A note on $M$-groups

Abstract

Every finite solvable group is only a subgroup of an $M$-group and all $M$-groups are solvable. Supersolvable group is an $M$-group and also subgroups of solvable or supersolvable groups are solvable or supersolvable. But a subgroup of an $M$-group need not be an $M$-group. It has been studied that whether a normal subgroup or Hall subgroup of an $M$-group is an $M$-group or not. In this note, we investigate some historical research background on the $M$-groups and also we give some conditions that a normal subgroup of an $M$-group is an $M$-group and show that a solvable group is an $M$-group.

0. Introduction

An irreducible complex character $\chi$ of a finite group $G$ is monomial if it is induced from a linear (i.e. degree 1) character of some subgroup of $G$. A finite group $G$ is $M$-group if all its irreducible characters are monomial. Let $\text{Irr}(G)$ be the set of all irreducible complex character of a finite group $G$.

One of the remaining mysteries about $M$-group is whether of not normal subgroups of odd $M$-groups must, themselves, be $M$-groups. In [3], Dade constructed an example of an $M$-group of order $2^3 \cdot 7$ which has a non $M$-normal subgroup of index 2. A normal subgroup of an $M$-group must not be an $M$-group. I. Chubarov[1] proved that odd normal subgroups of $M$-groups are $M$-groups. Let $G$ be an $M$-group and suppose $N \triangleleft G$. If $N$ is an $M$-group then all of its primitive characters are linear. The converse of this statement is easily seen to be false.

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In [7], if $G$ is an $M$-group and $N \lhd G$ with either $|N|$ or $|G:N|$ odd, then $N$ is an $M$-group. In [6], Isaacs proved that if $G$ is an $M$-group and suppose $S \lhd \lhd G$ is a subnormal subgroup of odd index then every primitive character of $S$ is linear. Two are still the main problems on $M$-groups: are Hall subgroups of $M$-groups $M$-group? Under certain addness hypothesis are normal subgroups of $M$-groups $M$-group? In both cases there is evidence that this could be the case: the primitive characters of the subgroups in question are the linear characters.

Recently, some idea appears to take form. In [13], T. Okuyama proved that if $G$ is an $M$-group and $P$ is a Sylow $p$-subgroup of $G$, then $N_G(P)/P$ is an $M$-group. In [8], M. Isaacs showed that if $H$ is a Hall subgroup of an $M$-group then $N_G(H)/H'$ is also an $M$-group. In [12], G. Navarro proved that if $H$ is a Hall subgroup of an $M$-group $G$ and $\varphi \in \text{Irr}(N_G(H))$ is primitive then $\varphi$ is linear. In [10], M. Lewis proved that if $H$ is a maximal subgroup of an $M$-group $G$ such that $|G:H|$ is odd, and $\varphi \in \text{Irr}(H)$ is primitive then $\varphi(1)^2$ divides $|G:H|$. In [10], he proved that if $S$ is a subgroup of an $M$-group of $G$ that is reachable by primes, $H$ is a Hall subgroup of $S$ and $\varphi \in \text{Irr}(H)$ is primitive, then $\varphi(1)$ is a power of 2. Furthermore, if $|G:S|$ is odd, then $\varphi(1)=1$.

Recall that $M$-groups are necessarily solvable(Takeda, [16]).

A group $G$ is said to be supersolvable if there is a normal subgroup series

$$G = G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n = 1$$

with cyclic factor of prime order where each $G_i \lhd G$.

Since supersolvable groups are $M$-group[3], we have

\{nilpotents\} $\subset$ (supersolubles) $\subset$ ($M$-groups) $\subset$ (solvables).

The following remarks are clear([4], [14], [15]).

1. A subgroup of a supersolvable group is supersolvable.
2. Any factor group of a supersolvable group is also supersolvable.
3. A minimal normal subgroup of a supersolvable group is of prime order.
4. The index of maximal subgroup of a supersolvable group is a prime order.

Let $\text{Irr}(G/\theta)$ be the set of all irreducible constituents of $\theta^G$ where $\theta^G$ is the induced characters of $G$ for a character $\theta$ of normal subgroup $N$, and let for a character $\chi$ of $G$, $\chi_N$ be the restriction of $\chi$ to a normal subgroup $N$.

In this note, we show that under certain hypothesis the normal subgroup of $M$-group and the solvable group are $M$-groups.
1. Normal subgroups

Proposition 1. Let $N \triangleleft G$ and assume that $G/N$ is solvable. If for $\chi \in \text{Irr}(G)$, $\theta$ is an irreducible constituent of $\chi_N$ then $\chi(1)/\theta(1)$ divides $|G:N|$. 

\textbf{proof.} We induct on $|G:N|$. If $|G:N|=1$ then $\chi = \theta$ and so $\chi(1)/\theta(1)=1$. Thus it is clear.

We assume that $N \triangleleft G$. Let $M$ is an maximal normal subgroup of $G$ containing $N$. Since $G/N$ is solvable, $|G:M|=p$ is prime.

Let $\varphi \in \text{Irr}(M)$ be a constituent of $\chi_M$ such that $\theta$ is a constituent of $\varphi_N$.

By inductive hypothesis, $\varphi(1)/\theta(1)$ divides $|M:N|$. 

Now we need $\chi(1)/\varphi(1)$ divides $|G:M|$. Hence since $|G:M|=p$. Thus we have either $\chi_M=\varphi$ is irreducible or $\chi_M=\sum_i \varphi_i[5]$.

If $\chi_M=\varphi$, then $\chi(1)=\varphi(1)$ and $\chi(1)/\varphi(1)=p$ divides $|G:M|$, otherwise $\chi(1)=p\varphi(1)$ and so $\chi(1)/\varphi(1)=p$ divides $|G:M|$. Hence the proof is complete.

Corollary 2. Let $N \triangleleft G$ and assume that $G/N$ is solvable. Let $\chi \in \text{Irr}(G)$ if $(\chi(1), |G:M|)=1$ then $\chi_N$ is irreducible.

\textbf{proof.} Let $\theta$ be an irreducible constituent of $\chi_N$. Then by Proposition 1, $\chi(1)/\varphi(1)$ divides $|G:M|$. 

Thus we have $\chi(1)/\theta(1)=1$ since $(\chi(1), |G:M|)=1$. So $\chi(1)=\varphi(1)$.

Thus $\theta_N=\theta$ is irreducible.

Theorem 3. Let $G$ be an $M$-Group and suppose that $N \triangleleft G$ with $(|N|, |G:M|)=1$. Then $N$ is an $M$-Group.

\textbf{proof.} Let $\theta \in \text{Irr}(N)$ and let $\chi$ be an irreducible constituent of $\theta^G$. Since $G$ is an $M$-Group, $\chi$ is monomial. So $\chi = \lambda^G$ where $\lambda \in \text{Irr}(N)$ is linear for some $H \subseteq G$.

Let $\varphi = \lambda^{NH}$. Then we have $\varphi^G = (\lambda^{NH})^G = \lambda^G = \chi \in \text{Irr}(G)$.

Thus $\varphi \in \text{Irr}(NH)$. Hence we obtain $\varphi(1)=\lambda^{NH}(1)=|NH:H|\lambda(1)=|NH:H|=|N:N \cap H|$.

This divides $|N|$. Since $|N|$ is coprime to $|G:N|$, $(\varphi(1), |G:N|)=1$. But since $|NH:N|$ divides $|G:N|$, we have $(\varphi(1), |NH:N|)=1$. Note that $M$-Group is solvable(Takeda, [5]). Hence $G$ is solvable. So $NH/N$ is solvable. Thus by corollary 2, $\varphi_N$ is irreducible.

But $\varphi_N=(\lambda^{NH})_N=(\lambda_{N\cap H})^N$. So $\varphi_N$ is monomial. Since $\varphi^G = \chi$, by Frobenius
Reciprocity $\varphi$ is a constituent of $\chi_{NH}$. Thus $\varphi_N$ is an irreducible constituent of $(\chi_{NH})_N = \chi_N$. Since $\varphi$ is irreducible constituent of $\chi_N$, by Clifford’s theorem $\theta = (\varphi_N)^k$ for some $g \in G$. Hence $\theta$ is a monomial. The proof is now complete.

2. Characters of solvables

**Theorem 4.** Let $N \triangleleft G$ and suppose that $G/N$ is supersolvable. Let $\chi \in \text{Irr}(G)$. Then

1. If $\chi_N$ is reducible then there exists a subgroup $H$ with $N \triangleleft H \triangleleft G$ such that $|G:H|$ is prime and $\chi$ is induced from irreducible character of $H$.
2. There exists a subgroup $U$ with $N \triangleleft U \triangleleft G$ and a character $\varphi \in \text{Irr}(U)$ such that $\varphi^G = \chi$ and $\varphi_N$ is irreducible.
3. If $G$ is metabelian ($G'' = 1$) then $G$ is an $M$-group.

**Proof.** (1) Let $L \triangleleft G$ be maximal with $N \triangleleft L \triangleleft G$ and $\chi_L$ is reducible. Then $G/N$ is supersolvable. If we take $K \triangleleft G$ such that $K/L$ is chief factor ($K/L$ is minimal normal subgroup of $G/L$), then by the supersolvability of $G/L$, $K/L$ is cyclic with order prime $p$ and $\chi_K \in \text{Irr}(G)$.

Since $(\chi_K)_L = \chi_L$ is reducible, we have

$$\chi_L = \varphi_1 + \varphi_2 + \cdots + \varphi_p,$$

where $\varphi_i \in \text{Irr}(L)$ are distinct [5].

On the other hand, $\chi_L = e \sum_{i=1}^r \theta_i$ where $\{\theta_1, \ldots, \theta_r\}$ is the conjugacy classes of $\theta = \theta_1$ via the action of $G$ on $\text{Irr}(G)$ and $t = |G : I_G(\theta)|$, where $I_G(\theta)$ is inertia group [5].

Hence we have $e = 1$ and $t = p$. If $H = I_G(\theta)$, then $N \triangleleft H \triangleleft G$ and $|G : H| = t = p$ prime. Since $|\chi_L, \theta| = 1 \neq 0$, $\chi \in \text{Irr}(G \mid \theta)$ and thus by Clifford’s correspondence, $\chi$ is induced from some irreducible character of $H$.

(2) Let $U \triangleleft G$ be minimal such that $N \triangleleft H \triangleleft G$ and $\chi$ is induced from some irreducible character of $U$. Let $\psi \in \text{Irr}(U)$ such that $\psi^G = \chi$. Assume that $\psi_N$ is reducible. Then by (1), there is a subgroup $V \triangleleft U$ with $N \triangleleft V \triangleleft U$, $|U : V|$ is prime and $\psi = \theta^V$ for some $\theta \in \text{Irr}(V)$. Thus we have $V \triangleleft U$ and $\chi = \psi^G = (\theta^V)^G = \theta^G$ which contradicts to the minimality of $U$. Hence $\psi_N$ is irreducible.

(3) Let $\chi \in \text{Irr}(G)$, we have $G' \triangleleft G$ and $G/G'$ is abelian. Thus by (2), there exists $U \triangleleft G$ with $G' \triangleleft U \triangleleft G$ and $\varphi \in \text{Irr}(U)$ such that $\chi = \varphi^G$ and $\phi_{G'} \in \text{Irr}(G')$.

But by hypotheses $G'' = 1$, that is, $G'$ is abelian.

Hence all irreducible characters are linear. In particular $\phi_{G'} = \lambda$ is linear. It follows
that \( \phi(1) = \phi \circ (1) = \lambda(1) = 1 \). Hence \( \phi \) itself was linear.

Note that \( G' \leq U \leq G \) implies \( U/G' \leq G/G' \) be abelian, so \( U/G' \triangleleft G/G' \) and so \( U \triangleleft G \) conclude that all \( \chi \in \text{Irr}(G) \) is induced from an irreducible character \( \phi \) of a normal subgroup \( U \triangleleft G \). Thus \( G \) is \( M \)-group.

**Lemma 5.** Let \( \chi \in \text{Irr}(G) \) be primitive and \( N \triangleleft G \). Then \( \chi_N \) is homogeneous.

**proof.** Let \( \theta \) be an irreducible constituent of \( \chi_N \) and \( T = I_G(\theta) \). Then there is \( \phi \in \text{Irr}(T | \theta) \) such that \( \phi^G = \chi \). Primitivity of \( \chi \) yields that \( T = G \). Hence \( \theta \) is invariant in \( G \), so \( \{ \theta \} \) is a \( G \)-orbit in \( \text{Irr}(N) \) and thus \( \theta \) is the only irreducible constituent of \( \chi_N \). Therefore \( \chi_N \) is homogeneous.

**Corollary 6.** Let \( \chi \in \text{Irr}(G) \) be primitive and \( A \triangleleft G \) be abelian. Then \( A \trianglelefteq Z(\chi) \).

**proof.** By Lemma 5, we have \( \chi_A = e \lambda \), where \( \lambda \in \text{Irr}(A) \) is linear. Thus we obtain \( e = \chi(1) \) and if \( a \in A \) then

\[
| \chi(a) | = | \chi(1) \lambda(a) | = \chi(1) | \lambda(a) | = \chi(1),
\]

hence \( A \trianglelefteq Z(\chi) \).

**Corollary 7.** Let \( \chi \in \text{Irr}(G) \) be primitive and \( N = \text{Ker} \chi \). Then every abelian normal subgroup of \( G/N \) is central and cyclic.

**Proof.** If \( N=1 \) (\( \Leftrightarrow \text{Ker} \chi = 1 \Leftrightarrow \chi \) is faithful), then by Corollary 6, \( A \trianglelefteq Z(\chi) = Z(G) \). But \( Z(\chi) \) is cyclic. Thus \( A \) is central and cyclic.

In general, let \( A/N \triangleleft G/N \) and let \( A/N \) be abelian, then \( A \triangleleft G \) and by Lemma 5, \( \chi_A = e \theta \) for \( \theta \in \text{Irr}(A) \). Hence we have \( \chi(1) = e \theta(1) \). If \( n \in N \), then we get \( \chi(1) = \chi(n) = e \theta(n) \) and thus we obtain \( \theta(n) = \theta(1) \). Hence \( N \trianglelefteq \text{Ker} \theta \). But \( \theta \) comes from some irreducible character of \( A/N \). Since \( A/N \) is abelian, \( \theta \) is linear. Thus we have \( \chi(a) = \chi(1) \) for \( a \in A \), so \( A \trianglelefteq Z(\chi) \). But \( Z(\chi)/N = Z(\chi)/\text{Ker} \chi \) is central and cyclic in \( G/N \) [5]. Hence \( A/N \) is central and cyclic in \( G/N \).

**Theorem 8.** Let \( G \) be a solvable. Suppose \( N \triangleleft G \) such that \( G/N \) is supersolvable and every Sylow subgroup of \( N \) for all prime is abelian. Then

1. There exists an abelian normal subgroup \( A \) of \( G \) such that \( A = C_G(A) \).
2. \( G \) is an \( M \)-group.

**proof.** (1) Let \( A \triangleleft G \) be abelian and maximal with the property. Write \( C = C_G(A) \). Then \( A \trianglelefteq C \). Assume that \( A \triangleleft C \). Then \( C/A \triangleleft G/A \). Let \( M/A \) be minimal normal in \( G/A \) with \( M/A \trianglelefteq C/A \). Then \( A \trianglelefteq M \trianglelefteq C \) and \( M/A \) is \( p \)-group, since \( G \) is solvable.
A note on $M$-groups

Case I. $M \subseteq NA$

$M=(M \cap N)A$, and also $M \cap N/A \cap N$ is $p$-group. Thus for some $S \in \text{Syl}_p(M \cap N)$, $M \cap N = S(A \cap N)$. Let $M = S(A \cap N)A = SA$. By hypothesis, $S$ is abelian. Since $S \subseteq M \subseteq C = C_0(A)$ and $A$ and $S$ are abelian, $M = AS$ is abelian and also $M < G$, $M > A$. Hence it contradicts to the maximality of $A$.

Case II. $M \not\subseteq NA$

$NA \cap M < G$ and also $A \subseteq NA \cap M \subseteq M$. By minimality of $M/A$, we have $NA \cap M = A$ and $NA/M = NM$. Claim that $NM/NA$ is minimal normal subgroup of $G/NA$. But $G/NA$ is a homomorphic image of $G/N$. So it is supersolvable. It follows that $NM/NA \cong M/A$ has prime order and is hence cyclic. Thus $M = A(m)$ for $m \in M$. Note that $<m> \subseteq M \subseteq C_0(A)$ and $<m>$, $A$ are abelian. Hence $M$ is abelian which contradicts to the maximality of $A$. Therefore, $A = C = C_0(A)$ and the proof is complete.

(2) Let $\chi \in \text{Irr}(G)$ for a group $G$. Then there is $N \leq G$ such that for some $\psi \in \text{Irr}(N)$, $\psi^G = \chi$ and $\psi$ is primitive. But $N$ is a subgroup of $G$ with the hypothesis. We put $K = \text{Ker}\psi$. Then $N/K$ satisfies the hypothesis. Hence $N/K$ has the property that all of its abelian normal subgroup are central and cyclic. By (1), $N/K$ is abelian. Since $K = \text{Ker}\psi$, $\psi$ comes from an irreducible character of the abelian group $N/K$ and thus $\psi(1) = 1$.

References