The Korean Communications
in Statistics Vol. 6, No. 3, 1999
pp. 947-956

Maximum Likelihood Estimation of Multinomial Parameters with Known or Unknown Crossing Point

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Abstract

We define a crossing point x_0 such that $f(x) \geq g(x)$ for $x \leq x_0$ and $f(x) \leq g(x)$ for $x > x_0$ where f and g are probability density functions. We may encounter such situation when we compare two histograms from two independent observations. For example, two contingency tables where initially admitted students and actually enrolled students are classified according to their high school ranking may show such situation. In this paper we consider maximum likelihood estimation of cell probabilities when a crossing point exists. We first assume a known crossing point and find an estimator. The estimation procedure for the case of unknown crossing point is just a straightforward extension. A real data is analyzed for an illustrative purpose.

1. Introduction

The area of statistics involving contingency tables is rich with problems in which restrictions on the parameter space can be exploited. Most of such restrictions are related to various types of dependence concepts and studied by many researchers. Interested readers may refer to Cohen and Sackrowitz(1991), Douglas, *et al.*(1990), Grove(1980), Hirotsu(1982), Nguyen and Sampson(1987), Oh(1995, 1996, 1998), Patefield(1982) among others. In this paper we consider a different type of dependence concept which is related to a restriction concerning two probability density functions.

First consider the two random variables-continuous or discrete, X and Y with the same support. Let f and g be the probability density functions of X and Y, respectively. Suppose for some x_0, either known or unknown, we have

$$f(x) \geq g(x) \text{ for } x \leq x_0 \text{ and } f(x) \leq g(x) \text{ for } x > x_0.$$

We call x_0 a crossing point of the two probability density functions. We note that $x_0 = \inf\{x : f(x) \leq g(x)\}$. In a 2 by k contingency table, suppose the first row has multinomial distribution with parameter $(n\,;\, q_1, \ldots, q_k)$ and the second row has multinomial

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distribution with parameter $(m\ ;\ p_1, \ldots, p_k)$. where $j=1, \ldots, k$, $\sum_{j=1}^{k} q_j = \sum_{j=1}^{k} p_j = 1$. And then q_j and p_j satisfy the following restriction (R1), for some j_0,

$$p_j - q_j \le 0 \text{ for } j=1, \ldots, j_0 \text{ and } p_j - q_j \ge 0 \text{ for } j = j_0+1, \ldots, k.$$

This type of restriction is often found in practice. Suppose a university official classifies newly admitted students according to their high school percentile ranks. Not all of them will attend the university and the vacancy will be filled by the students who applied for the university but not admitted initially. Then one can easily expect that the proportion of enrolled students with high high-school rank will be decreased while the proportion of students with low high-school rank is increased. In this sense one may want to know where the crossing point is. Borrowing the term of reliability we may say that ''new is not better than used."

Similar works have been done by some researchers. Among them Hawkins and Kochar (1991) studied inference for the crossing point of two continuous cumulative distribution functions. But we are not aware that there is substantial amount of research works concerning inference for the crossing point of two probability density functions.

In this paper we are going to study the maximum likelihood estimation of multinomial parameters under known or unknown crossing point. The main purpose of this paper is to provide the explicit forms of the maximum likelihood estimates of cell probabilities under aforementioned restriction (R1) with known crossing point j_0 using the techniques developed in order restricted statistical inference. For the case of unknown crossing point we can find maximum likelihood estimate of the crossing point and corresponding cell probabilities by repeated use of the estimation procedure for the case of known crossing point. The details will be given at the end of Section 3.

In this paper we consider three types of sampling schemes, which are full multinomial, one and two sample product multinomial models. We state these sampling schemes briefly. The full multinomial assumes the joint distribution of cell counts given the total count. If we fix row(column) totals, each row(column) is distributed as multinomial distribution. We will call this model row(column) product multinomial model. Note that the column product multinomial model in a 2 by k contingency table is just a product of k independent binomial models. We will call simply product multinomial model for row product multinomial model. In section 2, we consider the full multinomial model. In section 3, we consider the product multinomial models in three ways; one- and two-sample product multinomial models and binomial model. For each of the sampling scheme the explicit forms of maximum likelihood estimates of cell probability are given. In section 4, we analyze a real data to illustrate the estimation procedure discussed in this paper. In section 5, we briefly discuss the possible extension of the result obtained in this paper and the likelihood ratio tests for given crossing point.

2. Full Multinomial Model

Let $0 < p_{ij} < 1$ for $i=1,2, j=1,2,\ldots,k$ and $\sum_{i,j} p_{ij}=1$. The likelihood function is proportional to $L_1 = \prod_{i=1}^{2}\prod_{j=1}^{k} p_{ij}^{n\bar{p}_{ij}}$. We would like to find p_{ij}'s which maximize L_1 subject to the following restriction (R2)

$$p_{1j}-p_{2j} \geq 0 \text{ for } j=1,\ldots,j_0 \text{ and } p_{1j}-p_{2j} \leq 0 \text{ for } j=j_0+1,\ldots,k.$$

It is convenient to define a one-to-one transformation of the parameter space by introducing new parameters θ_j and δ_j defined by $\theta_j = p_{1j}/(p_{1j}+p_{2j})$ and $\delta_j = p_{1j}+p_{2j}$, $j=1,\ldots,k$. Then $p_{1j}=\theta_j\delta_j$, $p_{2j}=\delta_j(1-\theta_j)$, $j=1,\ldots,k$, and the likelihood function L_1 is rewritten as

$$\left[\prod_{j=1}^{k}\theta_j^{n\bar{p}_{1j}}(1-\theta_j)^{n\bar{p}_{2j}}\right]\cdot\left[\prod_{j=1}^{k}\delta_j^{n(\bar{p}_{1j}+\bar{p}_{2j})}\right], \tag{2.1}$$

where $0 < \theta_j,\ \delta_j < 1,\ \sum_{j=1}^{k}\delta_j = 1$.

If restriction (R2) is expressed in terms of θ_j's and δ_j's, we have

$$\theta_j \geq \frac{1}{2} \text{ for } j=1,\ldots,j_0 \text{ and } \theta_j \leq \frac{1}{2} \text{ for } j=j_0+1,\ldots,k.$$

The above restriction does not include δ_j's. This suggests that we can maximize (2.1) by maximizing the two parts(one involves only θ_j's and the other involves only δ_j's) separately. Note that the unrestricted maximum likelihood estimate for $\widehat{\theta}_j$'s and $\widehat{\delta}_j$'s are given by, for $j=1,\ldots,k$,

$$\hat{\theta}_j = \frac{\hat{p}_{1j}}{\hat{p}_{1j}+\hat{p}_{2j}}, \text{ and } \hat{\delta}_j = \hat{p}_{1j}+\hat{p}_{2j}.$$

Then the restricted maximum likelihood estimates of θ_j and δ_j are given by θ_j^* and δ_j^*, where

$$\theta_j^* = \begin{cases} \max\{\frac{1}{2}, \frac{\hat{p}_{1j}}{\hat{p}_{1j}+\hat{p}_{2j}}\} & \text{if } j=1,\ldots,j_0, \\ \min\{\frac{1}{2}, \frac{\hat{p}_{1j}}{\hat{p}_{1j}+\hat{p}_{2j}}\} & \text{if } j=j_0+1,\ldots,k, \end{cases}$$

$$\delta_j^* = \hat{p}_{1j}+\hat{p}_{2j}.$$

The estimation procedure for the case of unknown crossing point when the product multinomial model is assumed will be given at the end of next section. The estimation procedure for the full multinomial model with unknown crossing point is quite similar to that for the product multinomial model. Thus the estimation procedure will not be given here.

3. Product Multinomial Model

First, suppose q_j's are known and the second row in a 2 by k has multinomial distribution with $(n\ ;\ p_1, \ldots, p_k)$, $\sum_{j=1}^{k} q_j = \sum_{j=1}^{k} p_j = 1$. This model is called one sample product multinomial model. We consider the maximum likelihood estimation cell probabilties under (R1). The likelihood function of this model is proportional to $L_2 = \prod_{j=1}^{k} p_j^{n\widehat{p_j}}$. It is convenient to partition the whole index set $J=\{j\ :\ 1 \le j \le k\}$ into four subsets J_1, J_2, J_3 and J_4 as follows;

$$\begin{aligned} J_1 &= \{j: j=1, \ldots, j_0,\quad \hat{p}_j - q_j \le 0\}, \\ J_2 &= \{j: j=1, \ldots, j_0,\quad \hat{p}_j - q_j > 0\}, \\ J_3 &= \{j: j=j_0+1, \ldots, k,\quad \hat{p}_j - q_j \ge 0\}, \text{ and} \\ J_4 &= \{j: j=j_0+1, \ldots, k,\quad \hat{p}_j - q_j < 0\}. \end{aligned}$$

We note that violations to the restriction (R1) occur on J_2 and J_4. Let, for $i=1,\ldots,4$,

$$P_i = \sum_{j \in J_i} p_j, \quad Q_i = \sum_{j \in J_i} q_j.$$

Now we are going to show that the maximum likelihood estimate p_j^* of p_j under the restriction is determined by q_j if $j \in J_2 \cup J_4$. First we need to state the unimodality of multinomial likelihood function which is crucial to solve this maximization problem. We note that the likelihood function is unimodal with respect to each of p_j's. That is, for a given p_j the likelihood function is strictly increasing for $p_j < \hat{p}_j$ and strictly decreasing for $p_j > \hat{p}_j$ while others are fixed. Suppose $(x_1, x_2, \ldots, x_k)$ has multinomial distribution with parameter $(n\ ;\ p_1, \ldots, p_k)$. Then the likelihood function $f(x_1, x_2, \ldots, x_k)$ is, where $\sum_{i=1}^{k} x_i = n$,

$$\frac{n!}{\prod_{i=1}^{k} x_i!} \prod_{i=1}^{k} p_i^{x_i},$$

which is reexpressed by $f(x_1, x_2, \ldots, x_k) = f(x_2, x_3, \ldots, x_k | x_1) \cdot f(x_1)$, i.e.,

$$\left[\frac{(n-x_1)!}{\prod_{i=2}^{k} x_i!} \prod_{i=2}^{k} \left(\frac{p_i}{1-p_1} \right)^{x_i} \right] \cdot \left[\frac{n!}{(n-x_1)! x_1!} p_1^{x_1} (1-p_1)^{n-x_1} \right] \tag{3.1}$$

Now we find the maximum likelihood estimate of p_i, $i=2\ldots,k$ while p_1 is being fixed. Then the maximum likelihood estimates of p_i's also change their values but the first part of (3.1) does not change its value. Suppose p_1 is bounded by q_1 and $\hat{p}_1 > q_1$ then the restricted maximum likelihood estimate should be q_1. By the similar argument discussed above we can show that $p_j^* = q_j$ for $j \in J_2 \cup J_4$.

Fixing p_j by $p_j^* = q_j$ for $j \in J_2 \cup J_4$, L_2 can be rewritten as

$$P_1^{n\widehat{P_1}} P_3^{n\widehat{P_3}} \prod_{j\in J_2} q_j^{n\widehat{p_j}} \prod_{j\in J_4} q_j^{n\widehat{p_j}},$$

and we need to maximize it subject to

$$P_1 + P_3 = Q_1 + Q_3, \quad P_1 - Q_1 \le 0, \text{and} P_3 - Q_3 \ge 0.$$

We note that $Q_1 + Q_3 \neq 1$. It is convenient to define a one-to-one transformation of the parameter space by introducing new parameters S_1, S_3 defined by $S_1 = P_1/(Q_1 + Q_3)$, $S_3 = P_3/(Q_1 + Q_3)$. Then the restriction becomes

$$0 < S_1, S_3 < 1, S_1 + S_3 = 1, \text{ and } S_1 \le \frac{Q_1}{Q_1 + Q_3},$$

and the likelihood function is

$$S_1^{n\widehat{P_1}} S_3^{n\widehat{P_3}} = S_1^{n\widehat{P_1}} (1 - S_1)^{n\widehat{P_3}}. \tag{3.2}$$

Now we maximize (3.2) under the above restriction. Then we have

$$S_1^* = \min\{\frac{Q_1}{Q_1 + Q_3}, \frac{\hat{P}_1}{\hat{P}_1 + \hat{P}_3}\},$$
$$S_3^* = \max\{\frac{Q_3}{Q_1 + Q_3}, \frac{\hat{P}_3}{\hat{P}_1 + \hat{P}_3}\}.$$

and hence

$$P_1^* = (Q_1 + Q_3) \cdot \min\{\frac{Q_1}{Q_1 + Q_3}, \frac{\hat{P}_1}{\hat{P}_1 + \hat{P}_3}\},$$
$$P_3^* = (Q_1 + Q_3) \cdot \max\{\frac{Q_3}{Q_1 + Q_3}, \frac{\hat{P}_3}{\hat{P}_1 + \hat{P}_3}\}.$$

Finally we need to find p_j's for $j \in J_1 \cup J_3$ subject to $\sum_{j\in J_1} p_j = P_1^*$, $\sum_{j\in J_3} p_j = P_3^*$. This can be done by distributing P_1^* and P_3^* to p_j's proportional to p_j's within each index set J_1 and J_3, respectively. Thus we have

$$p_j^* = \begin{cases} q_j, & \text{for } j \in J_2 \cup J_4, \\ P_1^* \cdot \dfrac{\hat{p}_j}{\sum_{j\in J_1} \hat{p}_j}, & \text{for } j \in J_1, \\ P_3^* \cdot \dfrac{\hat{p}_j}{\sum_{j\in J_3} \hat{p}_j}, & \text{for } j \in J_3. \end{cases}$$

We next consider the two sample product multinomial model. Suppose the first row has multinomial distribution with $(n\ ;\ q_1, \ldots, q_k)$ and the second row has multinomial distribution with $(m\ ;\ p_1,\dots,p_k)$, where $\sum_{j=1}^{k} q_j = \sum_{j=1}^{k} p_j = 1$. We consider this problem under the restriction (R1). The likelihood function is proportional to $L_3 = \prod_{j=1}^{k} p_j^{m\widehat{p_j}} q_j^{n\widehat{q_j}}$. Similarly for the one-sample case, let

$$\begin{aligned} J_1 &= \{j: j=1,\dots,j_0,\quad \hat{p}_j - \hat{q}_j \le 0\},\\ J_2 &= \{j: j=1,\dots,j_0,\quad \hat{p}_j - \hat{q}_j > 0\},\\ J_3 &= \{j: j=j_0+1,\dots,k,\quad \hat{p}_j - \hat{q}_j \ge 0\}, \text{ and}\\ J_4 &= \{j: j=j_0+1,\dots,k,\quad \hat{p}_j - \hat{q}_j < 0\}, \end{aligned}$$

and, for $i=1,\dots,4$,

$$P_i = \sum_{j\in J_i} p_j,\quad Q_i = \sum_{j\in J_i} q_j.$$

There occur violations on index set J_2 and J_4. As we discussed earlier in the one-sample product multinomial model case we can show that $p_j^* = q_j^*$ for $j\in J_2\cup J_4$. Then we fix $p_j = q_j$ for $j\in J_2\cup J_4$.

Now L_3 can be rewritten as

$$\begin{aligned} &P_1^{m\widehat{P_1}} P_3^{m\widehat{P_3}} Q_1^{n\widehat{Q_1}} Q_3^{n\widehat{Q_3}} \cdot \prod_{j\in J_2} p_j^{m\widehat{p_j}} \prod_{j\in J_4} p_j^{m\widehat{p_j}} \prod_{j\in J_2} q_j^{n\widehat{q_j}} \prod_{j\in J_4} q_j^{n\widehat{q_j}}\\ &= P_1^{m\widehat{P_1}} P_3^{m\widehat{P_3}} Q_1^{n\widehat{Q_1}} Q_3^{n\widehat{Q_3}} \cdot \prod_{j\in J_2} p_j^{m\widehat{p_j}+n\widehat{q_j}} \prod_{j\in J_4} p_j^{m\widehat{p_j}+n\widehat{q_j}}, \end{aligned}$$

and the restrictions become

$$\begin{gathered} P_1 + P_3 + \sum_{j\in J_2} p_j + \sum_{j\in J_4} p_j = 1,\quad Q_1 + Q_3 + \sum_{j\in J_2} q_j + \sum_{j\in J_4} q_j = 1,\\ 0 < P_1, P_3 < 1,\quad 0 < Q_1, Q_3 < 1,\\ P_1 \le Q_1, P_3 \ge Q_3,\quad P_1 + P_3 = Q_1 + Q_3. \end{gathered}$$

Let

$$A = 1 - \sum_{j\in J_2} p_j - \sum_{j\in J_4} p_j \text{ and } P_1' = \frac{P_1}{A},\ P_3' = \frac{P_3}{A},\ Q_1' = \frac{Q_1}{A},\ Q_3' = \frac{Q_3}{A}.$$

Then we have $0 < P_1', P_3', Q_1', Q_3' < 1$ and $P_1' + P_3' = Q_1' + Q_3'$, $P_1' \le Q_1'$, $P_3' \ge Q_3'$.

The likelihood function L_3 can be written as

$$\begin{aligned} &(P_1')^{m\widehat{P_1}} (Q_1')^{n\widehat{Q_1}} (P_3')^{m\widehat{P_3}} (Q_3')^{n\widehat{Q_3}}\\ &\cdot \prod_{j\in J_2} p_j^{m\widehat{p_j}+n\widehat{q_j}} \prod_{j\in J_4} p_j^{m\widehat{p_j}+n\widehat{q_j}} A^{m(\widehat{P_1}+\widehat{P_3})+n(\widehat{Q_1}+\widehat{Q_3})}\\ &= (P_1')^{m\widehat{P_1}} (1-P_1')^{m\widehat{P_3}} (Q_1')^{n\widehat{Q_1}} (1-Q_1')^{n\widehat{Q_3}}\\ &\cdot \prod_{j\in J_2} p_j^{m\widehat{p_j}+n\widehat{q_j}} \prod_{j\in J_4} p_j^{m\widehat{p_j}+n\widehat{q_j}} A^{m(\widehat{P_1}+\widehat{P_3})+n(\widehat{Q_1}+\widehat{Q_3})}. \end{aligned}$$

The likelihood function is consisted of two parts; the first part is binomial problem and the second part is an ordinary multinomial problem. Moreover, the restriction does not relate the two parts to each other. Hence we can maximize the likelihood function by maximizing the two parts separately. The first part is just a bioassay problem as discussed in Example 1.5.1 of Robertson, Wright and Dykstra(1988). Then the restricted maximum likelihood estimates of P_1, P_3 and p_j, q_j for $j \in J_2 \cup J_4$ are given by P_1^*, P_3^* and p_j^*, q_j^*, where

$$p_j^* = q_j^* = \frac{m\,\hat{p}_j + n\,\hat{q}_j}{m+n} \quad \text{for } j \in J_2 \cup J_4$$

$$P_1' = \begin{cases} \dfrac{\hat{P}_1}{\hat{P}_1 + \hat{P}_3} & \text{if } P_1' > Q_1' \\ \dfrac{m\,\hat{P}_1 + n\,\hat{Q}_1}{m\,\hat{P}_1 + m\,\hat{P}_3 + n\,\hat{Q}_1 + n\,\hat{Q}_3} & \text{if } P_1' < Q_1' \end{cases}$$

$$P_3' = \begin{cases} \dfrac{\hat{P}_3}{\hat{P}_1 + \hat{P}_3} & \text{if } P_3' > Q_3' \\ \dfrac{m\,\hat{P}_3 + n\,\hat{Q}_3}{m\,\hat{P}_1 + m\,\hat{P}_3 + n\,\hat{Q}_1 + n\,\hat{Q}_3} & \text{if } P_3' < Q_3' \end{cases}$$

$$Q_1' = \begin{cases} \dfrac{\hat{Q}_1}{\hat{Q}_1 + \hat{Q}_3} & \text{if } P_1' < Q_1' \\ \dfrac{m\,\hat{P}_1 + n\,\hat{Q}_1}{m\,\hat{P}_1 + m\,\hat{P}_3 + n\,\hat{Q}_1 + n\,\hat{Q}_3} & \text{if } P_1' > Q_1' \end{cases}$$

$$Q_3' = \begin{cases} \dfrac{\hat{Q}_3}{\hat{Q}_1 + \hat{Q}_3} & \text{if } P_3' < Q_3' \\ \dfrac{m\,\hat{P}_3 + n\,\hat{Q}_3}{m\,\hat{P}_1 + m\,\hat{P}_3 + n\,\hat{Q}_1 + n\,\hat{Q}_3} & \text{if } P_3' > Q_3' \end{cases}$$

and

$$A^* = (1 - \sum_{j \in J_2} p_j^* - \sum_{j \in J_4} p_j^*), \text{ and}$$

$$P_1^* = P_1' \cdot A^*, \quad P_3^* = P_3' \cdot A^*, \quad Q_1^* = Q_1' \cdot A^*, \quad Q_3^* = Q_3' \cdot A^*.$$

Consequently,

$$p_j^* = \begin{cases} \dfrac{m\,\hat{p}_j + n\,\hat{q}_j}{m+n}, & \text{for } j \in J_2 \cup J_4, \\ P_1^* \cdot \dfrac{\hat{p}_j}{\sum_{j \in J_1} \hat{p}_j}, & \text{for } j \in J_1, \\ P_3^* \cdot \dfrac{\hat{p}_j}{\sum_{j \in J_3} \hat{p}_j}, & \text{for } j \in J_3, \end{cases}$$

and

$$q_j^* = \begin{cases} \dfrac{m\,\hat{p}_j + n\,\hat{q}_j}{m+n}, & \text{for } j \in J_2 \cup J_4, \\ Q_1^* \cdot \dfrac{\hat{q}_j}{\sum_{j\in J_1} \hat{q}_j}, & \text{for } j \in J_1, \\ Q_3^* \cdot \dfrac{\hat{q}_j}{\sum_{j\in J_3} \hat{q}_j}, & \text{for } j \in J_3, \end{cases}$$

Finally we consider the binomial model which can be considered as column product multinomial model. Then the restriction becomes

$$p_j \geq \frac{1}{2} \text{ for } j = 1, \ldots, j_0, \text{ and } p_j \leq \frac{1}{2} \text{ for } j = j_0 + 1, \ldots, k.$$

The estimation procedure is very straightforward and then we have

$$p_j^* = \begin{cases} \max\{\hat{p}_j, \frac{1}{2}\} & \text{for } j = 1, \ldots, j_0, \\ \min\{\hat{p}_j, \frac{1}{2}\} & \text{for } j = j_0 + 1, \ldots, k. \end{cases}$$

Now we discuss about the estimation procedure for the case of unknown crossing point. We assume two-sample product multinomial model. Let $j_1 = \inf\{j: \hat{p}_j \geq \hat{q}_j\}$ and $j_2 = \inf\{j: \hat{p}_j \leq \hat{q}_j\}$. We note that $j_1 \leq j_2$ and for some $j \in \{j_1, \ldots, j_2\}$ we may have $\hat{p}_j < \hat{q}_j$ *or* $\hat{p}_j > \hat{q}_j$. Let $L_{3,j}$ be the likelihood when the crossing point is assumed to be j. Then the maximum likelihood estimator, $\hat{j}_0$, for j_0 is given by $\hat{j}_0 = \arg\max_{j_1 \leq j \leq j_2}\{L_{3,j}\}$. Then the maximum likelihood estimation of cell probabilities can be obtained by assuming $j_0 = \hat{j}_0$ and appeal to the estimation procedure given above. For other sampling models we may applied the similar procedure.

4. An Example

We analyze a real data to illustrate the estimation procedure discussed in sections 2 and 3. 131 prospective students are admitted initially to the College of Engineering, Pusan University of Foreign Studies on regular admission program and tabulated in Table 1 according to their high school ranks. There are 15 scales in high school ranks. The smaller the number, the higher the school rank. The second column of Table 1 shows the number of enrolled students for each high school rank. A school official claims that large portion of the newly admitted student with high-school rank 6 or higher tends not to attend the university. We use the same notation as in sections 2 and 3. Then $j_0 = 6$, we have $J_1 = \{1,2,3,4,5\}$, $J_2 = \{6\}$ $J_3 = \{7,8,9,10\}$, and $J_4 = \phi$.

Since violation occurs on $J_2 = \{6\}$, we have $p_6^* = q_6^* = \dfrac{30+33}{131+129} = 0.242308$. We note that

$$\begin{cases} \hat{P}_1 = \dfrac{(7+8+13+18+29)}{131}, & \hat{P}_3 = \dfrac{(24+2)}{131}, \\ \hat{Q}_1 = \dfrac{(4+3+3+7+23)}{129}, & \hat{Q}_3 = \dfrac{(45+7+3+1)}{129}. \end{cases}$$

Now need to find P_1, P_3, Q_1, and Q_3 which maximize

$$P_1^{131\hat{P}_1} P_3^{131\hat{P}_3} Q_1^{129\hat{Q}_1} Q_3^{129\hat{Q}_3}$$

subject to

$$P_1 + P_3 = Q_1 + Q_3 = 1 - 0.242308 \text{ and } P_1 \geq Q_1, \ P_3 \leq Q_3.$$

We can show that

$$P_1^* = 0.562643, \ P_3^* = 0.195050, \ Q_1^* = 0.315705, \text{ and } Q_3^* = 0.441987.$$

Finally p_j^*'s and q_j^*'s on J_1 and J_3 are listed in Table 1.

Table 1. Computational Details

	m_j	n_j	p	q	p^*	q^*
1	7	4	0.053435115	0.031007752	0.052513328	0.031570513
2	8	3	0.061068702	0.023255814	0.060015232	0.023677885
3	13	3	0.099236641	0.023255814	0.097524752	0.023677885
4	18	7	0.137404580	0.054263566	0.135034273	0.055248397
5	29	23	0.221374046	0.178294574	0.217555217	0.181530449
6	30	33	0.229007634	0.255813953	0.242307692	0.242307692
7	24	45	0.183206107	0.348837209	0.180045697	0.355168269
8	2	7	0.015267176	0.054263566	0.015003808	0.055248397
9		3	0	0.023255814	0	0.023677885
10		1	0	0.007751938	0	0.007892628
total	131	129	1	1	1	1

5. Concluding Remarks

Frequently we may impose some restrictions on probability density functions such as monotonicity. Finding a crossing point with monotonicity restriction might be of great interest and even challenging. On the other hand, one possible generalization of the result obtained in this paper is extending to the problem of general probability density functions.

In this paper we are not dealing with testing problems. Possible interests might be put on testing $H_0 : j_0 = J$ (*J is known*) against $H_1 : j_0 > J$ (*or* $j_0 < J$, $j_0 \neq J$). This testing problem is quite similar to the testing problem of unimodality with unknown peak. See Shi(1989).

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