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Estimations of the Minimum and Maximum for Two Generalized Uniform Scale Parameters

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Abstract

We shall derive several estimators for the minimum and maximum of two generalized uniform scale parameters with a common known shape parameter when the order of the scales is unknown and sample sizes are equal. Also, we shall obtain the biases and mean squared errors for the proposed several estimators, and compare numerically performances for the proposed several estimators.

1. Introduction

A generalized uniform distribution is given by

$$f(x: \alpha, \beta) = \frac{\alpha+1}{\beta^{\alpha+1}} x^{\alpha}, \quad 0 < x < \beta, \ -1 < \alpha, \tag{1.1}$$

where α and β are referred as the shape and scale parameters, respectively. It is denoted it by $X \sim GUNIF(\alpha, \beta)$. Proctor(1987) and Tiwari, Yang & Zalkikar(1996) introduced the generalized uniform distribution. Here, we shall investigate the problem of estimating the minimum and maximum of two generalized uniform scale parameters with a common known shape parameter when the order of the scales is unknown.

Elfessi & Pal(1992) and Misra, Anand & Singh(1993) considered the estimation of the smaller and larger of the two uniform scale parameters under the ordering among two scale parameters is unknown. Mitra, Kundu, Dhariyal & Misra(1994) considered the problem of estimating the ratio of the smaller and larger sclae parameters of two uniform distributions. Misra & Dhariyal(1995) studied the problem of estimating ordered restricted scale parameters of k uniform distributions under the assumption that the correct ordering among the parameters is known apriori. Woo & Lee(1995) introduced estimators of the smaller and larger of two Pareto scale parameters with a common known shape parameter and compared efficiences for proposed estimators.

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In this paper, we shall derive several estimators of the minimum and maximum for two generalized uniform scale parameters with a common known shape parameter when the order of the scales is unknown and sample sizes are equal. Also, we obtain the biases and mean squared errors(MSE) for the proposed estimators, and compare numerically performances for the proposed several estimators

2. Estimations of the minimum and maximum scale parameters

Let $X_{i1},\cdots,X_{in}$, $i=1,2$ be independent random samples from two independent populations which are generalized uniformly distributed with scale parameters β_i, $i=1,2$ and a common known shape parameter α. Define $\theta_1=\min(\beta_1,\beta_2)$ and $\theta_2=\max(\beta_1,\beta_2)$. Our problem is estimation of θ_1 and θ_2.

Note that complete and sufficient statisitcs for β_i, i=1,2, are $X_{i(n)}=\max\{X_{i1},\cdots,X_{in}\}$. So our goal is to estimate θ_1 and θ_2 based on $X_{1(n)}$ and $X_{2(n)}$. Define $Z_1=\min\{X_{1(n)},X_{2(n)}\}$ and $Z_2=\max\{X_{1(n)},X_{2(n)}\}$. Since the MLE's for β_1 and β_2 are $X_{1(n)}$ and $X_{2(n)}$, respectively, the MLE's for θ_1 and θ_2 are

$$\hat{\theta}_1^{(1)}=Z_1 \quad \text{and} \quad \hat{\theta}_2^{(1)}=Z_2, \qquad \text{respectively.} \tag{2.1}$$

Since $X_{i(n)}$ follows $GUNIF(n\alpha,\beta_i)$ and $X_{1(n)}$ and $X_{2(n)}$ are independent, we can obtain the joint density of (Z_1,Z_2) and marginal densities of Z_1 and Z_2 as follows :

$$\text{(a)} \quad f(z_1,z_2)=\begin{cases} 2(n(\alpha+1))^2(\theta_1\cdot\theta_2)^{-n(\alpha+1)}(z_1\cdot z_2)^{n(\alpha+1)-1}, & 0\le z_1\le z_2\le\theta_1 \\ (n(\alpha+1))^2(\theta_1\cdot\theta_2)^{-n(\alpha+1)}(z_1\cdot z_2)^{n(\alpha+1)-1}, & 0\le z_1\le\theta_1\le z_2\le\theta_2. \end{cases}$$

$$\text{(b)} \quad \begin{aligned} f(z_1)= & \ n(\alpha+1)(\theta_1^{-n(\alpha+1)}+\theta_2^{-n(\alpha+1)})z_1^{n(\alpha+1)-1} \\ & -2n(\alpha+1)(\theta_1\cdot\theta_2)^{-n(\alpha+1)}z_1^{2n(\alpha+1)-1}, \quad 0\le z_1\le\theta_1, \end{aligned} \tag{2.2}$$

$$f(z_2)=\begin{cases} 2n(\alpha+1)(\theta_1\cdot\theta_2)^{-n(\alpha+1)}\cdot z_2^{2n(\alpha+1)-1}, & 0\le z_2\le\theta_1 \\ n(\alpha+1)\theta_2^{-n(\alpha+1)}z_2^{n(\alpha+1)-1}, & \theta_1\le z_2\le\theta_2. \end{cases}$$

From (2.2), we can obtain biases and MSE's for the $\hat{\theta}_1^{(1)}$ and $\hat{\theta}_2^{(1)}$ as follows :

$$BIAS[\hat{\theta}_1^{(1)}]=-\left(\frac{1}{n(\alpha+1)+1}+\frac{n(\alpha+1)}{(n(\alpha+1)+1)(2n(\alpha+1)+1)}\delta^{n(\alpha+1)}\right)\cdot\theta_1,$$

$$BIAS[\ \widehat{\theta}_2^{(1)}] = -\left(\frac{1}{n(\alpha+1)+1} - \frac{n(\alpha+1)}{(n(\alpha+1)+1)(2n(\alpha+1)+1)}\delta^{n(\alpha+1)+1}\right)\cdot\theta_2,$$

$$MSE[\ \widehat{\theta}_1^{(1)}] = \frac{2}{(n(\alpha+1)+1)(n(\alpha+1)+2)}\theta_1^2 + \frac{3n(\alpha+1)}{(n(\alpha+1)+1)(n(\alpha+1)+2)(2n(\alpha+1)+1)}\theta_1^2\delta^{n(\alpha+1)}, \tag{2.3}$$

$$MSE[\ \widehat{\theta}_2^{(1)}] = \frac{2}{(n(\alpha+1)+1)(n(\alpha+1)+2)}\theta_2^2 - \frac{2n(\alpha+1)}{(n(\alpha+1)+1)(2n(\alpha+1)+1)}\theta_2^2\delta^{n(\alpha+1)+1} + \frac{n(\alpha+1)}{(n(\alpha+1)+1)(n(\alpha+1)+2)}\theta_2^2\delta^{n(\alpha+1)+2},$$

where $\delta = \theta_1/\theta_2$.

We have known in Woo & Lee(1998) that $(n(\alpha+1)+2)X_{i(n)}/(n(\alpha+1)+1)$, $i=1,2$, are minimum risk estimators(MRE) for β_i. So we shall propose the following estimators for β_1 and β_2 as following :

$$\widehat{\theta}_1^{(2)} = \frac{n(\alpha+1)+2}{n(\alpha+1)+1}Z_1 \quad \text{and} \quad \widehat{\theta}_2^{(2)} = \frac{n(\alpha+1)+2}{n(\alpha+1)+1}Z_2. \tag{2.4}$$

From (2.2), the biases and MSE's for the $\widehat{\theta}_1^{(2)}$ and $\widehat{\theta}_2^{(2)}$ are

$$BIAS[\ \widehat{\theta}_1^{(2)}] = -\frac{1}{(n(\alpha+1)+1)^2}\theta_1 - \frac{n(\alpha+1)(n(\alpha+1)+2)}{(n(\alpha+1)+1)^2(2n(\alpha+1)+1)}\theta_1\cdot\delta^{n(\alpha+1)},$$

$$BIAS[\ \widehat{\theta}_2^{(2)}] = -\frac{1}{(n(\alpha+1)+1)^2}\theta_2 + \frac{n(\alpha+1)(n(\alpha+1)+2)}{(n(\alpha+1)+1)^2(2n(\alpha+1)+1)}\theta_2\cdot\delta^{n(\alpha+1)+1},$$

$$MSE[\ \widehat{\theta}_1^{(2)}] = \frac{1}{(n(\alpha+1)+1)^2}\theta_1^2 + \frac{n(\alpha+1)(n(\alpha+1)+2)}{(n(\alpha+1)+1)^3(2n(\alpha+1)+1)}\theta_1^2\delta^{n(\alpha+1)}, \tag{2.5}$$

$$MSE[\ \widehat{\theta}_2^{(2)}] = \frac{1}{(n(\alpha+1)+1)^2}\theta_2^2 - \frac{2n(\alpha+1)(n(\alpha+1)+2)}{(n(\alpha+1)+1)^2(2n(\alpha+1)+1)}\theta_2^2\delta^{n(\alpha+1)+1} + \frac{n(\alpha+1)(n(\alpha+1)+2)}{(n(\alpha+1)+1)^3}\theta_2^2\delta^{n(\alpha+1)+2}.$$

Now we shall consider the class of estimators of θ_i given as

$$T_i = \{\ \widehat{\theta}_{i(c)} = c\cdot Z_i\ ;\ c>0\},\quad i=1,2. \tag{2.6}$$

From (2.2), the biases and MSE's for $\widehat{\theta}_{i(c)}$, $i=1,2$, are

$$BIAS[\ \hat{\theta}_{1(c)}] = \theta_1\Big[\frac{cn(a+1)}{n(a+1)+1} - 1 - \frac{cn(a+1)}{(n(a+1)+1)(2n(a+1)+1)} \cdot \delta^{n(a+1)}\Big],$$

$$BIAS[\ \hat{\theta}_{2(c)}] = \theta_2\Big[\frac{cn(a+1)}{n(a+1)+1} - 1 + \frac{cn(a+1)}{(n(a+1)+1)(2n(a+1)+1)} \cdot \delta^{n(a+1)+1}\Big],$$

$$\begin{aligned} MSE[\ \hat{\theta}_{1(c)}] = & \Big[\frac{c^2n(a+1)}{(n(a+1)+2)} - \frac{2cn(a+1)}{(n(a+1)+1)} + 1\Big]\theta_1^2 \\ & + \Big[\frac{2}{(2n(a+1)+1)} - \frac{c}{(n(a+1)+2)}\Big]\frac{cn(a+1)}{(n(a+1)+1)}\theta_1^2\delta^{n(a+1)}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} MSE[\ \hat{\theta}_{2(c)}] = & \Big[\frac{c^2n(a+1)}{(n(a+1)+2)} - \frac{2cn(a+1)}{(n(a+1)+1)} + 1\Big]\theta_2^2 \\ & - \frac{2cn(a+1)}{(n(a+1)+1)(2n(a+1)+1)}\theta_2^2\delta^{n(a+1)+1} \\ & + \frac{c^2n(a+1)}{(n(a+1)+1)(n(a+1)+2)}\theta_2^2\delta^{n(a+1)+2}. \end{aligned}$$

From (2.7), it is impossible to find a $c_o > 0$ such that $|BIAS(\ \hat{\theta}_{i(c_o)})|$ is minimum in the class T_i for all δ. A minimax approach can be taken to find an optimal value of $c_o\,(>0)$ so that

$$\max_{0<\delta\le 1} |BIAS(\ \hat{\theta}_{i(c_o)})| = \min_{c>0}\ \max_{0<\delta\le 1} |BIAS(\ \hat{\theta}_{i(c)})|. \tag{2.8}$$

Then the minimax absolute bias estimators for θ_1 and θ_2 are given as follows ;

$$\hat{\theta}_1^{(3)} = \frac{2(n(a+1)+1)(2n(a+1)+1)}{n(a+1)(4n(a+1)+1)}Z_1, \quad \hat{\theta}_2^{(3)} = \frac{2(n(a+1)+1)(2n(a+1)+1)}{n(a+1)(4n(a+1)+3)}Z_2. \tag{2.9}$$

From (2.7), the biases and MSE's for $\hat{\theta}_1^{(3)}$ and $\hat{\theta}_2^{(3)}$ can be obtained as follows ;

$$BIAS[\ \hat{\theta}_1^{(3)}] = \frac{1}{(4n(a+1)+1)}\theta_1 - \frac{2}{(4n(a+1)+1)}\theta_1\delta^{n(a+1)},$$

$$BIAS[\ \hat{\theta}_2^{(3)}] = -\frac{1}{(4n(a+1)+3)}\theta_2 + \frac{2}{(4n(a+1)+3)}\theta_2\delta^{n(a+1)+1},$$

$$\begin{aligned} MSE[\ \hat{\theta}_1^{(3)}] = & \Big\{\frac{4(n(a+1)+1)^2(2n(a+1)+1)^2}{n(a+1)(n(a+1)+2)(4n(a+1)+1)^2} - \frac{4n(a+1)+3}{4n(a+1)+1}\Big\}\theta_1^2 \\ & + \frac{4}{4n(a+1)+1}\Big\{1 - \frac{(n(a+1)+1)(2n(a+1)+1)^2}{n(a+1)(n(a+1)+2)(4n(a+1)+1)}\Big\}\theta_1^2\delta^{n(a+1)}, \end{aligned} \tag{2.10}$$

$$MSE[\ \widehat{\theta}_2^{(3)}] = \left\{\frac{4(n(\alpha+1)+1)^2(2n(\alpha+1)+1)^2}{n(\alpha+1)(n(\alpha+1)+2)(4n(\alpha+1)+3)^2} - \frac{4n(\alpha+1)+1}{4n(\alpha+1)+3}\right\}\theta_2^2$$
$$-\frac{4}{4n(\alpha+1)+3}\theta_2^2\delta^{n(\alpha+1)+1} + \frac{4(n(\alpha+1)+1)(2n(\alpha+1)+1)^2}{(n(\alpha+1)+2)(4n(\alpha+1)+3)^2}\theta_2^2\delta^{n(\alpha+1)+2}.$$

Again, there does not exist any $c_o > 0$ such that $MSE(\ \widehat{\theta}_{i(c_o)})$ is minimum in the class T_i for all δ. Therefore we can choose $c_o(>0)$ such that

$$\max_{0<\delta\le 1} MSE(\ \widehat{\theta}_{i(c_o)}) = \min_{c>0}\ \max_{0<\delta\le 1} MSE(\ \widehat{\theta}_{i(c)}). \tag{2.11}$$

Then the minimax mean squared error estimators for θ_1 and θ_2 are

$$\widehat{\theta}_1^{(4)} = \frac{2(n(\alpha+1)+2)}{2n(\alpha+1)+1} Z_1 \quad \text{and} \quad \widehat{\theta}_2^{(4)} = \frac{n(\alpha+1)+2}{n(\alpha+1)+1} Z_2, \tag{2.12}$$

where $\widehat{\theta}_2^{(4)}$ is the same estimator given by (2.4).

From (2.7), the bias and MSE for $\widehat{\theta}_1^{(4)}$ are

$$BIAS[\ \widehat{\theta}_1^{(4)}] = \frac{n(\alpha+1)-1}{(n(\alpha+1)+1)(2n(\alpha+1)+1)}\theta_1 - \frac{2n(\alpha+1)(n(\alpha+1)+2)}{(n(\alpha+1)+1)(2n(\alpha+1)+1)^2}\theta_1\delta^{n(\alpha+1)},$$

$$MSE[\ \widehat{\theta}_1^{(4)}] = \left\{1 - \frac{4n^2(\alpha+1)^2(n(\alpha+1)+2)}{(n(\alpha+1)+1)(2n(\alpha+1)+1)^2}\right\}\theta_1^2. \tag{2.13}$$

From results (2.5), (2.7), (2.10) and (2.13), the estimators $\widehat{\theta}_1^{(i)}$ and $\widehat{\theta}_2^{(i)}, i=1,2,3,4,$ are asymptotically unbiased and consistent estimators for θ_1 and θ_2, respectively. Also, from the results (2.5), (2.7), (2.10) and (2.13) and from Tables, we get the following results.

a) For $\alpha=0$, our results are the same results in Elfessi and Pal(1992).

b) For $\alpha=1/2$, $\widehat{\theta}_1^{(2)}$ and $\widehat{\theta}_2^{(2)}$ for minimum and maximum of two scale parameters are more efficient than the other estimators for small values of δ, respectively. But as δ approaches at 1, $\widehat{\theta}_1^{(3)}$ and $\widehat{\theta}_2^{(3)}$ for minimum and maximum of two scale parameters are more efficient than the other estimators, respectively.

c) For $\alpha=-1/2$, $\widehat{\theta}_1^{(2)}$ for minimum of two scale parameters is more efficient than the other estimators for small values of δ. But as δ approaches at 1, $\widehat{\theta}_1^{(4)}$ is more efficient than the other estimators. And $\widehat{\theta}_2^{(3)}$ for maximum of two scale parameters is more efficient than the other estimators for all values of δ.

Table 1.1 The MSE's for the proposed estimators of the minimum of the two generalized uniform scale parameters ($\alpha = 1/2$).

δ	n	MSE			
		$\hat{\theta}_1^{(1)}$	$\hat{\theta}_1^{(2)}$	$\hat{\theta}_1^{(3)}$	$\hat{\theta}_1^{(4)}$
0.90	10	0.00684	0.00349	0.00362	0.00400
	15	0.00300	0.00153	0.00162	0.00184
	20	0.00168	0.00086	0.00092	0.00106
	25	0.00108	0.00055	0.00059	0.00068
0.95	10	0.00886	0.00436	0.00421	0.00446
	15	0.00386	0.00189	0.00190	0.00205
	20	0.00210	0.00104	0.00106	0.00118
	25	0.00131	0.00065	0.00068	0.00076
0.99	10	0.01170	0.00552	0.00487	0.00484
	15	0.00539	0.00249	0.00223	0.00223
	20	0.00305	0.00140	0.00127	0.00128
	25	0.00194	0.00089	0.00081	0.00083

Table 1.2 The MSE's for the proposed estimators of the minimum of the two generalized uniform scale parameters ($\alpha = -1/2$).

δ	n	MSE			
		$\hat{\theta}_1^{(1)}$	$\hat{\theta}_1^{(2)}$	$\hat{\theta}_1^{(3)}$	$\hat{\theta}_1^{(4)}$
0.90	10	0.05410	0.01919	0.02835	0.02900
	15	0.02646	0.01387	0.01365	0.01433
	20	0.01532	0.00790	0.00795	0.00851
	25	0.00987	0.00505	0.00517	0.00563
0.95	10	0.06564	0.03535	0.03197	0.03232
	15	0.03305	0.01694	0.01561	0.01596
	20	0.01952	0.00977	0.00916	0.00948
	25	0.01272	0.00627	0.00599	0.00627
0.99	10	0.07693	0.04095	0.03512	0.03510
	15	0.04010	0.02015	0.01741	0.01734
	20	0.02444	0.01190	0.01036	0.01030
	25	0.01638	0.00782	0.00686	0.00681

Table 2.1 The MSE's for the proposed estimators of the maximum of the two generalized uniform scale parameters ($\alpha = 1/2$).

		MSE		
δ	n	$\widehat{\theta_2}^{(1)}$	$\widehat{\theta_2}^{(2)}$	$\widehat{\theta_2}^{(3)}$
0.90	10	0.00534	0.00238	0.00241
	15	0.00293	0.00137	0.00143
	20	0.00184	0.00089	0.00094
	25	0.00125	0.00062	0.00065
0.95	10	0.00379	0.00165	0.00151
	15	0.00212	0.00089	0.00087
	20	0.00140	0.00060	0.00061
	25	0.00100	0.00044	0.00046
0.99	10	0.00233	0.00166	0.00121
	15	0.00115	0.00074	0.00054
	20	0.00070	0.00041	0.00030
	25	0.00048	0.00026	0.00020

Table 2.2 The MSE's for the proposed estimators of the maximum of the the two generalized uniform scale parameters ($\alpha = -1/2$).

		MSE		
δ	n	$\widehat{\theta_2}^{(1)}$	$\widehat{\theta_2}^{(2)}$	$\widehat{\theta_2}^{(3)}$
0.90	10	0.02403	0.01133	0.01062
	15	0.01386	0.00614	0.00584
	20	0.00937	0.00408	0.00399
	25	0.00690	0.00302	0.00302
0.95	10	0.01937	0.01099	0.00972
	15	0.01050	0.00540	0.00464
	20	0.00684	0.00325	0.00283
	25	0.00493	0.00222	0.00198
0.99	10	0.01593	0.01238	0.01048
	15	0.00792	0.00611	0.00479
	20	0.00478	0.00361	0.00272
	25	0.00322	0.00236	0.00174

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