1.96	2.08	2.39	3.98

Table 1 : Normal case

n \ a_{12}	-10/11	-8/11	-6/11	-4/11	-2/11	0	2/11	4/11	6/11	8/11	10/11
2	65.17	69.17	40.30	77.59	40.63	59.67	67.45	51.30	46.26	67.55	59.80
6	30.10	26.90	22.67	14.05	20.95	21.66	20.66	22.02	22.44	22.74	29.83
10	23.44	13.70	12.21	13.14	17.49	10.56	17.79	10.79	17.53	13.75	12.77
14	17.50	10.94	12.79	8.27	7.65	7.51	8.32	11.88	11.35	8.50	21.54
18	14.74	7.80	10.66	6.40	7.18	6.09	7.24	6.60	8.32	4.95	13.15
22	16.08	8.40	6.38	7.39	7.39	7.27	4.05	5.89	5.69	9.46	16.79
26	7.50	9.49	6.40	4.29	5.34	6.39	4.04	4.09	4.65	5.95	7.55
30	8.38	6.32	5.84	4.93	4.68	4.66	5.71	4.93	5.76	5.76	8.23
34	8.24	5.05	2.59	3.91	3.60	4.93	3.65	3.33	4.00	5.19	8.57
38	4.82	4.96	3.76	3.08	3.81	3.42	2.30	3.28	3.18	5.11	6.89
42	5.02	3.94	3.98	2.27	2.46	2.36	3.62	3.16	4.32	3.37	5.89
46	8.04	2.90	2.51	2.91	3.80	3.00	2.79	3.63	3.32	3.60	6.87
50	6.91	3.40	2.91	2.85	1.62	3.01	2.70	2.13	2.64	3.14	5.53
54	5.22	3.40	2.29	2.50	2.23	2.52	2.76	3.03	3.04	3.78	6.23
58	4.27	1.37	2.91	1.98	2.44	1.03	2.72	2.18	2.90	2.54	3.74
62	6.04	0.89	2.32	3.28	1.87	3.14	2.39	2.83	2.47	3.62	5.34
66	3.73	2.03	1.86	2.13	1.94	1.80	1.99	1.35	2.30	3.21	4.33
70	4.27	2.62	2.14	3.25	1.58	1.86	2.26	2.03	2.29	3.73	3.31

Table 2 : Simulation for normal case

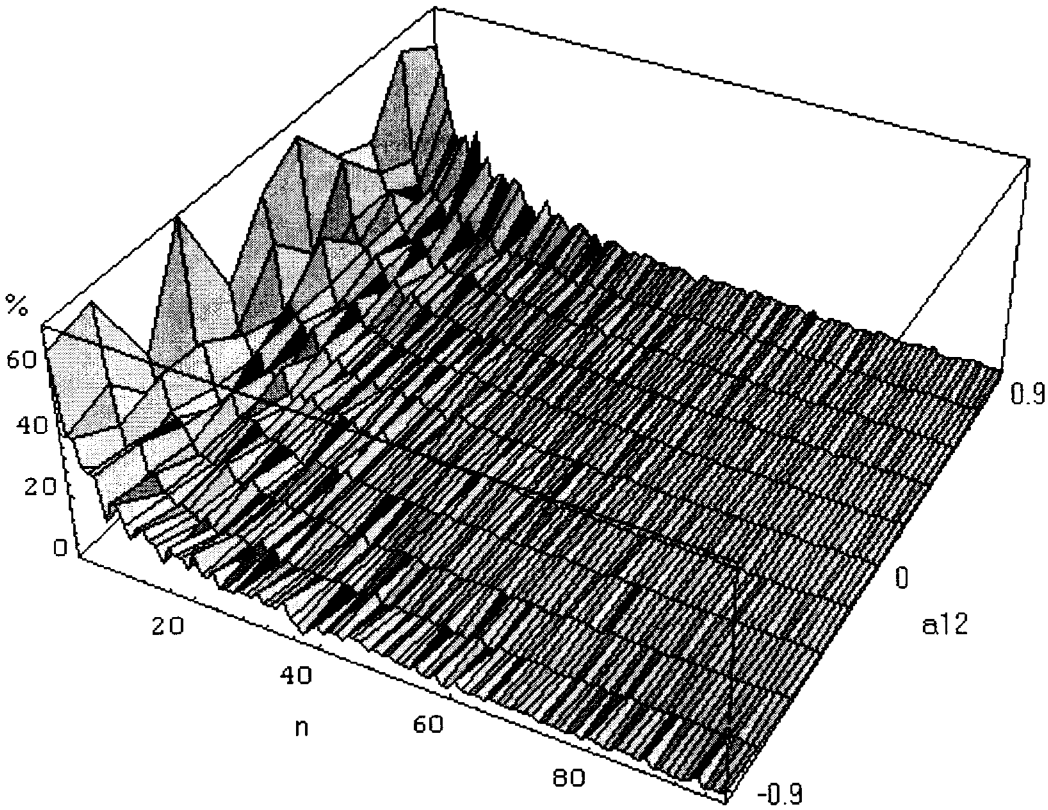

Figure 3 : Simulation for normal case

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