Journal of the Korean Statistical Society (1999), **28**: *1, pp 21–34*

Limit Theorems for Fuzzy Martingales†

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ABSTRACT

In this paper, conditional expectation of a fuzzy random variable is introduced and its properties are investigated. Using this, we introduce the concept of fuzzy martingales and prove some convergence theorems which generalize the corresponding results for the classical martingales.

Keywords: Fuzzy numbers; Fuzzy random variables; Fuzzy conditional expectations; Fuzzy martingales

1. INTRODUCTION

The concept of a fuzzy random variable was introduced as a natural generalization of a set-valued random variable in order to represent relationships between the outcomes of a random experiment and inexact data. By inexactness here we mean non-statistical inexactness due to the subjectivity and imprecision of human knowledge. Conditional expectations and martingales of set-valued random variables have received much attention in recent years because of its usefulness in several applied fields such as mathematical economics, optimal control theory and system sciences. In particular, Hiai (1985), Hiai and Umegaki (1977), Korvin and Kleyle (1985), Papageorgiou (1985a, 1985b, 1987, 1993) established various convergence theorems for set-valued martingales. These results have been extended to the case of fuzzy-valued random variables by Ban (1990), Lushu(1995), Puri and Ralescu (1991), Stojakovic (1992, 1994), and so on.

In this paper, we introduce the notion of conditional expectations and martingales for fuzzy valued functions slightly different from those in the above works and prove some limit theorems which generalize the results for classical martingales. Section 2 is devoted to describe some basic concepts of fuzzy random variables. In section 3 we discuss fuzzy valued measures and introduce the conditional expectation for a fuzzy random variable. Finally, in section 4 we define fuzzy martingales and prove some convergence theorems for fuzzy martingales.

†The authors wish to acknowledge the financial support of the Korea Research Foundation made in the program year of 1997.

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2. PRELIMINARIES

In this section, we describe some basic concepts of fuzzy numbers and fuzzy random variables.

Let R denote the real line. A fuzzy number is a fuzzy set $\tilde{u} : R \to [0,1]$ with the following properties;

(1) $\tilde{u}$ is normal, i.e, there exists $x \in R$ such that $\tilde{u}(x) = 1$.
(2) $\tilde{u}$ is upper semicontinuous.
(3) supp $\tilde{u} = cl\{x \in R : \tilde{u}(x) > 0\}$ is compact.
(4) $\tilde{u}$ is a convex fuzzy set, i.e., $\tilde{u}(\lambda x + (1-\lambda)y) \geq \min(\tilde{u}(x), \tilde{u}(y))$ for $x, y \in R$ and $\lambda \in [0,1]$.

We denote the family of all fuzzy numbers by $F(R)$. For a fuzzy set $\tilde{u}$, the α-level set of $\tilde{u}$ is defined by

$$L_\alpha \tilde{u} = \begin{cases} \{x : \tilde{u}(x) \geq \alpha\} & \text{if } 0 < \alpha \leq 1 \\ \text{supp } \tilde{u} & \text{if } \alpha = 0 \end{cases}$$

Then it follows that $\tilde{u}$ is a fuzzy number if and only if $L_1\tilde{u} \neq \emptyset$ and $L_\alpha \tilde{u}$ is a closed bounded interval for each $\alpha \in [0,1]$. From this characterization of fuzzy numbers, a fuzzy number $\tilde{u}$ is completely determined by the end points of the intervals $L_\alpha \tilde{u} = [u^1_\alpha, u^2_\alpha]$.

Theorem 2.1 (Goetschel and Voxman(1986)) *For $\tilde{u} \in F(R)$, denote $u^1(\alpha) = u^1_\alpha$ and $u^2(\alpha) = u^2_\alpha$ by considering as functions of $\alpha \in [0,1]$. Then the followings hold.*

(1) u^1 is a bounded increasing function on [0,1].
(2) u^2 is a bounded decreasing function on [0,1].
(3) $u^1(1) \leq u^2(1)$.
(4) u^1 and u^2 are left continuous on (0,1] and right continuous at 0.
(5) If v^1 and v^2 satisfy above $(1)-(4)$, then there exists a unique $\tilde{v} \in F(R)$ such that $L_\alpha \tilde{v} = [v^1(a), v^2(\alpha)]$.

The above theorem implies that we can identify a fuzzy number $\tilde{u}$ with the parametrized representation $\{(u^1_\alpha, u^2_\alpha) | \; 0 \leq \alpha \leq 1\}$, where u^1 and u^2 satisfy $(1)-(4)$ of Theorem 2.1. Suppose now that $\tilde{u}, \tilde{v} \in F(R)$ are fuzzy numbers whose representations are $\{(u^1_\alpha, u^2_\alpha) | \; 0 \leq \alpha \leq 1\}$ and $\{(v^1_\alpha, v^2_\alpha) | \; 0 \leq \alpha \leq 1\}$, respectively. If we define

$$(\tilde{u}+\tilde{v})(z) = \sup_{x+y=z}\min(\tilde{u}(x), \tilde{v}(y))$$

$$(\lambda\tilde{u})(z) = \begin{cases} \tilde{u}(z/\lambda), & \lambda \neq 0 \\ \tilde{0}, & \lambda = 0 \end{cases}$$

where $\tilde{0} = I_{\{0\}}$ is the indicator function of $\{0\}$, then

$$\tilde{u}+\tilde{v} = \{(u_\alpha^1 + v_\alpha^1, u_\alpha^2 + v_\alpha^2)| \quad 0 \leq \alpha \leq 1\}$$

$$\lambda\tilde{u} = \begin{cases} \{(\lambda u_\alpha^1, \lambda u_\alpha^2)| \quad 0 \leq \alpha \leq 1\} \text{ if } \lambda \geq 0 \\ \{(\lambda u_\alpha^2, \lambda u_\alpha^1)| \quad 0 \leq \alpha \leq 1\} \text{ if } \lambda < 0 \end{cases}.$$

Now we define the metric d on $F(R)$ by

$$d(\tilde{u}, \tilde{v}) = \sup_{0\leq\alpha\leq1} d_H(L_\alpha\tilde{u}, L_\alpha\tilde{v}) \tag{2.1}$$

where d_H is the Hausdorff metric defined as

$$d_H(L_\alpha\tilde{u}, L_\alpha\tilde{v}) = \max(|u_\alpha^1 - v_\alpha^1|, |u_\alpha^2 - v_\alpha^2|).$$

Also, the norm $\|\tilde{u}\|$ of fuzzy number $\tilde{u}$ will be defined as

$$\|\tilde{u}\| = d(\tilde{u}, \tilde{0}) = \max(|u_0^1|, |u_0^2|).$$

It is well-known that $F(R)$ is complete but nonseparable with repect to d (See Klement, Puri and Ralescu(1986)).

Now, we review the definition of fuzzy random variables. let $(\Omega, \sum, P)$ denote a complete probability space. For a fuzzy number valued function $\tilde{X} : \Omega \to F(R)$ and a subset B of R, $\tilde{X}^{-1}(B)$ denotes the fuzzy subset of Ω defined by

$$\tilde{X}^{-1}(B)(\omega) = \sup_{\omega\in B}\tilde{X}(\omega)(x).$$

for every $\omega \in \Omega$. The function $\tilde{X} : \Omega \to F(R)$ is called $\sum$-measurable if for every closed subset B of R, the fuzzy set $\tilde{X}^{-1}(B)$ is $\sum$-measurable when considered as a function from Ω to [0,1]. A $\sum$-measurable function $\tilde{X} : \Omega \to F(R)$ is called a fuzzy random variable. If we denote

$$\tilde{X}(\omega) = \{(X_\alpha^1(\omega), X_\alpha^2(\omega))| \quad 0 \leq \alpha \leq 1\},$$

then it is well known that $\tilde{X}$ is a fuzzy random variable if and only if for each $\alpha \in [0,1], X_\alpha^1$ and X_α^2 are random variables in the usual sense. A fuzzy random variable $\tilde{X} = \{(X_\alpha^1, X_\alpha^2)|\ \ 0 \le \alpha \le 1\}$ is called integrable if for each $\alpha \in [0,1]$, X_α^1 and X_α^2 are integrable , equivalently, $\int \|\tilde{X}\| dP < \infty$. In this case, the expectation of $\tilde{X}$ is defined by

$$E\tilde{X} = \int \tilde{X} dP = \{(\int X_\alpha^1 dP, \int X_\alpha^2 dP)|\ \ 0 \le \alpha \le 1\}.$$

We denote the space of all integrable fuzzy random variables $\tilde{X} : \Omega \to F(R)$ by $\Lambda(R)$, where two fuzzy random variables $\tilde{X}, \tilde{Y} \in \Lambda(R)$ are considered to be identical if $\tilde{X} = \tilde{Y}$ a.s.. If we define

$$\Delta(\tilde{X}, \tilde{Y}) = \int_\Omega d(\tilde{X}, \tilde{Y}) dP \quad for \quad \tilde{X}, \tilde{Y} \in \Lambda(R), \tag{2.2}$$

then $\Lambda(R)$ is a complete metric space with respect to the metric Δ.(For details, see Kim and Ghil (1997)).

3. FUZZY CONDITIONAL EXPECTIONS

In this section, we introduce the concept of fuzzy number valued measures in the sense of Kim and Ghil (1997) and prove the existence of conditional expectation for fuzzy random variables.

Definition 3.1. *Let $\{\tilde{u}_n\}$ be a sequence of fuzzy numbers in $F(R)$ and $\tilde{u} \in F(R)$. The series $\sum_{n=1}^\infty \tilde{u}_n$ is said to converge to $\tilde{u}$ if $d(\sum_{i=1}^n \tilde{u}_i, \tilde{u}) \to 0$ as $n \to \infty$. In this case, $\tilde{u}$ is called the sum of the series $\sum_{n=1}^\infty \tilde{u}_n$ and denoted by $\tilde{u} = \sum_{n=1}^\infty \tilde{u}_n$.*

Definition 3.2. *A set function $\tilde{\mu} : \sum \to F(R)$ is called a fuzzy number valued measure if*

(1) $\tilde{\mu}(\emptyset) = \tilde{0}$

(2) *$\tilde{\mu}(\cup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \mu(A_n)$ for every sequence $\{A_n\}$ of pairwise disjoint elements of $\sum$.*

Note that a set function $\tilde{\mu} : \sum \to F(R), \tilde{\mu}(A) = \{(\mu_\alpha^1(A), \mu_\alpha^2(A)|0 \le \alpha \le 1\}$ is a fuzzy number valued measure if and only if

(a) for each $\alpha \in [0,1], \mu_\alpha^1$ and μ_α^2 are real-valued measures.

(b) the family $\{\mu_\alpha^1, \mu_\alpha^2|\ \ 0 \le \alpha \le 1\}$ of measures is uniformly countably additive, that is, the convergence of $\mu_\alpha^i(\cup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \mu_\alpha^i(A_n)$ is uniform in $\alpha \in [0,1], i = 1, 2$.

Theorem 3.1. *If $\tilde{X} : \Omega \to F(R)$ is an integrable fuzzy random variable, then the set function $\tilde{\mu}$ defined by*

$$\tilde{\mu}(A) = \int_A \tilde{X} dP, \quad A \in \sum$$

is a fuzzy number valued measure which is absolutely continuous with respect to P in the following sense:

$$P(A) = 0 \Rightarrow \tilde{\mu}(A) = \tilde{0}.$$

Proof: See Theorem 4.4 of Kim and Ghil (1997) □

The next theorem is the converse of the above theorem which generalize the classical Random-Nikodym theorem.

Theorem 3.2. *If a fuzzy number valued measure $\tilde{\mu} : \sum \to F(R)$ is absolutely continuous whit respect to P, then there exists a unique integrable fuzzy random variable $\tilde{X} : \Omega \to F(R)$ such that*

$$\tilde{\mu}(A) = \int_A \tilde{X} dP \ for \ all \ A \in \sum$$

Proof: See Theorem 4.5 of Kim and Ghil (1997). □

We are now ready to define the conditional expectation of a fuzzy random variable relative to a sub-σ-algebra $\sum_0$ of $\sum$.

Theorem 3.3. *Let $\tilde{X} : \Omega \to F(R)$ be a fuzzy random variable with $\tilde{X} \in \Lambda(R)$ and $\sum_0$ a sub-σ-algebra of $\sum$. Then there exists a unique fuzzy random variable $\tilde{Y} \in \Lambda(R)$ such that*

(1) *$\tilde{Y}$ is $\sum_0$ - measurable*

(2) *$\int_A \tilde{X} dP = \int_A \tilde{Y} dP$ for every $A \in \sum_0$.*

Proof: It follows easily from Theorem 3.1 and 3.2. □

The above $\tilde{Y}$ of theorem 3.3 is called the conditional expectation of $\tilde{X}$ relative to $\sum_0$ and denoted by $E(\tilde{X}| \sum_0)$. It follows immediately from the proof of Theorem 4.5 in Kim and Ghil(1997) that if $\tilde{X} = \{(X_\alpha^1, X_\alpha^2)| \ \ 0 \leq \alpha \leq 1\}$, then

$$E(\tilde{X}| \textstyle\sum_0) = \{(E(X_\alpha^1| \sum_0), E(X_\alpha^2| \sum_0))| \ \ 0 \leq \alpha \leq 1\}$$

Theorem 3.4. *Let $\tilde{X}$, $\tilde{Y} \in \Lambda(R)$. Then the followings hold;*

(1) *If λ_1 and λ_2 are real numbers, then*

$$E(\lambda_1\tilde{X} + \lambda_2\tilde{Y}| \textstyle\sum_0) = \lambda_1 E(\tilde{X}| \sum_0) + \lambda_2 E(\tilde{Y}| \sum_0).$$

(2) $E[E(\tilde{X}| \sum_0)] = E(\tilde{X})$.

(3) *If $\tilde{X}$ is $\sum_0$ - measurable, then $E(\tilde{X}| \sum_0) = \tilde{X}$.*

(4) *If $\sum_0 \subset \sum_1$ are two sub-σ-algebras of $\sum$, then*

$$E[E(\tilde{X}| \textstyle\sum_1)| \sum_0] = E(\tilde{X}| \sum_0).$$

(5) $d(E(\tilde{X}| \sum_0), E(\tilde{Y}| \sum_0)) \leq E(d(\tilde{X}, \tilde{Y})| \sum_0)$.
In particular, $\int \|E(\tilde{X}| \sum_0)\| dP \leq \int \|\tilde{X}\| dP$.

Proof: (1)-(4) are trivial. To prove (5), we first note that for each $\alpha \in [0, 1]$

$$|E(X_\alpha^i| \textstyle\sum_0) - E(Y_\alpha^i| \sum_0)| \leq E(|X_\alpha^i - Y_\alpha^i|| \sum_0) \leq E(d(\tilde{X}, \tilde{Y})| \sum_0).$$

Hence the desired result is obtained if we take the supermum with respect to α over a countable dense subset of [0,1]. □

Now we conclude this section by giving a convergence theorem for conditional expectations.

Theorem 3.5. *Let $\tilde{X}$ and $\{\tilde{X}_n\}_{n\in N}$ be fuzzy random variables in $\Lambda(R)$ such that*

$$\tilde{X}_n \xrightarrow{d} \tilde{X} \ \ a.s.$$

If there exists an integral function $g : \Omega \to R$ such that

$$\|\tilde{X}_n\| \leq g \ \ a.s. \ for \ every \ n,$$

then $E(\tilde{X}_n| \sum_0) \xrightarrow{d} E(\tilde{X}| \sum_0)$ a.s.

Proof: First we note that

$$d(\tilde{X}_n, \tilde{X}) \leq \|\tilde{X}_n\| + \|\tilde{X}\| \leq g + \|\tilde{X}\| \ \ a.s..$$

Thus by the Lebesque dominated convergence theorem for conditional expectation in the classical case, we have

$$d(E(\tilde{X}_n| \textstyle\sum_0), E(\tilde{X}| \sum_0)) \leq E(d(\tilde{X}_n, \tilde{X})| \sum_0) \to 0 \ \ a.s..$$ □

4. FUZZY MARTINGALES AND CONVERGENCE THEOREMS

In this section, we introduce the concept of fuzzy martingales and prove some convergence theorems which generalize the corresponding results for classical martingales. First, we start with a sequence $\{\tilde{X}_n\}_{n\in N}$ of fuzzy random variables and an increasing sequence $\{\sum_n\}_{n\in N}$ of sub-σ-algebras of $\sum$.

Definition 4.1. *The sequence $\{\tilde{X}_n, \sum_n\}_{n\in N}$ of fuzzy random variables and σ -algebras is called a fuzzy martingale if for each n,*

(1) *$\tilde{X}_n$ is $\sum_n$ -measurable with $E\|\tilde{X}_n\| < \infty$.*
(2) *$E(\tilde{X}_{n+1}|\sum_n) = \tilde{X}_n$.*

Puri and Ralescu (1991) obtained convergence theorem for fuzzy martingale $\{\tilde{X}_n, \sum_n\}_{n\in N}$ which $\tilde{X}_n$ assume values in $F_L(R)$ which is defined as a subspace of $\tilde{u} \in F(R)$ with the property that the function $\alpha \to L_\alpha \tilde{u}$ is Lipschitz ; i.e., there exists a constant $C > 0$ such that for every $\alpha, \beta \in [0, 1]$,

$$d_H(L_\alpha \tilde{u}, L_\beta \tilde{u}) \leq C|\alpha - \beta|$$

where d_H is the Hausdorff metric.

Now we wish to obtain converge theorems for which $\tilde{X}_n$ takes values in a subspace $F_C(R)$ of $F(R)$ that includes $F_L(R)$. Let $F_C(R)$ be the space of $\tilde{u} \in F(R)$ with the property that u^1_α and u^2_α are continuous when considered as functions of α. Then it is well-known that $\tilde{u} \in F_C(R)$ if and only if for every $\beta \in (0, 1)$, there exist at most two different $x_1, x_2 \in R$ such that $\tilde{u}(x_1) = \tilde{u}(x_2) = \beta$ (see Theorem 5.1 of Congxin and Ming (1992)). Note that $F_C(R)$ is a closed subspace of $F(R)$ with respect to the metric d defined by (2.1)

Lemma 4.1. *$(F_C(R), d)$ is separable.*

Proof: Let $F_0(R)$ be the family of $\tilde{u} \in F(R)$ which for some positive integer k, there exist rational points $a_0 \leq \cdots \leq a_k \leq b_k \leq \cdots \leq b_0$ such that

$$\tilde{u}(a_i) = \tilde{u}(b_i) = i/k, i = 0, 1, \cdots, k,$$

and $\tilde{u}$ is linear in between, where we adopt the conventions as follows;

(1) If $i_0 = \min\{n|a_i = a_n\}, i_1 = \max\{n|a_i = a_n\}$,
then $\tilde{u}(a_i) = i_1/k, \tilde{u}(a_i^-) = i_0/k$.

(2) If $i_0' = \min\{n | b_i = b_n\}, i_1' = \max\{n | b_i = b_n\}$,
then $\tilde{u}(b_i) = i_1'/k, \tilde{u}(b_i^+) = i_0'/k$.

Then $F_0(R)$ is exactly same as the family of $\tilde{u} \in F(R)$ which for some positive integer k, there exist rational points $a_0 \leq \cdots \leq a_k \leq b_k \leq \cdots \leq b_0$ such that $u^1(i/k) = a_i, u^2(i/k) = b_i, i = 0, 1, \cdots, k$, and u^1, u^2 are linear on each interval $[\frac{i-1}{k}, \frac{i}{k}], i = 1, 2, \cdots, k$. Now it is easy to show that $F_0(R)$ is a countable dense subset of $(F_C(R), d)$. □

Let $\Lambda_C(R)$ denote the set of all $F_C(R)$-valued fuzzy random variable $\tilde{X} \in \Lambda(R)$. Then $\Lambda_C(R)$ is a closed supspace of $\Lambda(R)$ with respect to the metric Δ defined by (2.2). Furthermore, if $\tilde{X} \in \Lambda_C(R)$, then $E(\tilde{X} | \sum_0) \in \Lambda_C(R)$.

Theorem 4.1. *If $\tilde{X} \in \Lambda_C(R)$, then there exists a sequence $\{\tilde{Z}_n\}$ of simple fuzzy random variables in $\Lambda_C(R)$ such that*

$$\tilde{Z}_n \xrightarrow{d} \tilde{X} \quad a.s..$$

Proof: Let $F_0(R) = \{\tilde{u}_n\}$ be a countable dense subset of $(F_C(R), d)$. For $\varepsilon > 0$, if we define

$$A_n = \{w \in \Omega : d(\tilde{X}(w), \tilde{u}_n) < \varepsilon\}$$

$$B_n = A_n - \bigcup_{k=1}^{n-1} A_k$$

$$\tilde{Y} = \sum_{n=1}^{\infty} \tilde{u}_n I_{B_n}(w).$$

Then $d(\tilde{X}(w), \tilde{Y}(w)) < \varepsilon$ for all $w \in \Omega$. Thus it follows that for each $n \in N$, there exist

$$\tilde{Y}_n = \sum_{k=1}^{\infty} \tilde{u}_{nk} I_{B_{nk}} \text{ with } \tilde{u}_{nk} \in F_0(R), B_{ni} \cap B_{nj} = \emptyset \text{ for } i \neq j$$

such that

$$d(\tilde{X}(w), \tilde{Y}_n(w)) < 1/n \text{ for all } w \in \Omega.$$

Now we choose k_n so that

$$P(\bigcup_{k=k_n}^{\infty} B_{nk}) < 1/2^n$$

and let $\tilde{Z}_n(w) = \sum_{k=1}^{k_n} \tilde{u}_{nk} I_{B_{nk}}(w)$.
Then $\tilde{Z}_n$ is a $F_C(R)$-valued simple fuzzy random variable and

$$\sum_{n=1}^{\infty} P(\tilde{Y}_n \neq \tilde{Z}_n) < \infty.$$

Hence, $P(\tilde{Y}_n \neq \tilde{Z}_n$ infinitely often$)= 0$ which implies

$$d(\tilde{X}, \tilde{Z}_n) \to 0 \quad a.s.. \qquad \square$$

Theorem 4.2. *Let $\tilde{X} \in \Lambda_C(R)$ and $\{\sum_n\}_{n\in N}$ be an increasing sequence of sub-σ-algebras of $\sum$. If we define $\tilde{X}_n = E(\tilde{X}|\sum_n)$ for each n, then*

$$d(\tilde{X}_n, \tilde{X}) \to 0 \quad a.s.$$

and

$$\Delta(\tilde{X}_n, \tilde{X}) \to 0.$$

Proof: First we note that since $\{\tilde{X}_n, \sum_n\}_{n\in N}$ is a fuzzy martingale, we have $\{\|\tilde{X}_n\|, \sum_n\}_{n\in N}$ is positive real submartingale by (5) of Theorem 3.4. Since $\sup_n E\|\tilde{X}_n\| \leq E\|\tilde{X}\| < \infty$, it follows that

$$\lim_{n\to\infty} \|\tilde{X}_n\| = Y \quad a.s.$$

for some positive random variable Y with $EY < \infty$.
Futhermore, since $\{\|\tilde{X}_n\|\}_{n\in N}$ is uniformly integrable, we have

$$E(Y) = \lim_{n\to\infty} E\|\tilde{X}_n\| \leq E\|\tilde{X}\|. \tag{4.1}$$

But if we denote $\tilde{X}_n = \{(X^1_{n\alpha}, X^2_{n\alpha})|0 \leq \alpha \leq 1\}$, then for each $\alpha \in [0,1]$,

$$E(X^i_\alpha|\sum n) = X^i_{n\alpha}, \quad i = 1,2.$$

Hence, the convergence theorem for real-valued submartingale implies that

$$\lim_{n\to\infty} \int |X^i_{n\alpha} - X^i_\alpha| = 0$$

and

$$E|X^i_\alpha| \leq \lim_{n\to\infty} E|X^i_{n\alpha}|$$

for each $\alpha \in [0,1]$ and $i = 1,2$. Thus,

$$E\|\tilde{X}\| \leq \lim_{n\to\infty} E\|\tilde{X}_n\| = EY$$

which, together with (4.1), implies $Y = \|\tilde{X}\|$ a.s.
Similarily, it can be proved that for each $\tilde{u} \in F_C(R)$,

$$\lim_{n\to\infty} d(\tilde{X}_n, \tilde{u}) = d(\tilde{X}, \tilde{u}) \quad a.s.$$

Now let $F_0(R)$ be a countable dense subset of $F_C(R)$. Then there exists $A \in \sum$ with $P(A) = 0$ such that for each $w \notin A$,

$$\lim_{n\to\infty} d(\tilde{X}_n(w), \tilde{u}) = d(\tilde{X}(w), \tilde{u}) \text{ for all } \tilde{u} \in F_0(R).$$

Therefore, for each $w \notin A$,

$$\lim_{n\to\infty} d(\tilde{X}_n(w), \tilde{X}(w)) = 0. \tag{4.2}$$

Finally, $\Delta(\tilde{X}_n, \tilde{X}) \to 0$ follows from (4.2) and the uniform integrability of $\{d(\tilde{X}_n, \tilde{X})\}_{n\in N}$. □

Theorem 4.3. *Let $\{\tilde{X}_n, \sum_n\}_{n\in N}$ be a $F_C(R)$-valued fuzzy martingale. Then $\{\tilde{X}_n\}$ converge in the metric Δ if and only if there exists a $\tilde{X} \in \Lambda_C(R)$ such that for each $A \in \bigcup_n \sum_n$,*

$$\int_A \tilde{X}_n dP \stackrel{d}{\to} \int_A \tilde{X} dP.$$

Proof: If $\Delta(\tilde{X}_n, \tilde{X}) \to 0$ for some $\tilde{X} \in \Lambda_C(R)$, then for each $A \in \bigcup_n \sum_n$

$$\begin{aligned} d(\int_A \tilde{X}_n dP, \int_A \tilde{X} dP) &\le \int_A d(\tilde{X}_n, X) dP \\ &\le \Delta(\tilde{X}_n, \tilde{X}) \to 0. \end{aligned}$$

To prove the converse, suppose that there exists $\tilde{X} \in \Lambda_C(R)$ such that

$$d(\int_A \tilde{X}_n dP, \int_A \tilde{X} dP) \to 0 \text{ for all } A \in \bigcup_n \sum_n .$$

Let $\sum_\infty$ be the σ-algebra generated by $\bigcup_n \sum_n$ and set

$$\tilde{X}_\infty = E(\tilde{X} | \sum_\infty).$$

Then $\tilde{X}_\infty \in \Lambda_C(R)$ and since $\int_A X_n dP = \int_A X_m dP$ for $m \ge n$ and $A \in \sum_n$, we have

$$\int_A \tilde{X}_\infty dP = \int_A \tilde{X} dP = \int_A \tilde{X}_n dP \; for \; A \in \sum_n .$$

Hence, $E(\tilde{X}_\infty | \sum_n) = \tilde{X}_n$ for each n. Therefore, $\Delta(\tilde{X}_n, \tilde{X}_\infty) \to 0$ by Theorem 4.2. □

Corollary 4.1. *Let $\{X_n, \sum_n\}_{n\in N}$ be a $F_C(R)$-valued fuzzy martingale. Then $\{\tilde{X}_n\}_{n\in N}$ converges in the metric Δ if and only if there exists a $\tilde{X} \in \Lambda_C(R)$ such that*

$$E(\tilde{X} | \textstyle\sum_n) = \tilde{X}_n \quad for\ all \quad n.$$

Proof: It follows from Theorems 4.2 and 4.3. As a final result, we prove that

Δ-convergence is stronger than d-convergence for fuzzy martingales. First we need the following lemma. □

Lemma 4.2. *Let $\{\tilde{X}_n, \sum_n\}_{n\in N}$ be a fuzzy martingale and $\tilde{X} \in \Lambda(R)$. Then, for each $\delta > 0$,*

$$P(\sup_n\ d(\tilde{X}_n, \tilde{X}_1) > \delta) \leq \frac{1}{\delta} \sup_n\ \Delta(\tilde{X}_n, \tilde{X}_1).$$

Proof: Let $A = \{w : \sup_n\ d(\tilde{X}_n(w), \tilde{X}_1(w)) > \delta\}$
$A_k = \{w : d(\tilde{X}_k(w), \tilde{X}_1(w)) > \delta \text{ and } d(\tilde{X}_j(w), \tilde{X}_1(w)) \leq \delta\ for\ j < k\}$.

Then $\{A_k\}_{k\geq 2}$ is a disjoint sequence of sets and $A = \cup_{k=2}^{\infty} A_k$. Thus,

$$\begin{aligned} P(A) &= \sum_{k=2}^{\infty} P(A_k) \leq \frac{1}{\delta} \sum_{k=2}^{\infty} \int_{A_k} d(\tilde{X}_k, \tilde{X}_1) dP \\ &= \frac{1}{\delta} \limsup_{n\to\infty} \lim_{m\to\infty} \sum_{k=2}^{m} \int_{A_k} d(\tilde{X}_k, \tilde{X}_1) dP. \end{aligned}$$

But if m is fixed and $n \geq m$, then

$$E(\tilde{X}_n | \textstyle\sum_k) = \tilde{X}_k \ for\ k = 2, 3, \cdots, m.$$

Hence, by (5) of theorem 3.4,

$$\int_{A_k} d(\tilde{X}_k, \tilde{X}_1) dP \leq \int_{A_k} d(\tilde{X}_n, \tilde{X}_1) dP.$$

Therefore,

$$\begin{aligned} P(A) &\leq \frac{1}{\delta} \limsup_{n\to\infty} \sum_{k=2}^{\infty} \int_{A_k} d(\tilde{X}_n, \tilde{X}_1) dP \\ &\leq \frac{1}{\delta} \limsup_{n\to\infty} \Delta(\tilde{X}_n, \tilde{X}_1) \\ &\leq \frac{1}{\delta} \sup_n\ \Delta(\tilde{X}_n, \tilde{X}_1). \end{aligned}$$

□

Theorem 4.4. *Let* $\{\tilde{X}_n, \sum_n\}_{n\in N}$ *be a fuzzy martingale and* $\tilde{X} \in \Lambda(R)$. *If* $\Delta(\tilde{X}_n, \tilde{X}) \to 0$, *then*

$$d(\tilde{X}_n, \tilde{X}) \to 0 \quad a.s..$$

Proof: Let $\varepsilon, \delta > 0$ be arbitrary given. Then there exists n_0 such that

$$\Delta(\tilde{X}_n, \tilde{X}_m) < \varepsilon\delta \text{ for } n, m \geq n_0.$$

Now fix $m \geq n_0$. Then by Lemma 4.2, we have

$$\begin{aligned} P(\sup_{n\geq m} d(\tilde{X}_n, \tilde{X}_m) > \delta) &\leq \frac{1}{\delta} \sup_{n\geq m} \Delta(\tilde{X}_n, \tilde{X}_m) \\ &< \varepsilon \text{ for } n \geq n_0. \end{aligned}$$

It follows immediately that $\{\tilde{X}_n\}$ is almost uniformly cauchy with respect to d. As in the case of real-valued r.v., it can be proved that there exists a fuzzy r.v. $\tilde{Y}$ such that

$$d(\tilde{X}_n, \tilde{Y}) \to 0$$

almost uniformly. Since $\Delta(\tilde{X}_n, \tilde{X}) \to 0$, it is clear that $\tilde{X} = \tilde{Y}$ a.s.. Therefore, $d(\tilde{X}_n, \tilde{X}) \to 0$ a.s.. □

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