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Notes on Parametric Estimations in a Power Function Distribution

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Abstract

We shall propose the MME, MLE and UMVUE for the mean parameter and the right-tail probability in a power function distribution and obtain the mean squared errors for the proposed estimators. And we shall compare numerically efficiencies of the MME, MLE and UMVUE of the mean parameter and the right-tail probability in a power function distribution.

1. Introduction

A power function distribution is given by

$$f(x;\theta)=\frac{\theta}{1-\theta}\, x^{\frac{2\theta-1}{1-\theta}},\quad 0<x<1,\quad \text{where } 0<\theta<1, \tag{1.1}$$

which has mean θ and variance $\theta(1-\theta)^2/(2-\theta)$ (see p.386 of Rohatgi(1976)).

The power function distribution is a special case of the Beta distribution with $\alpha=\theta/(1-\theta)$ and $\beta=1$. The power function distribution is a uniform distribution over $(0,1)$ if $\theta=1/2$, and the density function is decreasing (or increasing) if $0<\theta<1/2$ (or $1/2<\theta<1$, respectively). This distribution is one version of a power function distribution (see p.235 of Johnson & Kotz(1995)). The reparametrizaton of this version plays a role in representing mean in the distribution with the pdf(1.1).

An example of some importance is the use of a power function distribution to fit the distribution of certain likelihood ratios in statistical tests. If the likehood ratio is based on n-independent identically distributed random variables, it is often found that a usefully good fit can be obtained by supposing (likelihood ratio)$^{2/n}$ to have a power function distribution.

Rider(1964) derived distributions of productions and quotients of maximum values in sample from a population with a power function and studied problems of estimating parameter in the

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power function distribution. Moments of order statistics for a power function distribution were calculated by Marik(1967). Lwin(1972) and Arnold & Press(1983) discussed Bayesian estimations for the scale parameter of the Pareto distribution using a power function prior. Moothathu(1984) studied characterizations of Lorenz curve in the power function distribution .

In this paper we shall derive the MME, MLE and UMVUE of the mean parameter and the right-tail probability in a power function distribution with the pdf (1.1) and obtain the mean squared errors(MSE) for the proposed estimators. And we shall compare numerically efficiencies of the MME, MLE and UMVUE of the mean parameter and the right-tail probability in a power function distribution with the pdf (1.1).

2. Mean Parameter Estimation

Let $X_1, X_2, \ldots, X_n$ be a simple random sample(SRS) from the power function distribution with the pdf (1.1). Then the likelihood function is given by

$$L(\theta; \boldsymbol{x}) = \left(\frac{\theta}{1-\theta}\right)^n e^{\frac{2\theta-1}{1-\theta}\sum_{i=1}^{n}\ln x_i}, \quad 0<\theta<1 .$$

Since $\{f_\theta(\underset{\sim}{x})\}$ is an one-parameter exponential family with $T(X_1, \ldots, X_n) = -\sum_{i=1}^{n} \ln X_i$, we can obtain the following facts.

Fact 1. Let $X_1, \cdots, X_n$ be a SRS form the power function distribution with the pdf (1.1).

(a) $T(X_1, \ldots, X_n) = -\sum_{i=1}^{n} \ln X_i$ is a complete sufficient statistic for the mean parameter θ.

(b) $T(X_1, \ldots, X_n) = -\sum_{i=1}^{n} \ln X_i$ has a gamma distribution with a shape parameter n and a scale parameter $(1-\theta)/\theta$.

Now, we consider the estimation for mean parameter θ in the power function distribution with the pdf (1.1). By the moment method, the moment estimator(MME) of the mean parameter θ is

$$\tilde{\theta} = \sum_{i=1}^{n} X_i / n,$$

which is an unbiased estimator of the mean parameter θ and its variance is

$$Var(\tilde{\theta}) = \frac{\theta(1-\theta)^2}{n(2-\theta)} . \qquad (2.1)$$

From the likelihood function of the distribution, the MLE of the mean parameter θ is

$$\hat{\theta} = (1 - \sum_{i=1}^{n} \ln X_i / n)^{-1}.$$

From the formula 3.353(5) of Gradshiteyn & Ryzhik(1965) and Fact 1(b), the expectation of the MLE $\hat{\theta}$ for the mean parameter θ can be obtained as follows :

$$E(\hat{\theta}) = \left(\frac{n\theta}{1-\theta}\right)^n \left[(-1)^{n-2} e^{n\theta/(1-\theta)} E_i(-\frac{n\theta}{1-\theta}) \right.$$

$$\left. + (-1)^{n-1} \sum_{k=1}^{n-1} \Gamma(k) \left(-\frac{n\theta}{1-\theta}\right)^{-k} \right] / \Gamma(n), \tag{2.2}$$

where $-E_i(-a) = \int_1^{\infty} e^{-at} / t \, dt$, $a > 0$ and $\Gamma(x)$ is a gamma function.

And from the formulas 3.351 (2) & (4) of Gradshiteyn & Ryzhik(1965), the variance of the MLE can be obtained as follows ;

$$\mathrm{Var}(\hat{\theta}) = \frac{1}{\Gamma(n)} \left(\frac{n\theta}{1-\theta}\right)^n e^{\frac{n\theta}{1-\theta}}$$

$$\cdot \left[(-1)^{n-1} (n - 1 + \frac{n\theta}{1-\theta}) E_i(-\frac{n\theta}{1-\theta}) + \frac{(-1)^{n-1}}{n} e^{-\frac{n\theta}{1-\theta}} \right.$$

$$\left. + e^{-\frac{n\theta}{1-\theta}} \sum_{k=2}^{n-1} \sum_{m=0}^{k-2} (-1)^{n-k-1} \binom{n-1}{k} \frac{(k-2)!}{m!} \left(\frac{n\theta}{1-\theta}\right)^{m-k+1} \right]$$

$$- E^2(\hat{\theta}), \tag{2.3}$$

From the results (2.1) through (2.3), we can show the following consistency :

Fact 2. The MME $\tilde{\theta}$ and MLE $\hat{\theta}$ of the mean parameter θ in the power function distribution with the pdf (1.1) are consistent.

Since the regular conditions are satisfied, the Frechet-Cramer-Rao lower bound(FCRLB) for an unbiased estimator of the mean parameter θ is

$$\theta^2 (1-\theta)^2 / n.$$

From Fact 1(a) and Lehmann-Scheffe Theorem, we can obtain the UMVUE of the mean parameter θ in the power function distribution with the pdf (1.1) and its variance as follows;

Fact 3. The UMVUE of the mean parameter θ in the power function distribution with the pdf (1.1) is

$$\hat{\theta}_U = \frac{(n-1)!\left[e^{\sum_{i=1}^{n}\ln X_i} - \sum_{m=0}^{n-2}\frac{(\sum_{i=1}^{n}\ln X_i)^m}{m!}\right]}{(\sum_{i=1}^{n}\ln X_i)^{n-1}},$$

and its variance is

$$\mathrm{Var}\,(\hat{\theta}_U) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}\frac{\Gamma(n)\cdot\Gamma(r+s+n)}{\Gamma(r+n)\cdot\Gamma(s+n)}\cdot\left(\frac{\theta}{1-\theta}\right)^{-(r+s)} - \theta^2, \tag{2.4}$$

which converges for $1/2 < \theta < 1$, and its variance can be represented by integral form for $0 < \theta \le 1/2$.

Proof. Since $T \equiv T(X_1, \ldots, X_n) = -\sum_{i=1}^{n}\ln X_i$ is a complete sufficient statistic for θ and X_1 is an unbiased estimator for θ, $U(X_1, \ldots, X_n) = E(X_1 \mid -\sum_{i=1}^{n}\ln X_i)$ is the UMVUE of θ from Lehmann-Scheffe Theorem,.
To evaluate the conditional expectation, we need to get the conditional pdf of X_1 given $T=t$;

$$f(x \mid t) = \frac{(n-1)(t+\ln x)^{n-2}}{x\cdot t^{n-1}}, \quad e^{-t} < x < 1. \tag{2.5}$$

From the formula 3.2 of Oberhettinger(1973), we can obtain the conditional expectation, that is, the UMVUE of the mean parameter θ. From the formula 3.351(3) of Gradshiteyn & Ryzhik(1965) and Taylor's expansion of the exponential function, the variance of the UMVUE can be obtained.

From the results of the means and the variances for $\hat{\theta}$, $\tilde{\theta}$ and $\hat{\theta}_U$, Table 1 shows the mean squared errors(MSE) of the MLE, MME and UMVUE for the mean parameter θ in the power function distribution with the pdf (1.1) when θ=0.25, 0.75 and n=10(5)30. From Table 1, when $\theta = 0.25$, the UMVUE $\hat{\theta}_U$ of the mean parameter θ in the power function distribution with the pdf (1.1) tends to be more efficient than the MLE $\hat{\theta}$ and the MME $\tilde{\theta}$ for the sample sizes n=10(5)30, but when $\theta = 0.75$. the MLE $\hat{\theta}$ of the mean parameter θ tends to be more efficient than the UMVUE $\hat{\theta}_U$ and the MME $\tilde{\theta}$ for the sample sizes n=10(5)30.

Next, we shall consider a confidence interval of the mean parameter θ in the power function distribution with the pdf (1.1). Since $-\frac{2\theta}{1-\theta}\sum_{i=1}^{n}\ln X_i$ is a pivotal quantity, which has a chi-square distribution with 2n degrees of freedom, a $(1-\alpha)100\%$ confidence interval of the mean parameter θ in the power function distribution with the pdf (1.1) is

$$\left(\frac{\chi^2_{2n,\frac{\alpha}{2}}}{\chi^2_{2n,\frac{\alpha}{2}} - 2\sum_{i=1}^{n} \ln X_i}, \frac{\chi^2_{2n,1-\frac{\alpha}{2}}}{\chi^2_{2n,1-\frac{\alpha}{2}} - 2\sum_{i=1}^{n} \ln X_i} \right)$$

where $\alpha = \int_0^{\chi^2_{2n,\alpha}} \chi^2_{2n}(t)dt$, $\chi^2_{2n}(t)$ is the pdf of chi-square distribution with d.f. 2n.

3. Right-tail Probability Estimation

Here we shall consider the estimation of the right-tail probability in the power function distribution with the pdf (1.1). The cumulative distribution function of the power function distribution with the pdf (1.1) is

$$F(x) = x^{\frac{\theta}{1-\theta}}, \quad 0 < x < 1,$$

and so the right-tail probability of the power function distribution with the pdf (1.1) is

$$R(t) = 1 - t^{\frac{\theta}{1-\theta}}, \quad 0 < t < 1.$$

By the conditions of regularity, the Frechet- Cramer-Rao Lower bound for an unbiased estimator of the right-tail probability R(t) is

$$t^{\frac{2\theta}{1-\theta}} \left(\frac{\theta}{1-\theta} \ln t \right)^2 / n \tag{3.1}$$

Using Lehmann-Scheffe Theorem based on the complete sufficient statistics, from the formulas 3.381(3) and 8.352 (2) & (3) of Gradshiteyn & Ryzhik(1965) and the conditional pdf (2.5), we can obtain the UMVUE of the right-tail probability and its variance.

Fact 4. The UMVUE of the right-tail probability in the power function distribution with the pdf (1.1) is

$$\widehat{R_U(t)} = 1 - \frac{(-\sum_{i=1}^{n} \ln X_i + \ln t)^{n-1}}{(-\sum_{i=1}^{n} \ln X_i)^{n-1}}, \quad -\sum_{i=1}^{n} \ln X_i > -\ln t.$$

and its variance is

$$\mathrm{Var}\,(\widehat{R_U(t)}) = \left(\ln t \cdot \frac{\theta}{1-\theta} \right)^{2n-2} \left[e^{\frac{\theta}{1-\theta}\ln t} \sum_{i=n}^{2n-2} \sum_{m=0}^{i-n+1} \binom{2n-2}{i} \left(\frac{\theta}{1-\theta} \ln t \right)^{-i} \right.$$

$$\cdot \frac{(i-n+1)!}{m!} \left(-\frac{\theta}{1-\theta} \ln t \right)^{m} \tag{3.2}$$

$$-E_i\left(\frac{\theta}{1-\theta}\ln t\right)\sum_{i=0}^{n-1}\binom{2n-2}{i}\left(\frac{\theta}{1-\theta}\ln t\right)^{-i}\frac{(-1)^{n-i-2}}{(n-i-2)!}$$

$$-e^{\frac{\theta}{1-\theta}\ln t}\sum_{i=0}^{n-1}\sum_{m=0}^{n-i-3}\binom{2n-2}{i}\left(\frac{\theta}{1-\theta}\ln t\right)^{-i}\frac{m!(-1)^{n+m-i-2}}{(n-i-2)!}$$

$$\cdot\left(-\frac{\theta}{1-\theta}\ln t\right)^{-m-1}\Bigg]\Big/\,\Gamma(n)-\ R^2(t).$$

From the MLE of the mean parameter θ, the MLE of the right-tail probability R(t) is

$$\widehat{R(t)}=1-t^{\frac{\widehat{\theta}}{1-\widehat{\theta}}}=\ 1\ -\ e^{-n\ln t/\sum_{i=1}^{n}\ln X_i},$$

From Fact 1(b) and using the formula 3.471(9) of Gradshiteyn & Ryzhik(1965), we can obtain the mean and the variance of the MLE of the right-tail probability as follows;

$$E(\widehat{R(t)})=1-2\left(-\frac{\theta\ln t}{1-\theta}n\right)^{\frac{n}{2}}\cdot K_n\left(2\sqrt{-\frac{\theta\ln t}{1-\theta}n}\ \right)\Big/\ \Gamma(n) \tag{3.3}$$

and

$$\mathrm{Var}\,(\widehat{R(t)})=2\left(-\frac{\theta\ln t}{1-\theta}2n\right)^{\frac{n}{2}}K_n\left(2\sqrt{-\frac{\theta\ln t}{1-\theta}2n}\ \right)\Big/\ \Gamma(n)$$

$$-\left[2\left(-\frac{\theta\ln t}{1-\theta}n\right)^{\frac{n}{2}}K_n\left(2\sqrt{-\frac{\theta\ln t}{1-\theta}n}\ \right)\Big/\ \Gamma(n)\right]^2, \tag{3.4}$$

where $K_n(x)$ is the modified Bessel function of order n.

From the results (3.1) through (3.4), Table 3 shows numerical values of the mean squared errors(MSE) for the UMVUE $\widehat{R_U(t)}$ and MLE $\widehat{R(t)}$ of the right-tail probability in the power function distribution with the pdf (1.1) when the sample sizes are n=10(5)30.

Since $t^{\theta/(1-\theta)}=1-R(t)$, the value $t^{\theta/(1-\theta)}$ (or $(\theta/(1-\theta))\ln t$) of the results (3.2), (3.3) and (3.4) depends only on $R(t)$. From Table 3, the UMVUE $\widehat{R_U(t)}$ of the right-tail probability in the power function distribution with the pdf (1.1) tends to be more efficient than the MLE $\widehat{R(t)}$ of the right-tail probability when the sample sizes are n=10(5)30.

Table 1. The Mean Squared Errors of the MME, MLE and UMVUE of the Mean Parameter in the Power Function Distribution with the pdf (1.1)

units are 10^{-3}

	θ=0.25			θ=0.75		
n	MLE	UMVUE	MME	MLE	UMVUE	MME
10	4.234	4.163	8.035	3.367	3.556	3.750
15	2.665	2.399	5.357	2.277	2.362	2.500
20	1.938	1.782	4.017	1.720	1.768	1.875
25	1.522	1.482	3.214	1.382	1.413	1.500
30	1.252	1.172	2.678	1.155	1.177	1.250

Table 2. The Frechet-Cramer-Rao Lower Bound for an Unbiased Estimator of the Mean Parameter in the Power Function Distribution with the pdf (1.1)

units are 10^{-3}

n	10	15	20	25	30
FCRLB	3.515	2.343	1.757	1.406	1.171

Table 3. The Mean Squared Errors of the UMVUE and MLE of the Right-Tail Probability in the Power Function Distribution with the pdf (1.1)

units are 10^{-4}

$R(t)$	n	MLE	UMVUE	FCRLB
	10	0.164	0.123	0.099
	15	0.092	0.076	0.066
0.01	20	0.063	0.055	0.049
	25	0.048	0.043	0.040
	30	0.039	0.035	0.033
	10	3.783	2.926	2.374
	15	2.158	1.811	1.583
0.05	20	1.498	1.311	1.187
	25	1.144	1.028	0.950
	30	0.924	0.845	0.791
	10	13.695	10.924	8.992
	15	7.949	6.801	5.994
0.1	20	5.559	4.936	4.496
	25	4.264	3.873	3.597
	30	3.454	3.187	2.997
	10	44.371	37.629	31.868
	15	26.631	23.692	21.245
0.2	20	18.911	17.280	15.934
	25	14.631	13.597	12.747
	30	11.921	11.207	10.623

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