The Korean Communications
in Statistics Vol. 6, No. 3, 1999
pp. 677-685

On the Moving Average Models with Multivariate Geometric Distributions[1)]

Jong-il Baek[2)]

Abstract

In this paper we introduce a class of moving-average(MA) sequences of multivariate random vectors with geometric marginals. The theory of positive dependence is used to show that in various cases the class of MA sequences consists of associated random variables. We utilize positive dependence properties to obtain weakly probability inequality of the multivariate processes.

1. Introduction

In time series analysis a primary stationary model is the $p\times 1$ moving average (MA) model given by,

$$X(n)=\sum_{j=-\infty}^{\infty}A(j)\varepsilon(n-j),\ n=0,\pm 1,\pm 2,\cdots \tag{1.1}$$

where $A(j), j=0,\pm 1,\pm 2,\cdots$, is a sequence of $p\times p$ parameter matrices such that $\sum_{j=-\infty}^{\infty}\|A(j)\| < \infty$, and $\varepsilon(n), n=0,\pm 1,\pm 2,\cdots$, is a sequence of uncorrelated $p\times 1$ random vectors with mean zero and common covariance matrix. It is well known that model emerges from many physically realizable systems(see, for example, Hann,E.J (1970), p.9).

Gaver and Lewis(1980) consider stationary autoregressive moving average($ARMA$) type where the random variables $X(n)$ have gamma distributions. Jacobs and Lewis(1983) construct $ARMA$-type model where the random variable $X(n)$ are discrete and assume values in a common finite set. The models mentioned above have been used in various fields of applied probability and time series analysis; for example, they have been used to model and analyze univariate point processes with correlated service and correlated interarrival times(see Jacobs(1983a).

In this paper we present a class of finite and infinite MA sequences of multivariate random vectors has geometric marginals. Within each class of models, the sequences are

---

1) This paper was partially supported by Wonkwang University Research Grant in 1999.
2) Professor, Division of Mathematical Science, Wonkwang University, Ik-San, 570-749, Korea

classified according to their order of dependence on the past. We use the theory of positive dependence to show that in a variety of cases the class of MA sequences is associated. We then apply positive dependence properties to establish weakly probability inequality of the multivariate processes.

In Section 2, we define the multivariate geometric distribution that it is the underlying distribution of our class, and present a variety of examples of such distribution. Furthermore, in Section 2, we define the concept of association and present geometric distributions that they are associated. In Section 3, we construct a class of MA sequences proving that it has geometric marginals and showing that if the underlying distribution is associated, so is the related MA sequence. Finally, in Section 4, we indicate how to relate multivariate point processes to the multivarite geometric MA processes discussed in Section 3. Also, in Section 4 we extend a useful concepts of Alzaid(1990)'s weakly bivariate dependence to weakly multivariate orthant dependence concepts and utilize positive dependence properties to obtain weak ly probability inequality of the multivariate processes.

2. Preliminaries

We start by stating the definition, examples, and prove basic result to be used in what follows.

Definition 2.1. Let $(X_1, \cdots, X_n)$ be a random vector assuming values in the set $\{1,2,\cdots\}$. We say that $(X_1, \cdots, X_n)$ has a multivariate geometric distribution (MVG) if the random variables $X_1, X_2, \cdots, X_n$ have geometrically distributed. Note that the $(k-1)$ dimensional marginals (hence k-dimensional marginals, $k=1,2,\cdots,n-1$) are MVG.

Examples 2.2. (a) Let $(X_1, \cdots, X_n)$ be independent geometric random vector. Then $(X_1, \cdots, X_n)$ has a multivariate geometric distribution.

(b) Let$(X_1, \cdots, X_{n+1})$ be independent geometric random variables and put $N_1 = \min(X_1, X_{n+1}), \cdots, N_n = \min(X_n, X_{n+1})$. Then$(N_1, \cdots, N_n)$ has a multivariate geometric distribution.

(c) Let $(M_1, \cdots, M_k)$ be multivariate geometric random vector and let $(N_1(j), \cdots, N_k(j))$, $j=1,2,\cdots$, be an $i.i.d.$ sequence of random vectors with multivariate geometric distributions which are independent of $(M_1, \cdots, M_k)$. Then $(\sum_{j=1}^{M_1} N_1(j), \cdots, \sum_{j=1}^{M_k} N_k(j))$ has a multivariate geometric distribution.

Next, we present a concept of positive dependence.

Definition 2.3. Let $X=(X_1,\cdots,X_n)$, $n=1,2,\cdots$ be a multivariate random vector. We say that the random variables $X_1,\cdots,X_n$ are associated if for all pairs of measurable bounded functions $f,g: R^n \rightarrow R$ both nondecreasing in each argument $Cov(f(X),g(X))\geq 0$.

Remark 2.4. Note that independent random variables are associated and nondecreasing functions of associated random variables are associated(cf. Barlow and Proschan(1975), pp.30-31). Thus the components of the random vector given in Examples 2.2 (b) are associated.

The following lemma provides sufficient conditions for some of the multivariate distributions presented in Examples 2.2 to be associated.

Lemma 2.5. Let $T=(T_1,\cdots,T_n)$ be a random vector with components assuming values in the set $\{1,2,\cdots\}$ and let $R(j)=(R_1(j),\cdots,R_n(j))$, $j=1,2,\cdots$, be an *i.i.d.* sequence of nonnegative random vectors independent of T. If $T=(T_1,\cdots,T_n)$ are associated and $R_1(j),\cdots,R_n(j)$ are associated, then $\sum_{j=1}^{T_1} R_1(j),\cdots,\sum_{j=1}^{T_n} R_n(j)$ are associated

Proof. Let $f,g: R^n \rightarrow R$ be measurable bounded functions nondecreasing in each argument and let $X_1=\sum_{j=1}^{T_1} R_1(j),\cdots,X_n=\sum_{j=1}^{T_n} R_n(j)$.

First note that

$$\begin{aligned} &Cov(f(X_1,\cdots,X_n),g(X_1,\cdots,X_n)) \\ &= E(Cov(f(X_1,\cdots,X_n),g(X_1,\cdots,X_n))\mid T) \\ &\qquad + Cov(Ef(X_1,\cdots,X_n)\mid T, Eg(X_1,\cdots,X_n)\mid T) \end{aligned}$$

Now, $Ef(X_1,\cdots,X_n)\mid T$, and $Eg(X_1,\cdots,X_n)\mid T$ are nondecreasing function of $T_1,\cdots,T_n$. Since $T_1,\cdots,T_n$ are associated

$$Cov(E[f(X_1,\cdots,X_n)\mid T], E[g(X_1,\cdots,X_n)\mid T])\geq 0.$$

Since $f[(X_1,\cdots,X_n)\mid T]$, and $g[(X_1,\cdots,X_n)\mid T]$ are nondecreasing function of $R_1(1),\cdots,R_1(T_1),\cdots,R_n(1),\cdots,R_n(T_n)$, these random variables are associated (cf. Barlow and Proschan(1975 a)). Thus

$$Cov(f(X_1,\cdots,X_n)\mid T, g(X_1,\cdots,X_n)\mid T)\geq 0.$$

Consequently, $Cov(f(X_1,\cdots,X_n),g(X_1,\cdots,X_n))\geq 0$ and $X_1,\cdots,X_n$ are associated.

3. Model Constructions and Notation

In this section we construct a class of MA sequences by multivariate random vectors. We denote the class of sequences by

$\{G(n, m)=(G_1(n,m),\cdots,G_k(n,m)),\ n=0,\pm1,\pm2,\cdots\},\ m=1,2,\cdots,\infty$.

We show that each random vector $G(n, m)$ has a multivariate geometric distribution with a vector mean independent of n or m. Within each class of sequences the order of dependence on the past is indicated by the parameter m. For each positive integer m, $G(n,m)$ depend only on the previous m variates $\{G(n-1,m),\cdots,G(n-m,m)\}$ and $G(n,\infty)$ depend on all the preceding random vector $\{G(n-1,\infty), G(n-2,\infty),\cdots\}$. After constructing the various models we conclude this section by present sufficient conditions for the random variables $\{G_l(n_j,m)\}$, $l=1,2,\cdots,k; j=1,2,\cdots,k$ to be associated, where $k=1,2,\cdots$ and $n_1<n_2<\cdots<n_k\in\{0,\pm1,\pm2,\cdots\}$. First, we construct the geometric class of sequences. Some notation is needed.

Notation. Let $p_1,\cdots,p_k$ be real numbers in $(0,1]$ and let $\alpha_1(n),\cdots,\alpha_k(n)$ be a sequence of parameters such that $p_j\le\alpha_j(n)\le1$, $j=1,2,\cdots,k$. Further, let $N(n)=(N_1(n),\cdots,N_k(n))$ be independent multivariate geometric vectors with mean vector $(p_1^{-1}\alpha_1(n),\cdots,p_k^{-1}\alpha_k(n))$ and let $M(n)=(M_1(n),\cdots,M_k(n))$ be $i.i.d.$ multivariate geometrics, independent of all $N(n)$, with mean vector $(p_1^{-1},\cdots,p_k^{-1})$. Finally, let $(J_1(n,j),\cdots,J_k(n,j))$ be independent random vectors, independent of all $M(n)$ and $N(n)$, such that $J_i(n,j)$ is Bernoulli with parameter $(1-\alpha_i(n))$, $i=1,2,\cdots,k$ and let $U_q(n,j)$ *be* a $n\times n$ random diagonal matrix

$$U_q(n,j)=diag\{\Pi_{k=q}^{j}J_1(n,k),\cdots,\Pi_{k=q}^{j}J_n(n,k)\},\ q\in\{1,2,\cdots,j\}.$$

To ease the notation we put $U_1(n,j)=U(n,j)$. We now present the class of geo -metric sequences. For $m=1,2,\cdots,$ and $n=0,\pm1,\pm2,\cdots,$ let

$$G(n,m)=\sum_{r=0}^{m}U(n,r)N(n-r)+U(n,m+1)M(n-m) \tag{3.1}$$

and

$$G(n,\infty)=\sum_{r=0}^{\infty}U(n,r)N(n-r) \tag{3.2}$$

Next, we show that $G(n,m)$ has a multivariate geometric distributions. First, the following Lemma is needed.

Lemma 3.1. For $n=0,\pm1,\pm2,\cdots,$ and $m,\ q=1,2,\cdots,$ let

$$H_q(n, m) = \sum_{r=0}^{m} U_q(n, r+q-1)N(n-r-q+1) + U_q(n, m+q)M(n-m-q+1).$$

Then for all n, m and q, $H_q(n, m)$ has a $k-$variate geometric distribution with mean vector $(p_1^{-1}, \cdots, p_k^{-1})$.

Proof. We prove the result of the lemma by an induction argument on m.

For $m=0$,

$$H_q(n, 0) = N(n-q+1) + U_q(n, q)M(n-q+1).$$

By computing the characteristic function of the components of $H_q(n, 0)$ we can verify that the lemma holds for all n, q. Assume now that the lemma holds for m, and all n, q.

Noting that $H_q(n, m+1) = N(n-q+1) + U_q(n, q)$

$$\times [\sum_{r=0}^{m} U_{q+1}(n, r+q)N(n-q-r) + U_{q+1}(n, m+q+1)M(n-m-q)],$$

we see that, by induction, the terms in the brackets are $k-$variate geometric with mean $(p_1^{-1}, \cdots, p_k^{-1})$. Since this term is independent of $N(n-q+1)$, it follows as in the case $m=0$ that $H_q(n, m+1)$ has a appropriate distribution for all n and q.

Thus, from the Lemma 3.1, we can obtain the following Corollary 3.2.

Corollary 3.2. For all n and m, $G(n, m)$ has a $k-$variate geometric with mean vector $(p_1^{-1}, \cdots, p_k^{-1})$.

Proof. Since $H_1(n, m) = \sum_{r=0}^{m} U_1(n, r)N(n-r) + U_1(n, m+1)M(n-m)$. $G(n, m)$ given in (3.1), $G(n, m) = H_1(n, m)$. Thus, we can obtain the result of Corollary 3.2 from Lemma 3.1.

Remark 3.3. For all n, $G(n, \infty)$ given in (3.2) has a k-variate geometric distribut ion with mean vector $(p_1^{-1}, \cdots, p_k^{-1})$.

Lemma 3.4. Suppose that $M_1(1), \cdots, M_k(1)$ are associated and that for all n, $N_1(n), \cdots, N_k(n)$ are associated. Then for all positive integers m, r and all integers $n_1 < n_2 < \cdots < n_r$, the random variables $\{G_i(n_j, m),\ i=1, \cdots, k;\ j=1, \cdots, r\}$ are associated.

Proof. By Barlow and Proschan((1975 b), Theorem 2.2, p.31 and Proschan 4, p.30) the random variables $M_i(n_i)$, $N_i(n_j, q), i=1, \cdots, k$; $j=1, \cdots, r$, and $q=1, \cdots, m$ are associated. We define the following the random variables $S_i = G_i(n_j, m)$, $i=1, \cdots, k$; $j=1, \cdots, r$ since $G_i(n_j, m)$, $i=1, \cdots, k, j=1, \cdots, r$ are nondecreasing functions of collection of associated random variables. If for $j=1, \cdots, r$, Γ and Δ are nondecreasing functions, then

$\Gamma(G_1(n_j,m),\cdots, G_k(n_j,m)), \Delta(G_1(n_j,m),\cdots, G_k(n_j,m))$ are nondecreasing functions of (n_j, m). Thus by definition of association (2.3), $Cov_S(\Gamma(S), \Delta(S)) = Cov_G(\Gamma(G), \Delta(G)) \geq 0$.

Lemma 3.5. Suppose that $M_1(1),\cdots,M_k(1)$ are associated and that for all n, $N_1(n),\cdots,N_r(n)$ are associated. Then for all positive integers k and all integers $n_1<n_2<\cdots<n_r$, the random variables $\{G_i(n_j,\infty),\ i=1,\cdots,k;\ j=1,\cdots,r\}$ are associated.

Proof. Let m be a positive integer. Since $\lim_{m\to\infty}(1-a_j(n))^m \leq \lim_{m\to\infty}(1-p_j)^m = 0$, $j=1,2,\cdots,k$, $\lim_{m\to\infty} G(n,m) =^p G(n,\infty)$. In particular since $G(n,m)\to G(n,\infty)$ converges in distribution as $m\to\infty$, the sequence

$$\{G_1(n_1,m),\cdots, G_k(n_1,m),\cdots, G_1(n_r,m),\cdots,G_k(n_r,m)\}$$

converges in distribution as $m\to\infty$ to

$$\{G_1(n_1,\infty),\cdots, G_k(n_1,\infty),\cdots, G_1(n_r,\infty),\cdots,G_k(n_r,\infty)\}.$$

By Lemma 3.4, the $\{G_i(n_j,m),\ i=1,2,\cdots,k,\ j=1,\cdots,r\}$ are associated for all m. Consequently, the result of the lemma follow by Esary et al.(1967), P_4.

4. Inequalities in Models

Throughout this section we fix m, $m=1,2,\cdots,\infty$, and hence suppress it from our notation, that is, $G(n, m)$ is denoted by $G(n)$.

In the point process theory of the models, the behavior of the vector of sums

$$T_G(r) = (T_{G_1}(r_1),\cdots, T_{G_k}(r_k))$$

where $T_{G_i}(r_i) = \sum_{n=1}^{r_i} G_i(n)$, $i=1,2,\cdots,k$ are of interest, $r_1,\cdots,r_k \in \{1,2,\cdots\}$.

For example, if $G(n)$ is a vector of $k-$variate geometric waiting times of a count process $N_G(r) = (N_{G_1}(r_1),\cdots,N_{G_k}(r_k))$ which are the number of occurences by trials $r_1,\cdots,r_k \in \{1,2,\cdots\}$, then $N_{G_i}(r_i) = T_{G_i}(r_i)$, $i=1,2,\cdots,k$.

We now utilize positive dependence properties to obtain weakly probability inequality for sum $T_G(r)$. First, we define the positive orthant dependence and new concepts of weakly positive orthant dependence.

Definition 4.1. Let $k=2,3,\cdots$, and let $X=(X_1,\cdots,X_k)$ be a random vector. We say that X is positively upper(lower) orthant dependent (PUOD(PLOD)) if for all real number $t_1,\cdots,t_k$,

$$P(X_1 > t_1,\cdots,X_k > t_k) \geq P(X_1 > t_1)\cdots P(X_k > t_k) \tag{4.1}$$

$$[P(X_1 \le t_1,\cdots,X_k \le t_k) \ge P(X_1 \le t_1)\cdots P(X_k \le t_k)] \tag{4.1)'$$

and we say that X is positively orthant dependent(POD) if they satisfy both PUOD and PLOD.

Definition 4.2. Let $k=2,3,\cdots$, and let $X=(X_1,\cdots,X_k)$ be a random vector. We say that X is weakly positive upper(lower) orthant dependent of the first type (WPUOD1(WPLOD1)) if for all real number $t_1,\cdots,t_k$,

$$\int_{x_1}^{\infty}\cdots\int_{x_k}^{\infty}[P(X_1 > t_1,\cdots,X_k > t_k) - P(X_1 > t_1)\cdots P(X_k > t_k)]dt_k\cdots dt_1 \ge 0 \tag{4.2}$$

$$\left[\int_{x_1}^{\infty}\cdots\int_{x_k}^{\infty}[P(X_1 \le t_1,\cdots,X_k \le t_k) - P(X_1 \le t_1)\cdots P(X_k \le t_k)]dt_k\cdots dt_1 \ge 0\right] \tag{4.2)'$$

and we say that X is weakly positive upper(lower) orthant dependent of the second type (WPUOD2(WPLOD2)) if for all real number $t_1,\cdots,t_k$,

$$\int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_k}[P(X_1 > t_1,\cdots,X_k > t_k) - P(X_1 > t_1)\cdots P(X_k > t_k)]dt_k\cdots dt_1 \ge 0 \tag{4.3}$$

$$\left[\int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_k}[P(X_1 \le t_1,\cdots,X_k \le t_k) - P(X_1 \le t_1)\cdots P(X_k \le t_k)]dt_k\cdots dt_1 \ge 0\right] \tag{4.3)'$$

Also, we say that the random vector X is weakly positive upper(lower) orthant dependent(WPUOD(WPLOD)) if they satisfy both (4.2(4.2)') and (4.3(4.3)'). Moreover, we say that the random vector X is weakly positive orthant dependent(WPOD) if they satisfy both WPUOD and WPLOD.

Remark 4.3. (a) If $X=(X_1,\cdots,X_k)$ is associated, then clearly X is POD, (b) X is POD, then X is WPOD but WPOD does not imply POD and (c) Let $f_1,\cdots,f_k$: $(-\infty,\infty)\rightarrow[0,\infty)$ be measurable nondecreasing(nonincreasing) functions and let $X=(X_1,\cdots,X_k)$ be PUOD(PLOD), then $E\prod_{i=1}^{k}f_i(X_i) \ge \prod_{i=1}^{k}Ef_i(X_i)$, (d) Let the random variable $\{G_i(n),\ i=1,2,\cdots,k\ ;n=1,2,\cdots,q\}$, $q=1,2,\cdots$ are associated. Then for $r_1,\cdots,r_k\in\{1,2,\cdots,k\}$ $\{T_{G_i}(r_i),i=1,2,\cdots\}$ or $\{N_{G_i}(r_i),i=1,2,\cdots\}$ is associated.

We obtain the following theorem for the sums $T_G(r)$

Theorem 4.4. Assume that $a_1(n),\cdots,a_k(n)$ are equal to $a_1,\cdots,a_k$, respectively, for all n. Let $NM_i(r_i,\theta_i)$, $i=1,2,\cdots,k$ be negative multinomial random variables with parameters $(r_i,\ \theta_i)$. Then

$$T_{G_i}(r_i) \ge NM_i(r_i,p_i a_i^{-1}),\ i=1,2,\cdots,k$$

If in addition, the random variables $\{G_i(n),\ i=1,2,\cdots,k,\ n=1,2,\cdots,q\}$, $q=1,2,\cdots$, are associated, then for $a_1 > r_1,\cdots,a_k > r_k$,

$$\int_{t_1}^{\infty}\cdots\int_{t_k}^{\infty}[P(T_{G_1}(r_1) > a_1,\cdots,T_{G_k}(r_k) > a_k)-\prod_{i=1}^{k}P(NM_i(r_i,\ p_ia_i^{-1)}>a_i)]\,da_k\cdots da_1\geq 0 \quad (4.4)$$

$$\left[\int_{t_1}^{\infty}\cdots\int_{t_k}^{\infty}[P(T_{G_1}(r_1)\leq a_1,\cdots,T_{G_k}(r_k)\leq a_k)-\prod_{i=1}^{k}P(NM_i(r_i,\ p_ia_i^{-1})\leq a_i)]\,da_k\cdots da_1\geq 0\right] (4.4)'$$

and

$$\int_{-\infty}^{t_1}\cdots\int_{-\infty}^{t_k}[P(T_{G_1}(r_1) > a_1,\cdots,T_{G_k}(r_k) > a_k)-\prod_{i=1}^{k}P(NM_i(r_i,\ p_ia_i^{-1}))>a_i)]\,da_k\cdots da_1\geq 0 \quad (4.5)$$

$$\left[\int_{-t_1}^{t_1}\cdots\int_{-\infty}^{t_k}[P(T_{G_1}(r_1)\leq a_1,\cdots,T_{G_k}(r_k)\leq a_k)-\prod_{i=1}^{k}P(NM_i(r_i,p_ia_i^{-1})\leq a_i)]\,da_k\cdots da_1\geq 0\right] (4.5)'$$

Proof. From the equations (3.1) or (3.2) we see that $G_i(n)\geq N_i(n)$, $i=1,\cdots,k$; $n=1,2,\cdots$. Hence

$$\begin{aligned}P(T_{G_i}(r_i)\geq a_i)&=P(\sum_{n=1}^{r_i}G_i(n)\geq a_i)\\&\geq P(\sum_{n=1}^{r_i}N_i(n)\geq a_i)\\&=P(NM_i(r_i,p_ia_i^{-1})\geq a_i),\ i=1,2,\cdots,k\end{aligned}$$

and the first assertion is proved. Secondly, let $f_i=\chi\{T_{G_i}(r_i)\leq a_i\}, i=1,\cdots,k$, where χ is the indicator function. Then f_i, $i=1,\cdots,k$ are associated since the random variables $T_{G_1}(r_1),\cdots,T_{G_k}(r_k)$are associated(Remark 4.3(c)). Hence

$$0\leq Cov(f_1,\prod_{i=2}^{k}f_i)=Ef_1\prod_{i=2}^{k}f_i-Ef_1E\prod_{i=2}^{k}f_i$$

Repeated applications of this argument yield

$$\begin{aligned}0&\leq E\prod_{i=1}^{k}f_i-\prod_{i=1}^{k}Ef_i\\&=E\prod_{i=1}^{k}\chi\{T_{G_i}(r_i)\leq a_i\}-\prod_{i=1}^{k}E\chi\{T_{G_i}(r_i)\leq a_i\}\\&=P(T_1(r_1))>a_1,\cdots,T_k(r_k))>a_k)-P(T_1(r_1))>a_1)\cdots P(T_k(r_k))>a_k)\end{aligned}$$

Thus the random variables

$T_{G_1}(r_1),\cdots,T_{G_k}(r_k)$ are PUOD and the proof of PLOD is similar to the proof of PUOD. Consequently, equation (4.4) and (4.5) follow from the first assertion and the fact that since $T_{G_1}(r_1),\cdots,T_{G_k}(r_k)$ are POD they are WPOD (Remark(4.3(b)).

References

[1] Alzaid, A.A.(1990). A weak quadrant dependence concept with Applications, *Communications in Statistics Stochastic Models*, 6(2), 353-363

[2] Barlow, R.E. and Proschan, F.(1975). *Statistical Theory of Reliability and Life-Testing* : Probabilaty Models, Holt, Rinehart and Winston, New York.

[3] Esary, J.D., Proschan, F. and Wallup, D.W.(1967). Association of random variables with applications, *Annals of Mathematical Statistics*, 38,1466-1474.

[4] Gaver, D.P. and Lewis, P.A.W.(1980). First-order autoregressive gamma sequences and point processes, *Advances in Applied Probability*, 12, 727-745.

[5] Hannan, E.J.(1970). *Multiple Time Series*, Wiley, New York.

[6] Jacobs, P.A. and Lewis, P.A.W.(1983). Stationary discrete autoregressive moving average time series generated by mixtures, *Journal of Time Series Analysis*, 4, 18-36.